SHARP LIPSCHITZ CONSTANTS FOR THE DISTANCE RATIO METRIC

SLAVKO SIMIĆ, MATTI VUORINEN and GENDI WANG

Abstract

We study expansion/contraction properties of some common classes of mappings of the Euclidean space R^n, $n \geq 2$, with respect to the distance ratio metric. The first main case is the behavior of Möbius transformations of the unit ball in R^n onto itself. In the second main case we study the polynomials of the unit disk onto a subdomain of the complex plane. In both cases sharp Lipschitz constants are obtained.

1. Introduction

Conformal invariants and conformally invariant metrics have been some of the key notions of geometric function theory and of quasiconformal mapping theory for several decades. One of the modern trends is to extend this theory to Euclidean spaces of higher dimension or to more general metric spaces and new methods have been invented. It has turned out that we cannot any more expect the same invariance properties as in the classical cases, but still some type of "nearinvariance" or "quasi-invariance" is a desirable feature. The quasihyperbolic metric of a domain $G \subset \mathsf{R}^n$, $n \geq 2$ is one of these new notions and it is known to be quasi-invariant in the sense described below and there are many reasons why one may regard it as a spatial version of the hyperbolic metric. This notion is also applicable in the wider context of metric spaces. For example, J. Väisälä's theory of quasiconformal maps in infinite dimensional Banach spaces is entirely based on the quasihyperbolic metric [12]. Another crucial notion, which is the object of the present study, is the distance ratio metric. It is often applied to study quasihyperbolic geometry. Very little is known about the geometric properties of balls in these two spaces: only recently it was proved for instance that in some special cases for small radii the balls are convex [7], [8], [10], [13], [11] no matter where the center is. This convexity property is expected to hold in general, but it has not been proved yet.

We next define these two metrics.

Distance ratio metric. For a proper subset G of R^n and for all $x, y \in G$, the distance-ratio metric j_G or j-metric is defined as

$$j_G(x, y) = \log\left(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}}\right),$$

where $d(x, \partial G)$ denotes the Euclidean distance from the point x to the boundary ∂G of the domain G. The distance ratio metric was introduced by F. W. Gehring and B. P. Palka [5] and in the above simplified form by M. Vuorinen [14]. It is frequently used in the study of hyperbolic type metrics [6], geometric theory of functions [15], and quasiconformality in Banach spaces [12].

Let $p \in G$, then for all $x, y \in G \setminus \{p\}$

$$j_{G\setminus\{p\}}(x, y) = \log\left(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G), |x - p|, |y - p|\}}\right).$$

This formula shows that the j-metric highly depends on the boundary of the domain.

Quasihyperbolic metric. Let G be a proper subdomain of R^n. For all $x, y \in G$, the quasihyperbolic metric k_G is defined as

$$k_G(x, y) = \inf_\gamma \int_\gamma \frac{1}{d(z, \partial G)} \, |dz|,$$

where the infimum is taken over all rectifiable arcs γ joining x to y in G.

It should be noted that these metrics can be estimated one in terms of the other and the hyperbolic metrics, ρ_{B^n} and ρ_{H^n}, of the unit ball B^n and the half space H^n, resp. For instance, below we shall apply the following basic inequalities [15, Lemma 2.41] and [1, Lemma 7.56]:

(1.1) $\dfrac{1}{2}\rho_{\mathsf{B}^n}(x, y) \leq j_{\mathsf{B}^n}(x, y) \leq k_{\mathsf{B}^n}(x, y) \leq \rho_{\mathsf{B}^n}(x, y), \forall\, x, y \in \mathsf{B}^n,$

(1.2) $\dfrac{1}{2}\rho_{\mathsf{H}^n}(x, y) \leq j_{\mathsf{H}^n}(x, y) \leq k_{\mathsf{H}^n}(x, y) \equiv \rho_{\mathsf{H}^n}(x, y), \forall\, x, y \in \mathsf{H}^n.$

The hyperbolic metric in the unit ball or half space is Möbius invariant. However, neither the quasihyperbolic metric nor the distance ratio metric is invariant under Möbius transformations. Therefore, it is natural to ask what the Lipschitz constants are for these metrics under conformal mappings or Möbius transformations in higher dimension. F. W. Gehring, B. G. Osgood, and B. P. Palka proved that these metrics are not changed by more than a factor 2 under Möbius transformations, see [4, proof of Theorem 4] and [5, Corollary 2.5]:

THEOREM 1.3. *If D and D' are proper subdomains of R^n and if f is a Möbius transformation of D onto D', then for all $x, y \in D$*

$$\frac{1}{2} m_D(x, y) \le m_{D'}(f(x), f(y)) \le 2 m_D(x, y),$$

where $m \in \{j, k\}$.

R. Klén, M. Vuorinen, and X.-H. Zhang studied the sharpness of the constant 2 in Theorem 1.3. They got the sharp bilipschitz constant $1 + |a|$ for the quasihyperbolic metric under Möbius self-mappings of the unit ball [9, Theorem 1.4], and proposed a conjecture for the distance ratio metric.

CONJECTURE 1.4 ([9, Conjecture 2.3]). *Let $a \in \mathsf{B}^n$ and $h : \mathsf{B}^n \to \mathsf{B}^n = h\mathsf{B}^n$ be a Möbius transformation with $h(a) = 0$. Then*

$$\sup_{\substack{x, y \in \mathsf{B}^n \\ x \ne y}} \frac{j_{\mathsf{B}^n}(h(x), h(y))}{j_{\mathsf{B}^n}(x, y)} = 1 + |a|.$$

The following Theorem 1.5 shows that the solution to Conjecture 1.4 is in the affirmative and thus the Gehring-Osgood-Palka Theorem 1.3 admits a refinement for Möbius transformations of the unit ball onto itself.

THEOREM 1.5. *Let $a \in \mathsf{B}^n$ and $f : \mathsf{B}^n \to \mathsf{B}^n = f\mathsf{B}^n$ be a Möbius transformation with $f(a) = 0$. Then for all $x, y \in \mathsf{B}^n$*

$$\frac{1}{1 + |a|} j_{\mathsf{B}^n}(x, y) \le j_{\mathsf{B}^n}(f(x), f(y)) \le (1 + |a|) j_{\mathsf{B}^n}(x, y),$$

and the constants $\frac{1}{1+|a|}$ and $1 + |a|$ are both the best possible.

We also study how the j-metric behaves under polynomials, or more generally, analytic functions of the unit disk onto a subdomain of the complex plane. In this direction, our main results are the following two theorems.

THEOREM 1.6. *Let $p \in \mathsf{N}$ and $\{a_l\}$ be a sequence of complex numbers with $\sum_{l=1}^{p} |a_l| \le 1$. Let $f : \mathsf{B}^2 \to \mathsf{B}^2$ with $f(\mathsf{B}^2 \setminus \{0\}) \subset \mathsf{B}^2 \setminus \{0\}$ and $f(z) = \sum_{l=1}^{p} a_l z^l$ and $f(0) = 0$. Then for all $x, y \in \mathsf{B}^2 \setminus \{0\}$*

$$j_{\mathsf{B}^2 \setminus \{0\}}(f(x), f(y)) \le p j_{\mathsf{B}^2 \setminus \{0\}}(x, y),$$

and the constant p is sharp.

THEOREM 1.7. *Let $\{a_l\}$ be a sequence of complex numbers and $\sum_{l=0}^{\infty} |a_l| \le 1$. Let $f : \mathsf{B}^2 \to \mathsf{B}^2$ be a non-constant analytic function defined by $f(z) =$*

$\sum_{l=0}^{\infty} a_l z^l$. *Then for all* $x, y \in \mathsf{B}^2$

$$j_{\mathsf{B}^2}(f(x), f(y)) \le j_{\mathsf{B}^2}(x, y),$$

and this inequality is sharp.

2. Lipschitz Constants under some special mappings

We recall here some basic facts about Möbius transformations from [3], [15].

Möbius transformations. The group of Möbius transformations in $\overline{\mathsf{R}^n}$ is generated by transformations of two types:

(1) reflections in the hyperplane $P(a, t) = \{x \in \mathsf{R}^n : x \cdot a = t\} \cup \{\infty\}$

$$f_1(x) = x - 2(x \cdot a - t)\frac{a}{|a|^2}, \qquad f_1(\infty) = \infty;$$

(2) inversions (reflections) in the sphere $S^{n-1}(a, r) = \{x \in \mathsf{R}^n : |x-a| = r\}$

$$f_2(x) = a + \frac{r^2(x - a)}{|x - a|^2}, \qquad f_2(a) = \infty, f_2(\infty) = a.$$

If $G \subset \overline{\mathsf{R}^n}$ we denote by $\mathcal{GM}(G)$ the group of all Möbius transformations which map G onto itself.

We denote $a^* = \frac{a}{|a|^2}$ for $a \in \mathsf{R}^n \setminus \{0\}$, and $0^* = \infty$, $\infty^* = 0$. For fixed $a \in \mathsf{B}^n \setminus \{0\}$, let

(2.1) $$\sigma_a(z) = a^* + r^2(z - a^*)^*, \qquad r^2 = |a|^{-2} - 1$$

be an inversion in the sphere $S^{n-1}(a^*, r)$ orthogonal to S^{n-1}. Then $\sigma_a(a) = 0$, $\sigma_a(a^*) = \infty$ and by [3, 3.1.5], [15, (1.5)]

(2.2) $$|\sigma_a(x) - \sigma_a(y)| = \frac{r^2|x - y|}{|x - a^*||y - a^*|}.$$

Lipschitz mappings. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be continuous and let $L \ge 1$. We say that f is L-lipschitz if

$$d_Y(f(x), f(y)) \le L d_X(x, y), \qquad \text{for} \quad x, y \in X,$$

and L-bilipschitz if f is a homeomorphism and

$$d_X(x, y)/L \le d_Y(f(x), f(y)) \le L d_X(x, y), \qquad \text{for} \quad x, y \in X,$$

A 1-bilipschitz mapping is called an isometry.

The next lemma, so-called *monotone form of l'Hôpital's rule*, has found recently numerous applications in proving inequalities. See the extensive bibliography of [2].

LEMMA 2.3. [1, Theorem 1.25] *For* $-\infty < a < b < \infty$, *let* $f, g : [a, b] \to$ R *be continuous on* $[a, b]$, *and be differentiable on* (a, b), *and let* $g'(x) \neq 0$ *on* (a, b). *If* $f'(x)/g'(x)$ *is increasing (deceasing) on* (a, b), *then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad and \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

THEOREM 2.4. *Let* $f : \mathsf{H}^2 \to \mathsf{B}^2$ *with* $f(z) = \frac{z-i}{z+i}$.

(1) *For all* $x, y \in \mathsf{H}^2$

$$j_{\mathsf{B}^2}(f(x), f(y)) \leq 2 j_{\mathsf{H}^2}(x, y),$$

and the constant 2 is the best possible.

(2) *For all* $x, y \in \mathsf{H}^2 \setminus \{i\}$

$$j_{\mathsf{B}^2 \setminus \{0\}}(f(x), f(y)) \leq 2 j_{\mathsf{H}^2 \setminus \{i\}}(x, y),$$

and the constant 2 is the best possible.

PROOF. It suffices to show the sharpness of the inequalities by Theorem 1.3.
(1) Let $x = t + ai$ and $y = ai$, where $a > 0$ and $t > 0$. Then

$$\lim_{t \to \infty} \frac{j_{\mathsf{B}^2}(f(t+ai), f(ai))}{j_{\mathsf{H}^2}(t+ai, ai)}$$

$$= \lim_{t \to \infty} \frac{\log\left(1 + \frac{t}{2a(a+1)}\left(\sqrt{t^2 + (a+1)^2} + \sqrt{t^2 + (a-1)^2}\right)\right)}{\log(1 + \frac{t}{a})}$$

$$= \lim_{t \to \infty} \frac{\log\left(\frac{2t^2}{2a(a+1)}\right)}{\log(\frac{t}{a})} = 2.$$

Hence the constant 2 is the best possible.

(2) Let $x = t + ai$ and $y = ai$, where $0 < a < \frac{1}{3}$ and $t > 1$. Then

$$\frac{j_{\mathsf{B}^2 \setminus \{0\}}(f(x), f(y))}{j_{\mathsf{H}^2 \setminus \{i\}}(x, y)} = \frac{j_{\mathsf{B}^2}(f(x), f(y))}{j_{\mathsf{H}^2}(x, y)}.$$

By (1) we obtain that the constant 2 is the best possible.

THEOREM 2.5. *Let* $f : \mathsf{B}^2 \to \mathsf{H}^2$ *with* $f(z) = i\frac{1+z}{1-z}$.

(1) *For all* $x, y \in \mathsf{B}^2$

$$j_{\mathsf{H}^2}(f(x), f(y)) \leq 2 j_{\mathsf{B}^2}(x, y),$$

and the constant 2 *is the best possible.*

(2) *For all* $x, y \in \mathsf{B}^2 \setminus \{0\}$

$$j_{\mathsf{H}^2 \setminus \{i\}}(f(x), f(y)) \leq 2 j_{\mathsf{B}^2 \setminus \{0\}}(x, y),$$

and the constant 2 *is the best possible.*

PROOF. By Theorem 1.3, we only need to show that 2 is the best possible.

(1) For $t = x = -y, t \in (0, 1)$, we get

$$j_{\mathsf{H}^2}(f(x), f(y)) = \log\left(1 + \frac{4t}{(1-t)^2}\right) = 2\log\left(1 + \frac{2t}{1-t}\right) = 2 j_{\mathsf{B}^2}(x, y).$$

Therefore, the constant 2 is the best possible.

(2) Since for $t = x = -y$ and $\frac{1}{2} < t < 1$, we have

$$j_{\mathsf{H}^2 \setminus \{i\}}(f(x), f(y)) = j_{\mathsf{H}^2}(f(x), f(y)) = 2 j_{\mathsf{B}^2}(x, y) = 2 j_{\mathsf{B}^2 \setminus \{0\}}(x, y),$$

the constant 2 is the best possible.

THEOREM 2.6. *Let* $a \in \mathsf{H}^2$ *and* $f : \mathsf{H}^2 \to \mathsf{B}^2 = f\mathsf{H}^2$ *be a Möbius transformation with* $f(a) = 0$.

(1) *For all* $x, y \in \mathsf{H}^2$

$$\frac{1}{2} j_{\mathsf{H}^2}(x, y) \leq j_{\mathsf{B}^2}(f(x), f(y)) \leq 2 j_{\mathsf{H}^2}(x, y),$$

and the constants $\frac{1}{2}$ *and* 2 *are both the best possible.*

(2) *For all* $x, y \in \mathsf{H}^2 \setminus \{a\}$

$$\frac{1}{2} j_{\mathsf{H}^2 \setminus \{a\}}(x, y) \leq j_{\mathsf{B}^2 \setminus \{0\}}(f(x), f(y)) \leq 2 j_{\mathsf{H}^2 \setminus \{a\}}(x, y),$$

and the constants $\frac{1}{2}$ *and* 2 *are both the best possible.*

PROOF. A Möbius transformation satisfying the condition in Theorem 2.6 is of the form

$$f(z) = e^{i\alpha}\frac{z - a}{z - \bar{a}}$$

and hence

$$f^{-1}(z) = \frac{a - \bar{a}e^{-i\alpha}z}{1 - e^{-i\alpha}z},$$

where α is a real constant. Since j-metric is invariant under translations, stretchings of H^2 onto itself and rotations of B^2 onto itself, we may assume that $a = i$ and $\alpha = 0$. Then we have

$$f(z) = \frac{z - i}{z + i} \qquad \text{and} \qquad f^{-1}(z) = i\frac{1 + z}{1 - z}.$$

By Theorem 2.4 and Theorem 2.5, we get the results immediately.

LEMMA 2.7. *Let $m \in (1, \infty)$, $\theta \in \left(0, \frac{\pi}{2m}\right)$, and $r \in (0, 1)$. Then*

(1) $f_1(\theta) \equiv \frac{m\sin\theta}{1 - \sin\theta} - \frac{\sin m\theta}{1 - \sin m\theta}$ *is decreasing from $\left(0, \frac{\pi}{2m}\right)$ onto $(-\infty, 0)$;*

(2) $f_2(r) \equiv \frac{1 - (1 - \sin\theta)r}{1 - (1 - \sin m\theta)r^m}$ *is decreasing from $(0, 1)$ onto $\left(\frac{\sin\theta}{\sin m\theta}, 1\right)$;*

(3) $f_3(r) \equiv \frac{\log\left(1 + \frac{1 - r^m}{r^m \sin m\theta}\right)}{\log\left(1 + \frac{1 - r}{r\sin\theta}\right)}$ *is decreasing from $(0, 1)$ onto $\left(\frac{m\sin\theta}{\sin m\theta}, m\right)$.*

PROOF. (1) By differentiation,

$$f_1'(\theta) = m(g(1) - g(m)),$$

where $g(m) = \frac{\cos m\theta}{(1 - \sin m\theta)^2}$. Since

$$g'(m) = \frac{\theta(1 - \sin m\theta + \cos^2 m\theta)}{(1 - \sin m\theta)^3} > 0,$$

we have $f_1'(\theta) < 0$ and hence f_1 is decreasing on $(0, \pi/2m)$. The limiting values are clear.

(2) By differentiation,

$$\begin{aligned}
\left(1 - (1 - \sin m\theta)r^m\right)^2 f_2'(r) &= -(m - 1)(1 - \sin\theta)(1 - \sin m\theta)r^m \\
&\quad + m(1 - \sin m\theta)r^{m-1} - (1 - \sin\theta) \\
&\equiv h(r).
\end{aligned}$$

Since

$$h'(r) = m(m - 1)(1 - \sin m\theta)r^{m-2}(1 - r(1 - \sin\theta)) > 0,$$

we have h is increasing on $(0, 1)$ and hence $h(r) \leq h(1)$. By (1),

$$h(1) = (1 - \sin\theta)(1 - \sin m\theta)f_1(\theta) < 0.$$

Therefore, $f_2'(r) < 0$ and hence f_2 is decreasing on $(0, 1)$. The limiting values are clear.

(3) Let $f_3(r) \equiv \frac{g_1(r)}{g_2(r)}$, here $g_1(r) = \log\left(1 + \frac{1-r^m}{r^m \sin m\theta}\right)$ and $g_2(r) = \log\left(1 + \frac{1-r}{r \sin \theta}\right)$. Then $g_1(1^-) = g_2(1^-) = 0$, by differentiation, we have

$$\frac{g_1'(r)}{g_2'(r)} = m f_2(r).$$

Therefore, f_3 is decreasing on $(0, 1)$ by Lemma 2.3. By l'Hôpital's rule and (2) we easily get the limiting values.

The planar angular domain is defined as

$$S_\varphi = \{re^{i\theta} \in \mathsf{C} : 0 < \theta < \varphi, r > 0\}.$$

LEMMA 2.8. *Let* $f : S_{\pi/m} \to \mathsf{H}^2$ *with* $f(z) = z^m$ $(m \geq 1)$. *Let* $x, y \in S_{\pi/m}$ *with* $\arg(x) = \arg(y) = \theta$. *Then for all* $x, y \in S_{\pi/m}$

$$\frac{m \sin \theta}{\sin m\theta} j_{S_{\pi/m}}(x, y) \leq j_{\mathsf{H}^2}(f(x), f(y)) \leq m j_{S_{\pi/m}}(x, y).$$

PROOF. Since $j-$metric is invariant under stretchings, we may assume that $x = re^{i\theta}$ and $y = e^{i\theta}, 0 < r < 1$. By symmetry, we also assume $0 < \theta \leq \frac{\pi}{2m}$. Then

$$\frac{j_{\mathsf{H}^2}(f(x), f(y))}{j_{S_{\pi/m}}(x, y)} = \frac{\log(1 + \frac{1-r^m}{r^m \sin m\theta})}{\log(1 + \frac{1-r}{r \sin \theta})}.$$

By Lemma 2.7(3), we get the result.

LEMMA 2.9. *Let* $n \in \mathsf{N}, 0 < \theta \leq \frac{\pi}{2n}$. *Then for* $x, y \in \mathsf{C} \setminus \{0\}$,

$$1 + \frac{|x^n - y^n|}{|x|^n \sin n\theta} \leq \left(1 + \frac{|x - y|}{|x| \sin \theta}\right)^n.$$

PROOF. The equality holds if $n = 1$.

Next, suppose that the inequality holds when $n = m$. Namely,

(2.10) $$1 + \frac{|x^m - y^m|}{|x|^m \sin m\theta} \leq \left(1 + \frac{|x - y|}{|x| \sin \theta}\right)^m,$$

where $0 < \theta \leq \frac{\pi}{2m}$.

Then, if $n = m + 1$, we have $0 < \theta \leq \frac{\pi}{2(m+1)} < \frac{\pi}{2m}$ and by (2.10),

$$\left(1 + \frac{|x-y|}{|x|\sin\theta}\right)^{m+1} \geq \left(1 + \frac{|x^m - y^m|}{|x|^m \sin m\theta}\right)\left(1 + \frac{|x-y|}{|x|\sin\theta}\right)$$

$$\geq 1 + \frac{|x-y|}{|x|\sin\theta} + \frac{|x^m - y^m|}{|x|^m \sin m\theta}\left(1 + \frac{|x-y|}{|x|}\right)$$

$$\geq 1 + \frac{|x^{m+1} - x^m y|}{|x|^{m+1}\sin\theta} + \frac{|x^m y - y^{m+1}|}{|x|^{m+1}\sin m\theta}$$

$$\geq 1 + \frac{|x^{m+1} - y^{m+1}|}{|x|^{m+1}\sin(m+1)\theta}.$$

This completes the proof by induction.

THEOREM 2.11. *Let $f : S_{\pi/m} \to \mathsf{H}^2$ with $f(z) = z^m$ ($m \in \mathsf{N}$). Then for all $x, y \in S_{\pi/m}$,*

$$j_{\mathsf{H}^2}(f(x), f(y)) \leq m j_{S_{\pi/m}}(x, y),$$

and the constant m is the best possible.

PROOF. By symmetry, we may assume that $d(f(x), \partial\mathsf{H}^2) \leq d(f(y), \partial\mathsf{H}^2)$. Hence by Lemma 2.9, we obtain

$$j_{\mathsf{H}^2}(f(x), f(y)) = \log\left(1 + \frac{|x^m - y^m|}{|x|^m \sin m\theta}\right)$$

$$\leq m \log\left(1 + \frac{|x-y|}{|x|\sin\theta}\right)$$

$$\leq m j_{S_{\pi/m}}(x, y),$$

where $0 < \theta = \min\left\{\arg(x), \frac{\pi}{m} - \arg(x)\right\} \leq \frac{\pi}{2m}$.

Let $x = re^{i\alpha}$ and $y = e^{i\alpha}$, where $0 < \alpha < \frac{\pi}{2m}$ and $0 < r < 1$. Letting $r \to 0$, and by Lemma 2.7(3) and Lemma 2.8, we know that the constant m is the best possible.

LEMMA 2.12. *For $z \in \mathsf{C}$ and $p \in \mathsf{N}$ we have*

$$\log(1 + |z^p - 1|) \leq p \log(1 + |z - 1|).$$

PROOF. Putting $z = u + 1$, we obtain

$$1 + |z^p - 1| = 1 + \left|\sum_{l=1}^{p} \binom{p}{l} u^l\right| \leq 1 + \sum_{l=1}^{p} \binom{p}{l} |u|^l = (1 + |u|)^p = (1 + |z - 1|)^p,$$

and the claim follows.

LEMMA 2.13. *Let Q_d^* denote the class of all polynomials with exact degree d ($d \geq 1$) which have no zeros inside the disk B^2.*
If $Q \in Q_d^*$ and $u, v \in \mathsf{B}^2$, then

$$\left| \frac{Q(u)}{Q(v)} - 1 \right| \leq \left(1 + \frac{|u - v|}{1 - |v|} \right)^d - 1.$$

PROOF. A polynomial $Q(z)$ with zeros $\{-a_l\}$ ($l = 1, 2, \ldots, d$) has a representation in the form

$$Q(z) = C \prod_{l=1}^{d} (z + a_l), \qquad C \neq 0.$$

Since $Q(z) \neq 0$ for $z \in \mathsf{B}^2$, we have $|a_l| \geq 1$.
Therefore,

$$\left| \frac{Q(u)}{Q(v)} - 1 \right| = \left| \prod_{l=1}^{d} \frac{u + a_l}{v + a_l} - 1 \right| = \left| \prod_{l=1}^{d} \left(1 + \frac{u - v}{a_l + v} \right) - 1 \right|$$

$$= \left| \sum \prod (\cdot)(\cdot) \right| \leq \sum \prod |(\cdot)||(\cdot)|$$

$$= \prod_{l=1}^{d} \left(1 + \frac{|u - v|}{|a_l + v|} \right) - 1 \leq \prod_{l=1}^{d} \left(1 + \frac{|u - v|}{|a_l| - |v|} \right) - 1$$

$$\leq \left(1 + \frac{|u - v|}{1 - |v|} \right)^d - 1.$$

REMARK 2.14. Note that, since $\left| \frac{Q(u)}{Q(v)} - 1 \right| \geq \left| \frac{Q(u)}{Q(v)} \right| - 1$, we also obtain

$$\left| \frac{Q(u)}{Q(v)} \right| \leq \left(1 + \frac{|u - v|}{1 - |v|} \right)^d.$$

2.15. *Proof of Theorem 1.6*
For $x, y \in \mathsf{B}^2 \setminus \{0\}$, we have

$$j_{\mathsf{B}^2 \setminus \{0\}}(x, y) = \log \left(1 + \frac{|x - y|}{\min\{|x|, |y|, 1 - |x|, 1 - |y|\}} \right)$$

and

$$j_{\mathsf{B}^2\setminus\{0\}}(f(x), f(y)) = \log\left(1 + \frac{|f(x) - f(y)|}{T}\right),$$

where $T = \min\{|f(x)|, |f(y)|, 1 - |f(x)|, 1 - |f(y)|\}$.

Case 1. $T = 1 - |f(x)|$. Since

$$|f(x) - f(y)| = |x - y|\left|\sum_{l=1}^{p} a_l\left(\sum_{\nu+\mu=l-1} x^\nu y^\mu\right)\right|$$

$$\leq |x - y|\sum_{l=1}^{\infty} |a_l|\left(\sum_{\nu=0}^{l-1} |x|^\nu\right)$$

and

$$1 - |f(x)| \geq \sum_{l=1}^{p} |a_l| - \sum_{l=1}^{p} |a_l||x|^l = (1 - |x|)\sum_{l=1}^{p} |a_l|\left(\sum_{\nu=0}^{l-1} |x|^\nu\right),$$

we obtain

$$j_{\mathsf{B}^2\setminus\{0\}}(f(x), f(y)) \leq \log\left(1 + \frac{|x - y|}{1 - |x|}\right) \leq j_{\mathsf{B}^2\setminus\{0\}}(x, y).$$

Case 2. $T = 1 - |f(y)|$. This case can be treated in the same way as Case 1.

Case 3. $T = |f(y)|$. Now we assume that 0 is m-th order zero of f. Since f has no other zeros in B^2 by the assumption of this theorem, we get $f(z) = z^m Q(z)$, $Q \in Q_d^*$, $m + d = p$, here Q_d^* is as in Lemma 2.13.

By Lemma 2.13 and Remark 2.14, it follows

$$\frac{|f(x) - f(y)|}{|f(y)|} = \left|\frac{x^m Q(x)}{y^m Q(y)} - 1\right| = \left|\left(\frac{x^m}{y^m} - 1\right)\frac{Q(x)}{Q(y)} + \frac{Q(x)}{Q(y)} - 1\right|$$

$$\leq \left|\frac{x^m}{y^m} - 1\right|\left|\frac{Q(x)}{Q(y)}\right| + \left|\frac{Q(x)}{Q(y)} - 1\right|$$

$$\leq \left(1 + \left|\frac{x^m}{y^m} - 1\right|\right)\left(1 + \frac{|x - y|}{1 - |y|}\right)^d - 1.$$

Therefore, by Lemma 2.12, we have

$$j_{\mathsf{B}^2\setminus\{0\}}(f(x), f(y)) = \log\left(1 + \frac{|f(x) - f(y)|}{|f(y)|}\right)$$

$$\leq d \log\left(1 + \frac{|x - y|}{1 - |y|}\right) + \log\left(1 + \left|\frac{x^m}{y^m} - 1\right|\right)$$

$$\leq d \log\left(1 + \frac{|x - y|}{1 - |y|}\right) + m \log\left(1 + \frac{|x - y|}{|y|}\right)$$

$$\leq p j_{\mathsf{B}^2\setminus\{0\}}(x, y),$$

and the proof for Case 3 is done.

Case 4. $T = |f(x)|$. This case can be treated in the same way as Case 3.

For the sharpness of the inequality, let $f(z) = z^p$ ($p \in \mathsf{N}$). For $s, t \in \left(0, \frac{1}{2}\right)$ and $s < t$, we have

$$j_{\mathsf{B}^2\setminus\{0\}}(f(t), f(s)) = \log\left(1 + \frac{t^p - s^p}{s^p}\right) = p \log\left(\frac{t}{s}\right) = p j_{\mathsf{B}^2\setminus\{0\}}(t, s).$$

Therefore the constant p is sharp.

2.16. *Proof of Theorem 1.7*

Let $r = \max\{|x|, |y|\}$ and suppose that $|f(x)| \geq |f(y)|$. Then

$$j_{\mathsf{B}^2}(x, y) = \log\left(1 + \frac{|x - y|}{1 - r}\right)$$

and

$$j_{\mathsf{B}^2}(f(x), f(y)) = \log\left(1 + \frac{|f(x) - f(y)|}{1 - |f(x)|}\right).$$

The next two inequalities follow by the proof of Theorem 1.6 Case 1 (replace p with ∞):

$$|f(x) - f(y)| \leq |x - y| \sum_{l=1}^{\infty} |a_l| \left(\sum_{\nu=0}^{l-1} |x|^\nu\right)$$

and

$$1 - |f(x)| \geq (1 - |x|) \sum_{l=1}^{\infty} |a_l| \left(\sum_{\nu=0}^{l-1} |x|^\nu\right).$$

Therefore,

$$j_{\mathsf{B}^2}(f(x), f(y)) \leq \log\left(1 + \frac{|x - y|}{1 - |x|}\right) \leq j_{\mathsf{B}^2}(x, y).$$

Obviously, the equality holds for the identity map and hence the sharpness is clear.

REMARKS 2.17. 1. Let $f : \mathsf{B}^2 \to \mathsf{B}^2$ be an analytic function. According to the Schwarz lemma, for all x, $y \in \mathsf{B}^2$ we have

$$\rho_{\mathsf{B}^2}(f(x), f(y)) \leq \rho_{\mathsf{B}^2}(x, y),$$

and further, by the inequality (1.1),

(2.18) $$j_{\mathsf{B}^2}(f(x), f(y)) \leq 2 j_{\mathsf{B}^2}(x, y).$$

Note that Theorem 1.7 gives a sufficient condition for the mapping to be a contraction, i.e., to have the Lipschitz constant at most 1.

2. Following [14, Example 3.10] consider the exponential function $f : \mathsf{B}^2 \to \mathsf{B}^2 \setminus \{0\} = f\mathsf{B}^2$ with $f(z) = \exp\left(\frac{z+1}{z-1}\right)$. Let $z_l = \frac{e^l-1}{e^l+1}$ ($l \in \mathsf{N}$). Then

$$j_{\mathsf{B}^2}(z_l, z_{l+1}) = \log \frac{e^{l+1} + 1}{e^l + 1} \to 1 \qquad (l \to \infty)$$

and

$$j_{f\mathsf{B}^2}(f(z_l), f(z_{l+1})) = e^l(e - 1) \to \infty \qquad (l \to \infty).$$

Therefore, in (2.18) $j_{\mathsf{B}^2}(f(x), f(y))$ can not be replaced with $j_{f\mathsf{B}^2}(f(x), f(y))$ if $f\mathsf{B}^2 \subset \mathsf{B}^2$ has isolated boundary points.

Furthermore, we have $|f(0)| + |f'(0)| = 3/e > 1$ and for $0 < t < 1$

$$\lim_{t \to 0} \frac{j_{\mathsf{B}^2}(f(0), f(t))}{j_{\mathsf{B}^2}(0, t)} = \lim_{t \to 0} \frac{\log\left(1 + \frac{e^{-1} - e^{(t+1)/(t-1)}}{1 - e^{-1}}\right)}{\log\left(1 + \frac{t}{1-t}\right)}$$

$$= \lim_{t \to 0} \frac{2}{1 - t} \frac{1}{e^{-(t+1)/(t-1)} - 1}$$

$$= \frac{2}{e - 1} > 1.$$

Therefore, in Theorem 1.7 the hypothesis $\sum_{l=0}^{\infty} |a_l| \leq 1$ cannot be removed.

THEOREM 2.19. *Let* $f(z) = a + r^2 \frac{(z-a)}{|z-a|^2}$ *be the inversion in* $S^{n-1}(a, r)$ *with* Im $a = 0$. *Then* $f(\mathsf{H}^n) = \mathsf{H}^n$ *and for all* x, $y \in \mathsf{H}^n$,

$$j_{\mathsf{H}^n}(f(x), f(y)) \leq 2 j_{\mathsf{H}^n}(x, y).$$

The constant 2 is the best possible.

PROOF. The inequality is clear by Theorem 1.3. Without loss of generality, we may assume that $a = 0$ and $r = 1$.

Putting $x = i$, $y = t + i$, $t > 0$, we have

$$\lim_{t \to \infty} \frac{j_{\mathsf{H}^n}(f(i), f(t+i))}{j_{\mathsf{H}^n}(i, t+i)} = \lim_{t \to \infty} \frac{\log(1 + t\sqrt{1 + t^2})}{\log(1 + t)} = \lim_{t \to \infty} \frac{\log t^2}{\log t} = 2.$$

Hence the constant 2 is the best possible.

3. Proof of Theorem 1.5

In order to prove Theorem 1.5, we first need some lemmas.

LEMMA 3.1 ([3, Theorem 3.5.1]). *Let f be a Möbius transformation and $f(\mathsf{B}^n) = \mathsf{B}^n$. Then*

$$f(x) = (\sigma x)A,$$

where σ is an inversion in some sphere orthogonal to S^{n-1} and A is an orthogonal matrix.

LEMMA 3.2. *Let $a, b \in \mathsf{B}^n$. Then*

(1) $|a|^2|b - a^*|^2 - |b - a|^2 = (1 - |a|^2)(1 - |b|^2)$;

(2) $\dfrac{||b| - |a||}{1 - |a||b|} \leq \dfrac{|b - a|}{|a||b - a^*|} \leq \dfrac{|b| + |a|}{1 + |a||b|}.$

PROOF. (1) By calculation, we have

$$\begin{aligned}
|a|^2|b - a^*|^2 - |b - a|^2 &= |a|^2\left(|b|^2 + \frac{1}{|a|^2} - \frac{2(b \cdot a)}{|a|^2}\right) \\
&\quad - \left(|b|^2 + |a|^2 - 2(b \cdot a)\right) \\
&= 1 + |a|^2|b|^2 - |a|^2 - |b|^2 \\
&= (1 - |a|^2)(1 - |b|^2).
\end{aligned}$$

(2) This can be directly obtained by [15, Exercise 2.52].

LEMMA 3.3. *Let $c, d \in (0, 1)$, $\theta \in (0, 1]$.*

(1) $f(\theta) \equiv \dfrac{\log\left(1 + \frac{2cd\theta}{1 - cd}\right)}{\log\left(1 + \frac{2d\theta}{1 - d}\right)}$ *is increasing. In particular,*

$$\frac{\log\left(1 + \frac{2cd\theta}{1 - cd}\right)}{\log\left(1 + \frac{2d\theta}{1 - d}\right)} \leq \frac{\log\left(1 + \frac{2cd}{1 - cd}\right)}{\log\left(1 + \frac{2d}{1 - d}\right)}.$$

(2) $g(\theta) \equiv \dfrac{\text{arth}(c\theta)}{\text{arth }\theta}$ *is decreasing. In particular,* $\dfrac{\text{arth}(c\theta)}{\text{arth }\theta} \leq c.$

$$(3) \quad \left(1 + \frac{2cd\theta}{1 - cd}\right)\left(1 + \frac{c(1-d)}{1+cd}\right) \leq 1 + \frac{c(1-d) + 2cd\theta}{1 - cd}.$$

PROOF. (1) Let $f_1(\theta) = \log\left(1 + \frac{2cd\theta}{1-cd}\right)$ and $f_2(\theta) = \log\left(1 + \frac{2d\theta}{1-d}\right)$. Then we have $f_1(0^+) = f_2(0^+) = 0$ and

$$\frac{f_1'(\theta)}{f_2'(\theta)} = 1 - \frac{1-c}{1 - cd + 2cd\theta},$$

which is clearly increasing in θ. Therefore, the monotonicity of f immediately follows by Lemma 2.3. The inequality follows by the monotonicity of f.

(2) Let $g_1(\theta) = \mathrm{arth}(c\theta)$ and $g_2(\theta) = \mathrm{arth}\,\theta$. Then we have $g_1(0^+) = g_2(0^+) = 0$ and

$$\frac{g_1'(\theta)}{g_2'(\theta)} = \frac{1}{c}\left(1 - \frac{1-c^2}{1 - c^2\theta^2}\right),$$

which is clearly decreasing in θ. Therefore, the monotonicity of g follows by Lemma 2.3. The inequality immediately follows by the monotonicity of g and l'Hôpital's rule.

(3) This inequality can be easily proved by direct calculation.

Now we are in a position to give a short proof of Theorem 1.5.

3.4. *Proof of Theorem 1.5*

The claim is trivial for $a = 0$, therefore, we only need to consider $a \neq 0$. Since j-metric is invariant under orthogonal transformations and by Lemma 3.1, for $x, y, a \in \mathsf{B}^n$, we have

$$j_{\mathsf{B}^n}(f(x), f(y)) = j_{\mathsf{B}^n}(\sigma_a(x), \sigma_a(y)),$$

where $\sigma_a(x)$ is an inversion in the sphere $S^{n-1}(a^*, \sqrt{|a|^{-2} - 1})$ orthogonal to S^{n-1}. Thus, it suffices to estimate the expression

$$J(x, y; a) \equiv \frac{j_{\mathsf{B}^n}(\sigma_a(x), \sigma_a(y))}{j_{\mathsf{B}^n}(x, y)} = \frac{\log\left(1 + \frac{|\sigma_a(x) - \sigma_a(y)|}{\min\{1 - |\sigma_a(x)|, 1 - |\sigma_a(y)|\}}\right)}{\log\left(1 + \frac{|x-y|}{\min\{1 - |x|, 1 - |y|\}}\right)}.$$

Let $r = \max\{|x|, |y|\}$ and suppose $|\sigma_a(x)| \geq |\sigma_a(y)|$. Then by (2.2), we have

$$\min\{1 - |\sigma_a(x)|, 1 - |\sigma_a(y)|\} = 1 - |\sigma_a(x)| = \frac{|a||x - a^*| - |x - a|}{|a||x - a^*|}.$$

We first prove the right-hand side of the inequality. By Lemma 3.2, we get

$$j_{\mathbb{B}^n}(\sigma_a(x), \sigma_a(y)) = \log\left(1 + \frac{(1 - |a|^2)|x - y|}{|a||y - a^*|(|a||x - a^*| - |x - a|)}\right)$$

$$= \log\left(1 + \frac{|x - y|(|a||x - a^*| + |x - a|)}{|a||y - a^*|(1 - |x|^2)}\right)$$

$$= \log\left(1 + \frac{|x - y||x - a^*|}{(1 - |x|^2)|y - a^*|}\left(1 + \frac{|x - a|}{|a||x - a^*|}\right)\right)$$

$$\leq \log\left(1 + \frac{|x - y|}{1 - r^2}\left(1 + \frac{|x - y|}{|y - a^*|}\right)\left(1 + \frac{|x| + |a|}{1 + |a||x|}\right)\right)$$

$$\leq \log\left(1 + \frac{|x - y|}{1 - r}\left(1 + \frac{|a||x - y|}{1 - |a|r}\right)\left(1 + \frac{|a|(1 - r)}{1 + |a|r}\right)\right).$$

Then

$$J(x, y; a) \leq \frac{\log\left(1 + \frac{|x-y|}{1-r}\left(1 + \frac{|a||x-y|}{1-|a|r}\right)\left(1 + \frac{|a|(1-r)}{1+|a|r}\right)\right)}{\log\left(1 + \frac{|x-y|}{1-r}\right)}$$

$$= \frac{\log\left(1 + \frac{2r\theta}{1-r}\left(1 + \frac{2|a|r\theta}{1-|a|r}\right)\left(1 + \frac{|a|(1-r)}{1+|a|r}\right)\right)}{\log\left(1 + \frac{2r\theta}{1-r}\right)},$$

where $\theta = \frac{|x-y|}{2r}$.

By Lemma 3.3, it follows

$$J(x, y; a) \leq \frac{\log\left(1 + \frac{2r\theta}{1-r}\left(1 + \frac{|a|(1-r)}{1-|a|r} + \frac{2|a|r\theta}{1-|a|r}\right)\right)}{\log\left(1 + \frac{2r\theta}{1-r}\right)}$$

$$= 1 + \frac{\log\left(1 + \frac{2|a|r\theta}{1-|a|r}\right)}{\log\left(1 + \frac{2r\theta}{1-r}\right)} \leq 1 + \frac{\log\left(1 + \frac{2|a|r}{1-|a|r}\right)}{\log\left(1 + \frac{2r}{1-r}\right)}$$

$$= 1 + \frac{\text{arth}(|a|r)}{\text{arth } r} \leq 1 + |a|.$$

Therefore, we get

$$j_{\mathbb{B}^n}(f(x), f(y)) \leq (1 + |a|) j_{\mathbb{B}^n}(x, y).$$

The sharpness of the upper bound $1 + |a|$ was proved in [9, Remark 3.4] by taking $x = ta/|a| = -y, t \in (0, 1)$, and letting $t \to 0^+$.

For the left-hand side of the inequality, we note that $f^{-1}(x) = A^{-1}\sigma_a^{-1}(x) = A^{-1}\sigma_a(x)$, here $\sigma_a(x)$ and A are as above. Note that because A is an orthogonal matrix, so is A^{-1}. Then by the above proof, for $x, y \in \mathsf{B}^n$, we get

$$\frac{j_{\mathsf{B}^n}(f^{-1}(x), f^{-1}(y))}{j_{\mathsf{B}^n}(x, y)} = \frac{j_{\mathsf{B}^n}(\sigma_a(x), \sigma_a(y))}{j_{\mathsf{B}^n}(x, y)} \leq 1 + |a|.$$

Therefore, we have

$$(3.5) \qquad j_{\mathsf{B}^n}(f(x), f(y)) \geq \frac{1}{1+|a|} j_{\mathsf{B}^n}(x, y).$$

This completes the proof.

CONJECTURE 3.6. *Let* $a \in \mathsf{B}^n$ *and* $f : \mathsf{B}^n \to \mathsf{B}^n = f\mathsf{B}^n$ *be a Möbius transformation with* $f(0) = a$. *Then for* $x, y \in \mathsf{B}^n \setminus \{0\}$

$$j_{\mathsf{B}^n \setminus \{a\}}(f(x), f(y)) \leq C(a) j_{\mathsf{B}^n \setminus \{0\}}(x, y),$$

where the constant $C(a) = 1 + \left(\log \frac{2+|a|}{2-|a|}\right)\big/\log 3$ *is the best possible.*

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MATHEMATICAL INSTITUTE SANU
KNEZA MIHAILA 36
11000 BELGRADE
SERBIA
E-mail: ssimic@turing.mi.sanu.ac.rs

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF TURKU
TURKU 20014
FINLAND
E-mail: vuorinen@utu.fi
genwan@utu.fi
wgdwang@gmail.com