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ALGEBRAS ON SURFACES

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(*Dedicated to the memory of Walter Rudin*)

Abstract

The first part of the paper is devoted to algebras on one-dimensional varieties in $\mathbb{C}^n$ that are bounded by finite unions of mutually disjoint rectifiable simple closed curves. The relevant Shilov boundaries are considered, and certain nonapproximation phenomena are exhibited. The second part of the paper is devoted to the study of uniform algebras whose maximal ideal spaces are smooth surfaces and that admit sets of smooth generators. Such algebras are shown to consist of functions holomorphic off their Shilov boundaries.

1. Introduction

It is known as a consequence work of Alexander [1], [19, p. 189, Cor. 4.5.6.] that if Δ is an analytic disc in $\mathbb{C}^n$ with $b\Delta$ a rectifiable curve γ, then $\bar{\Delta} = \Delta \cup \gamma$ is polynomially convex. In Section 2 of this note we investigate the more general situation in which the disc Δ is replaced by a one-dimensional analytic variety with boundary the disjoint union of finitely many rectifiable simple closed curves. In the brief Section 3 we consider an approximation question related to the work of Section 2. Section 4 of the paper is devoted to a discussion of the structure of certain uniform algebras whose spectra (or maximal ideal spaces) are compact surfaces, perhaps with boundary. Section 5 contains some open problems suggested by the work of the preceding sections.

2. The convex hulls

We use the standard notations that for a compact subset X of $\mathbb{C}^n$, $\mathscr{P}(X)$ is the algebra of continuous functions on X that can be approximated uniformly on X by polynomials and $\mathscr{R}(X)$ is the algebra of continuous functions on X that can be approximated uniformly on X by rational functions without poles on X.

For a compact set X in $\mathbb{C}^n$, $\widehat{X}$ denotes the polynomially convex hull of X and $\mathscr{R}$-hull X denotes the rationally convex hull of X. The former set is given by

$$\widehat{X} = \Big\{x \in \mathbb{C}^n : |P(x)| \le \sup_{y \in X} |P(y)| \text{ for all polynomials } P\Big\},$$

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the latter by

$$\mathscr{R}\text{-hull } X = \big\{x \in \mathsf{C}^n : \text{for all polynomials } P, \\ P(x) = 0 \text{ implies } P^{-1}(0) \cap X \neq \emptyset\big\}.$$

To fix ideas, let the context be this: Γ is the finite union $\gamma_1 \cup \ldots \cup \gamma_p$ of mutually disjoint rectifiable simple closed curves in C^n and Σ is a bounded one-dimensional complex subvariety of $\mathsf{C}^n \setminus \Gamma$ such that $\Gamma = \bar{\Sigma} \setminus \Sigma = b\Sigma$. We do not assume $\bar{\Sigma}$ to be a compact subset of an ambient one-dimensional variety, we do not assume Σ to be nonsingular, and we do not assume it to be irreducible. We shall see below that the variety Σ has at most p global branches.

In addition to the algebras $\mathscr{P}(\bar{\Sigma})$ and $\mathscr{R}(\bar{\Sigma})$, we have the algebra $A(\bar{\Sigma})$ that is the algebra of continuous functions on $\bar{\Sigma}$ that are holomorphic on Σ. Thus, $\mathscr{P}(\bar{\Sigma}) \subset \mathscr{R}(\bar{\Sigma}) \subset A(\bar{\Sigma})$.

The notation established in the last two paragraphs will be used consistently through the first three sections of the paper.

We consider below the questions of when $\bar{\Sigma}$ is polynomially convex or rationally convex. It turns out that $\bar{\Sigma}$ is always rationally convex and that, although $\bar{\Sigma}$ is not always polynomially convex, it is not difficult, given a certain amount of general theory, to describe its polynomially convex hull.

Before taking up this analysis of the hulls, it seems worthwhile to glance at some examples that suggest certain of the complications that arise in our general setting and to look at some simple facts about the structure of the sets we are considering.

EXAMPLE 2.1. Let $\gamma_1 = \{(e^{i\vartheta}, 0) \in \mathsf{C}^2 : \vartheta \in [-\pi, \pi]\}$ and $\gamma_2 = \{(0, 1 + e^{i\vartheta}) \in \mathsf{C}^2 : \vartheta \in [-\pi, \pi]\}$. Then $\Gamma = \gamma_1 \cup \gamma_2$ is the boundary of the reducible variety Σ that consists of the disc $\Delta_2 = \{(0, 1 + z_2) \in \mathsf{C}^2 : |z_2| < 1\}$ and the punctured disc $\Delta_1' = \{(z_1, 0) \in \mathsf{C}^2 : 0 < |z_1| < 1\}$. The point $(0, 0) \in \gamma_2$ is not a peak point for the algebra $A(\bar{\Sigma})$.

For the next examples, let $A^\infty(\bar{\mathsf{U}})$ be the algebra of all functions holomorphic on the open unit disc U whose derivatives of all orders are continuous on the closure of U.

EXAMPLE 2.2. A more complicated example in the same spirit is this. Let $\gamma_1 = \{(2e^{i\vartheta}, 0, 0) : \vartheta \in \mathsf{R}\}$ so that γ_1 is the boundary of the disc $\Delta_1 = \{(z_1, 0, 0) \in \mathsf{C}^3 : |z_1| < 2\}$. Let E be a compact subset in the unit circle in the plane that has zero length and that, in addition, satisfies the condition that if $\{J_k\}_{k=1,\ldots}$ are the intervals in the unit circle complementary to E and if ε_k is the length of J_k, then

$$-\infty < \sum_{j=1}^{\infty} \varepsilon_k \log \varepsilon_k.$$

Such sets are called *Carleson sets*. It is a result of Novinger [15] and, independently, of Taylor and Williams [20] that if E is a Carleson set, then there is a function $f \in A^\infty(\bar{\mathsf{U}})$ with $f^{-1}(0) = E$. Let $\gamma_2 = \{(e^{i\vartheta}, f(e^{i\vartheta}), e^{i\vartheta} f(e^{i\vartheta})) : \vartheta \in [-\pi, \pi]\}$, which is the boundary of the disc $\Delta_2 = \{(z, f(z), zf(z)) \in \mathsf{C}^3 : |z| < 1\}$. The curves γ_1 and γ_2 are rectifiable and disjoint, but the curve γ_2 meets the disc Δ_1 in the circular set $I = \{(\zeta, 0, 0) : \zeta \in E\}$, which can be a Cantor set.

EXAMPLE 2.3. Let $x_n = 1 - e^{-n^2}$, $n = 1, 2, \ldots$, so that $\sum_{n=1}^{\infty} n(1 - x_n) < \infty$, whence there are many functions in $A^\infty(\bar{\mathsf{U}})$ that vanish at all of the points x_n and, indeed, there is a rich supply of such functions that vanish to order n at x_n. See [20].

Let $f \in A^\infty(\bar{\mathsf{U}})$ have the set $\{1, x_1, x_2, \ldots\}$ as its zero set, and let $g \in A^\infty(\bar{\mathsf{U}})$ be the function given by $g(z) = zf(z)$. Let $\varphi : \bar{\mathsf{U}} \to \mathsf{C}^2$ be the map given by $\varphi = (f, g)$. The map φ is injective on bU, so $\gamma = \varphi(b\mathsf{U})$ is a simple closed curve, which is the boundary of the bounded subvariety $\Sigma = \varphi(\mathsf{U}) \setminus (0, 0)$ of $\mathsf{C}^2 \setminus \gamma$. We have $\varphi(1) = \varphi(x_1) = \varphi(x_2) = \cdots = (0, 0)$. Again the point $(0, 0)$ is not a peak point for the algebra $A(\bar{\Sigma})$.

EXAMPLE 2.4. This example is to show that distinct branches of Σ can intersect in an infinite set. Let $\gamma_1 = \{(2e^{i\vartheta}, 0) : \vartheta \in [-\pi, \pi]\} \subset \mathsf{C}^2$, and let f be as in the preceding example. The curve $\gamma_2 = \{(e^{i\vartheta}, f(e^{i\vartheta})) : \vartheta \in [-\pi, \pi]\}$ is smooth and disjoint from γ_1. The curve γ_1 bounds the disc Δ_1 in the z_1-plane, and the curve γ_2 bounds the variety $\Sigma_2 = \{(z, f(z)) : z \in \mathsf{C}, |z| < 1\}$. The union $\Gamma = \gamma_1 \cup \gamma_1$ is the boundary of the variety $\Sigma = \Sigma_1 \cup \Sigma_2$ if $\Sigma_1 = \Delta_1 \setminus (1, 0)$. The intersection $\Sigma_1 \cap \Sigma_2$ is the infinite set $\{(x_1, 0), (x_2, 0), \ldots\}$.

Example 2.2 shows that the topological type of the variety Σ need not be finite. However, Σ can have at most finitely many branches:

LEMMA 2.5. *The variety Σ has at most p global branches.*

PROOF. This is a consequence of the result [19, p. 213, Th. 4.7.1] that for a bounded purely one-dimensional variety V in C^n with boundary of finite length, the number of global branches of V does not exceed the rank of the cohomology group $\check{H}^1(bV; \mathsf{Z})$.

In the event that Γ consists of a single curve, the set $\bar{\Sigma}$ is polynomially convex: $\widehat{\Gamma} \setminus \Gamma$ is a variety, which contains the variety Σ. The number of global branches of the variety $\widehat{\Gamma} \setminus \Gamma$ does not exceed the rank of the group

$\check{H}^1(\Gamma; \mathsf{Z})$, which is one. In the case that Γ has more than one component, $\bar{\Sigma}$ need not be polynomially convex. There is, though, a simple description of the polynomially convex hull of $\bar{\Sigma}$. The path to this description leads through a discussion of the Shilov boundary for $\mathscr{P}(\bar{\Sigma})$.

Recall that the Shilov boundary for a uniform algebra B on a compact Hausdorff space X is the minimal closed subset E of X that satisfies $\max_{x\in X} |g(x)| \leq \max_{x\in E} |g(x)|$ for all $g \in B$. The Shilov boundary exists and is unique. When the underlying space X is metrizable, the Shilov boundary is the closure of the set of peak points for the algebra B.

The Shilov boundary for $\mathscr{P}(\bar{\Sigma})$ can perfectly well be a proper subset of Γ, as in the case of an annular domain in the plane.

LEMMA 2.6. *The Shilov boundary for $\mathscr{P}(\bar{\Sigma})$ is a union of some of the γ_js.*

Alternatively put, the Shilov boundary for $\mathscr{P}(\bar{\Sigma})$ is open and closed in Γ.

PROOF. Let E be the Shilov boundary for $\mathscr{P}(\bar{\Sigma})$. Then $\widehat{E} \supset \bar{\Sigma}$, and the maximum principle implies that $E \subset \Gamma$. Because E is contained in a connected set of finite length, the complementary set $\widehat{E} \setminus E$ is a variety, which we denote by V. If $E \cap \gamma_1$ is a proper, nonempty subset of γ_1, then the variety V continues through $E \cap \gamma_1$, because this set has finite length and satisfies $\check{H}^1(E \cap \gamma_1; \mathsf{Z}) = 0$. See [19, p.168, Cor.3.8.22]. It follows that no point of γ_1 is a peak point for the algebra $\mathscr{P}(\bar{\Sigma})$. Consequently, $E \cap \gamma_1$ is empty. Contradiction. Thus, if E meets γ_1, then E contains γ_1, and the lemma is proved.

THEOREM 2.7. *The Shilov boundary for $\mathscr{P}(\bar{\Sigma})$ is the union of those γ_js with the property that near none of their points does the hull $\widehat{\bar{\Sigma}}$ have the structure of a one-dimensional variety.*

PROOF. The point is that if $\bar{\Sigma}$ has the structure of a variety near a point $x \in \gamma_j$, then by the continuation process used in the preceding proof, $\bar{\Sigma}$ has the structure of a variety along the whole of γ_j.

This formulation is not entirely satisfactory in that it subordinates the determination of the Shilov boundary for $\mathscr{P}(\bar{\Sigma})$ to the determination of $\widehat{\bar{\Sigma}}$; one would prefer a way to determine directly the points of $\bar{\Sigma}$ that belong to the Shilov boundary or, equivalently, to determine directly the points of $\bar{\Sigma}$ that are peak points for $\mathscr{P}(\bar{\Sigma})$. How to make such a direct determination is not obvious.

THEOREM 2.8. *If the Shilov boundary for $\mathscr{P}(\bar{\Sigma})$ coincides with Γ, the set $\bar{\Sigma}$ is polynomially convex.*

A consequence of the theorem is the criterion:

COROLLARY 2.9. *The set $\bar{\Sigma}$ is polynomially convex if each boundary component γ_j contains a peak point for $\mathscr{P}(\Sigma)$.*

For the proof of the theorem, we need a lemma about the continuation of analytic curves, which is a simple application of a result of King [14], [19, p.157, Th.3.8.3].

LEMMA 2.10. *Let Ω be a domain in $\mathbb{C}^n$, and let γ be a rectifiable simple closed curve in Ω. If V and V' are distinct closed irreducible one-dimensional subvarieties of $\Omega \setminus \gamma$ such that $\bar{V} \cap \Omega = \gamma = \bar{V}' \cap \Omega$, then $V \cup V' \cup \gamma$ is a (one-dimensional) subvariety of Ω.*

PROOF. The varieties V and V' have finite area near γ, so the currents $[V]$ and $[V']$ of integration over V and V', respectively, acting on smooth compactly supported $(1, 1)$-forms on Ω are well defined. Moreover, by Stokes's Theorem, there are orientations on γ such that the current boundaries $b[V]$ and $b[V']$ are given by the condition that for all smooth compactly supported one-forms α on Ω

$$b[V](\alpha) = \int_{\gamma*} \alpha \qquad \text{and} \qquad b[V'](\alpha) = \int_{\gamma**} \alpha$$

in which $\gamma*$ and $\gamma**$ denote the curve γ with the orientations suitable for Stokes's Theorem to hold. We see then that $b[V] = \pm b[V']$. These boundaries cannot be equal for if they were, Stokes's Theorem would yield that for every smooth, compactly supported one-form α on Ω, $\int_V d\alpha = \int_{V'} d\alpha$. But this happens only if $V = V'$, which contradicts our hypothesis that V and V' are distinct. We therefore have that $b[V] + b[V'] = 0$, which allows us to invoke the theorem of King to conclude that $\overline{V \cup V'} \cap \Omega = V \cup \gamma \cup V'$ is a (one-dimensional) subvariety of Ω.

PROOF OF THEOREM 2.8. Suppose that the Shilov boundary for $\mathscr{P}(\bar{\Sigma})$ is Γ but that $\bar{\Sigma}$ is not polynomially convex. We have that $\widehat{\bar{\Sigma}} = \widehat{\Gamma}$. If $\bar{\Sigma}$ is not polynomially convex, the variety $\Sigma_o = \widehat{\Gamma} \setminus \Gamma$ has a global branch, say W, that is not a branch of Σ. The branch W is a bounded closed subvariety of $\mathbb{C}^n \setminus \Gamma$, so $bW \subset \Gamma$. Thus bW has vanishing two-dimensional Hausdorff measure. Accordingly, by [19, p. 164, Th. 3.8.15], the cohomology group $\check{H}^1(bW; \mathbb{Z})$ does not vanish, which implies [19, p. 213, Lem. 4.7.4] that bW contains a simple closed curve, say γ. Necessarily $\gamma \subset \Gamma$, so γ is one of the γ_js, say $\gamma = \gamma_1$. But then the varieties W and Σ abut along the rectifiable simple closed curve γ_1, whence, by Lemma 2.10, the union $W \cup \gamma_1 \cup \Sigma$ has the structure of a variety in the neighborhood of γ_1, and the elements of $\mathscr{P}(\Gamma)$ are holomorphic in this neighborhood. Consequently, no point of γ_1 can be a peak point for the algebra $\mathscr{P}(\Gamma)$, which implies that γ_1 is not contained in the Shilov boundary for $\mathscr{P}(\Gamma)$. Contradiction, and the theorem is proved.

In the case that $\bar{\Sigma}$ is not polynomially convex, its hull can be described in some detail. We have that $\widehat{\bar{\Sigma}} = \widehat{\Gamma}$ and that the set $\widehat{\Gamma} \setminus \Gamma$ is a variety, which we will denote by Σ_o. The Shilov boundary E for $\mathscr{P}(\Gamma)$ is a union of the γ_js, say $E = \gamma_1 \cup \ldots \cup \gamma_q$ for some q with $1 \leq q < p$. Each of the remaining γ_js, i.e., those with $q < j \leq p$ is contained in Σ_o. This can happen in either of two ways. It may be that $\Sigma \cup \gamma_j$ is itself a variety so that $\bar{\Sigma}$ has the structure of a variety along γ_j. An example is found by taking

$$\Sigma = \{z \in \mathbb{C} : 1 < |z| < 2 \text{ or } 2 < |z| < 3\},$$

and $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ with γ_j the circle of radius j centered at the origin. Here $\bar{\Sigma}$ has the structure of a variety along γ_2. Alternatively, with the same example, we see that $\widehat{\Gamma} = \widehat{\bar{\Sigma}} = \bar{\Sigma}_o = \{z : |z| \leq 3\}$, and $\widehat{\bar{\Sigma}}$ is obtained from $\bar{\Sigma}$ by adjoining the disc $\Delta = \{z : |z| < 1\}$ to $\bar{\Sigma}$.

In the general case, when $\bar{\Sigma}$ is not polynomially convex, the passage from $\bar{\Sigma}$ to $\widehat{\bar{\Sigma}}$ consists in adjoining to $\bar{\Sigma}$ certain varieties V with $bV \subset \gamma_1 \cup \ldots \cup \gamma_q$. Heuristically speaking, the passage from $\bar{\Sigma}$ to $\widehat{\bar{\Sigma}}$ amounts to filling in a holomorphic way certain holes in $\bar{\Sigma}$.

As concerns rational convexity, we have the following result.

THEOREM 2.11. *The set $\bar{\Sigma}$ is rationally convex.*

PROOF. As $\bar{\Sigma} \subset \mathscr{R}$-hull $\bar{\Sigma} \subset \widehat{\bar{\Sigma}}$, we need only show that no point of $\widehat{\bar{\Sigma}} \setminus \bar{\Sigma}$ lies in $\mathscr{R}$-hull $\bar{\Sigma}$.

We begin by treating the case that $n = 2$. In this case, we know $\check{H}^2(\widehat{\bar{\Sigma}}; \mathbb{Z}) = 0$: A polynomially convex subset X of $\mathbb{C}^n$ satisfies $\check{H}^n(X; \mathbb{Z}) = 0$. See [19, p. 96, Cor. 2.3.6].

Suppose then that $x \in \widehat{\bar{\Sigma}} \setminus \bar{\Sigma} = \widehat{\Gamma} \setminus \bar{\Sigma}$. There is an open ball $\mathbb{B}_x$ in $\mathbb{C}^2$ centered at the point x and small enough that it is disjoint from $\bar{\Sigma}$. Accordingly, $\widehat{\bar{\Sigma}} \cap \mathbb{B}_x$ is a one-dimensional subvariety of B_x, and it follows that there is a function $f \in \mathscr{O}(\mathbb{B}_x)$ such that x is an isolated point of $f^{-1}(0) \cap \widehat{\bar{\Sigma}} \cap \mathbb{B}_x$. By shrinking $\mathbb{B}_x$ we can suppose that $f^{-1}(0) \cap \widehat{\bar{\Sigma}} \cap \mathbb{B}_x = \{x\}$.

Let U be a neighborhood in $\mathbb{C}^2$ of $\widehat{\bar{\Sigma}} \setminus \{x\}$ that is disjoint from the closed set $f^{-1}(0)$ of $\mathbb{B}_x$. The set $\widehat{\bar{\Sigma}}$ is polynomially convex and so is the intersection of Stein domains in $\mathbb{C}^2$, so there is a Stein domain W with $\widehat{\bar{\Sigma}} \subset W \subset \mathbb{B}_x \cup U$. Consider the covering $\{W', W''\}$ of W given by $W' = \mathbb{B}_x \cap W$ and $W'' = U \cap W$ of W. Define $g' \in \mathscr{O}(W')$ and $g'' \in \mathscr{O}(W'')$ by $g' = f|W'$ and $g'' = 1$. On $W' \cap W''$, $g'/g'' = g'$ is zero-free. That is to say, we have constructed a set of Cousin II data on the Stein manifold W. There is no reason for this Cousin II problem to be solvable on W. However, $\widehat{\bar{\Sigma}}$ has a neighborhood basis consisting of Stein domains, and as $\check{H}^2(\widehat{\bar{\Sigma}}; \mathbb{Z}) = 0$, the continuity of Čech cohomology, implies that our Cousin II problem is solvable on a sufficiently thin neighborhood Ω of $\bar{\Sigma}$ that is contained in W. Thus, if Ω is thin enough, there is $F \in \mathscr{O}(\Omega)$ with g'/F holomorphic and zero-free on W' and g''/F holomorphic and zero-free on $\bar{\Sigma}$.

We have constructed therefore a function F holomorphic on Ω that vanishes at the point x and that does not vanish on $\bar{\Sigma}$. Because $\widehat{\bar{\Sigma}}$ is polynomially convex, the Oka-Weil approximation theorem implies that the function F can be approximated on $\widehat{\bar{\Sigma}}$ by polynomials, whence we can suppose F itself to be a polynomial. But this precludes the possibility that the point x lies in $\mathscr{R}$-hull $\bar{\Sigma}$. Thus, $\bar{\Sigma}$ is seen to be rationally convex.[1]

It remains to treat the case that $n > 2$. Again choose a point $x_0 \in \widehat{\bar{\Sigma}} \setminus \bar{\Sigma}$. As a set with two-dimensional measure zero, the set Γ is rationally convex [19, p. 52, Th. 1.6.7], so there is a polynomial P such that $P(x_0) = 0$ and $0 \notin P(\Gamma)$. Thus, the set $P^{-1}(0) \cap \widehat{\bar{\Sigma}}$ is a finite set, say

$$P^{-1}(0) \cap \widehat{\bar{\Sigma}} = \{x_0, x_1, \ldots, x_s\}.$$

Let Q be a polynomial with $Q(x_j) = j$, $j = 0, \ldots, s$. Define $\pi : \mathbb{C}^n \to \mathbb{C}^2$ by $\pi(z) = (P(z), Q(z))$. We have then that $\pi^{-1}(0,0) \cap \widehat{\bar{\Sigma}} = x_0$ and that $(0,0) \notin \pi(\Gamma)$.

The set $\pi(\Gamma)$ is the union of a finite number of rectifiable curves, not necessarily simple, and so is contained in a connected set of finite length. Accordingly, $\widehat{\pi(\Gamma)} \setminus \pi(\Gamma)$ is a one-dimensional subvariety of $\mathbb{C}^2 \setminus \pi(\Gamma)$. The variety $\widehat{\pi(\Gamma)} \setminus \pi(\Gamma)$ contains the origin. As above, there is a polynomial H on $\mathbb{C}^2$ that vanishes at $(0,0) = \pi(x_0)$ and nowhere else on $\pi(\widehat{\bar{\Sigma}})$. The polynomial A on $\mathbb{C}^n$ given by $A = H \circ \pi$ vanishes at x_0 and nowhere else on $\widehat{\bar{\Sigma}}$. In particular, it has no zero on $\bar{\Sigma}$. Thus, $x_0 \notin \mathscr{R}$-hull $\bar{\Sigma}$.

The theorem is proved.

We can now determine the Shilov boundary for $\mathscr{R}(\bar{\Sigma})$.

Theorem 2.12. *The Shilov boundary for $\mathscr{R}(\bar{\Sigma})$ is the union of those γ_js near no point of which does $\bar{\Sigma}$ have the structure of a one-dimensional variety.*

Proof. There is a function f defined and of class $\mathscr{C}^\infty$ on all of $\mathbb{C}^n$ with $f|\bar{\Sigma} \in \mathscr{R}(\bar{\Sigma})$ that generates $\mathscr{R}(\bar{\Sigma})$ over $\mathscr{P}(\bar{\Sigma})$, i.e., such that polynomials in f with coefficients in $\mathscr{P}(\bar{\Sigma})$ are dense in $\mathscr{R}(\bar{\Sigma})$. For the existence of such an f, see [19, p.9]. Then there is a natural identification of $\mathscr{R}(\bar{\Sigma})$ with $\mathscr{P}(\tilde{\Sigma})$ if $\tilde{\Sigma} \subset \mathbb{C}^{n+1}$ is the graph of the function $f|\bar{\Sigma}$. We have $\mathscr{P}(\tilde{\Sigma}) = \mathscr{P}(\tilde{\Gamma})$ if

[1] Notice that, in the n-dimensional case, if we knew the group $\check{H}^2(\widehat{\bar{\Sigma}}; \mathbb{Z})$ to vanish, the argument just given would yield the n dimensional case of the theorem we are proving. That, in general, $\check{H}^2(\widehat{\bar{\Sigma}}; \mathbb{Z})$ vanishes seems likely, but the author has found no proof.

$\tilde{\Gamma}$ is the graph of $f|\Gamma$. Because f is smooth $\tilde{\Gamma}$ consists of mutually disjoint rectifiable simple closed curves. In addition, f is holomorphic on every open set in $\bar{\Sigma}$ that has the structure of an analytic variety. The Shilov boundary for $\mathscr{P}(\tilde{\Sigma})$ is described by the Theorem 2.7, and that description is seen to yield the assertion of the present theorem.

THEOREM 2.13. *The Shilov boundary for $A(\bar{\Sigma})$ is the union of those γ_js near no point of which does $\bar{\Sigma}$ have the structure of a one-dimensional variety.*

That is, the Shilov boundaries for $\mathscr{R}(\bar{\Sigma})$ and $A(\bar{\Sigma})$ coincide.

PROOF. Let E be the Shilov boundary for $\mathscr{R}(\bar{\Sigma})$ and E' that for $A(\bar{\Sigma})$. We have $E \subset E' \subset \Gamma$.

We first observe that if $x \in \bar{\Sigma}$ is a point with a neighborhood U in $\bar{\Sigma}$ that has the structure of a one-dimensional variety, then either x is not in E' or else x is an isolated point of E': If $x \in \Sigma$, then the maximum principle applies immediately to yield that $x \notin E'$. If, on the other hand $x \in \gamma_j$, then we consider two cases. First, it may be that x is a regular point of U. In this case each $f \in A(\bar{\Sigma})$ continues holomorphically into some neighborhood of x in U. This is a consequence of the result that a function continuous on an open disc in the plane that is holomorphic off a rectifiable curve in it is actually holomorphic throughout the disc. For this classical removable singularity theorem due to Painlevé one can consult [5, p. 321, Exercise 9.28]. Thus, the only points of U that can lie in E' are those that are singular points of the variety U. These points constitute a discrete set. Thus, either x is not in E' or else x is an isolated point of E'. In fact, we will show x cannot be an isolated point of E'. Peak points are dense in the Shilov boundary, so an isolated point of the Shilov boundary must be a peak point.

A remark is in order here: The elements of $A(\bar{\Sigma})$ are holomorphic on the variety Σ by definition, and we have just seen that they extend holomorphically to the regular points of U if U is any open subset of $\bar{\Sigma}$ that has the structure of a variety. A priori this does not imply that they are holomorphic on the entire variety U; they might not be holomorphic at the singular points of U. This difficulty can be circumvented by the following legerdemain.

Let $(\tilde{U}, \eta)$ be the normalization of the space U so that $\tilde{U}$ is a not necessarily connected nonsingular space, i.e., a Riemann surface,[2] and $\eta : \tilde{U} \to U$ is a holomorphic map that effects a biholomorphism between $\tilde{U} \setminus \eta^{-1}(U_{\text{sing}})$ and $U_{\text{reg}} = U \setminus U_{\text{sing}}$ in which U_{sing} denotes the singular locus of the space U. The function $f \circ \eta$ is continuous on $\tilde{U}$ and is holomorphic off the finite set $\eta^{-1}(x)$. Thus, by the Riemann removable singularity theorem, $f \circ \eta$ is holomorphic on all of $\tilde{U}$. Consequently $|f|$ cannot attain its maximum at x.

[2] For the theory of the normalization, one can consult, e.g., [10].

We have now that $E' \cap U = \emptyset$.

To conclude: If a point $x \in \gamma_j$ has a neighborhood in $\bar{\Sigma}$ that has the structure of a variety, then again by [19, p. 168, Cor. 3.8.22] $\bar{\Sigma}$ has the structure of an analytic variety in a neighborhood of γ_1, and by the argument we have just given, E' must be disjoint from γ_j.

The theorem is proved.

We have shown that $\bar{\Sigma}$ is rationally convex, so the spectrum of the algebra $\mathscr{R}(\bar{\Sigma})$ is $\bar{\Sigma}$ itself. Similarly, the spectrum of the algebra $\mathscr{P}(\bar{\Sigma})$ is $\widehat{\bar{\Sigma}}$. What is not contained in the work above is a determination of the spectrum of the algebra $A(\bar{\Sigma})$.

EXAMPLE 2.14. So far we have dealt with varieties bounded by rectifiable curves. Without *some* restriction on the boundary, the situation can become much more complicated: If Δ is an analytic disc in $\mathbb{C}^n$ bounded by a simple closed curve, then the closure $\bar{\Delta}$ need not be polynomially convex; there are well-known examples. Let J be a simple closed curve in the plane that has locally positive measure. The curve J divides the Riemann sphere $\mathbb{C}^*$ into two domains D^+ and D^-. Constructions going back to Wermer and Rudin – see [19, pp.53-54] – show that there exist three functions, say g_1, g_2, g_3, that are continuous on $\mathbb{C}^*$ and holomorphic on $D^+ \cup D^-$ such that the map $G : \mathbb{C}^* \to \mathbb{C}^3$ defined by

$$G(z) = (g_1(z), g_2(z), g_3(z))$$

carries $\mathbb{C}^*$ homeomorphically onto the topological sphere $S = G(\mathbb{C}^*)$, which is contained in the polynomially convex hull of the simple closed curve $G(J)$. Thus the closure $G(J \cup D^+)$ of the analytic disc $G(D^+)$ is not polynomially convex.

There is a small technical point that should be noted. Without further ado, we do not know that the map G is regular, i.e., that it has nonvanishing differential, on $D^+ \cup D^-$. Thus, the discs $G(D^+)$ and $G(D^-)$ may have some singularities. The result of [16] implies that by choosing g_1, g_2, g_3 and then a fourth function g_4 suitably, we can obtain a map $\tilde{G} : \mathbb{C}^* \to \mathbb{C}^4$ that is assured to be regular on $D^+ \cup D^-$.

3. Approximation

A reasonable initial guess would be that in the context considered in the preceding section, the equality $\mathscr{R}(\bar{\Sigma}) = A(\bar{\Sigma})$ should obtain, and thus that, in the event that $\bar{\Sigma}$ is polynomially convex, we should have $\mathscr{P}(\bar{\Sigma}) = A(\bar{\Sigma})$. The situation is more complicated than this even in the case of discs, as shown by the following example, which is closely related to work of Dinh [7].

EXAMPLE 3.1. For this example, we shall use the standard factorization of elements of the disc algebra (or of functions of the Hardy class $H^p(\mathsf{U})$): If $f \in A(\bar{\mathsf{U}})$, then f admits a unique factorization of the form $f = BSF$ in which B is a Blaschke product, S is a singular inner function, and F is an outer function. This theory is developed in detail in the book of Hoffman [12].

For positive integers p and q, let $B_{p,q}$ be the closed subalgebra of the disc algebra generated by the functions f_p and f_q where, for a positive integer k, we understand f_k to be the function given by $f_k(z) = (z-1)^k H(z)$ in which H is the singular inner function given by $H(z) = \exp \frac{z+1}{z-1}$. Thus, $B_{p,q}$ is the uniform closure of the algebra of functions on the closed disc of the form

$$\sum_{\mu,\nu=0}^{N} c_{\mu,\nu}(z-1)^{p\mu+q\nu} H^{\mu+\nu}(z) \tag{1}$$

for constants $c_{\mu,\nu}$. We shall show that the algebra $B_{p,q}$ contains no nonconstant outer function that vanishes at the point 1.

Suppose, for the sake of contradiction, that the nonconstant outer function h, which is supposed to satisfy $h(1) = 0$, is the limit of the sequence $\{h_n\}_{n=1,\dots}$, with each h_n of the form (1). As $h(1) = 0$, it follows that $h_n(1) \to 0$, so we can suppose that each h_n has vanishing constant term whence

$$h_n(z) = \sum_{\substack{0 \le \mu,\nu \le N(n) \\ 1 \le \mu+\nu}} c_{\mu,\nu}(n)(z-1)^{p\mu+q\nu} H^{\mu+\nu}(z).$$

Multiply both sides of this equation by $\bar{H}$ and use the fact that H is continuous and unimodular on $b\mathsf{U} \setminus \{1\}$ to find that the sequence of functions

$$\bar{H}(z)h_n(z) = \sum_{\substack{0 \le \mu,\nu \le N(n) \\ 1 \le \mu+\nu}} c_{\mu,\nu}(n)(z-1)^{p\mu+q\nu} H^{\mu+\nu-1}(z)$$

converges uniformly on $b\mathsf{U}$ and so on $\bar{\mathsf{U}}$, say to G. We then have the factorization $h = GH$ with $G \in A(\bar{\mathsf{U}})$. The function h is thus seen to have a nontrivial singular inner factor. However, h is an outer function. We have reached a contradiction, and know, therefore that $h \notin B_{p,q}$.

Observe that this argument shows, in fact, that *every nonconstant element of $B_{p,q}$ that vanishes at the point* 1 *has a nontrivial inner factor*.

Now notice that if $p \ge 3$, then the derivative f_p' lies in $A(\bar{\mathsf{U}})$, so the restriction of f_p to $b\mathsf{U}$ is of class $\mathscr{C}^1$ and is, in particular, of bounded variation on $b\mathsf{U}$. For this derivative, we have

$$f_p'(z) = (z-1)^{p-2}(p(z-1)-2)\exp\frac{z+1}{z-1}. \tag{2}$$

Consider then f_p and f_{p+1} for $p \geq 3$. The derivative f_p' vanishes at the points $z = 1$ and $z = (p+2)/p$, so that f_p' has no zero on the punctured closed disc $\bar{\mathsf{U}} \setminus \{1\}$. Moreover, f_p and f_{p+1} separate points on $\bar{\mathsf{U}}$: Suppose $z, \zeta \in \bar{\mathsf{U}}$ and that

$$(3) \qquad f_p(z) = f_p(\zeta) \qquad \text{and} \qquad f_{p+1}(z) = f_{p+1}(\zeta).$$

The only point in $\bar{\mathsf{U}}$ at which either of f_p or f_{p+1} vanishes is the point 1. If $z \neq 1$, the equalities (3) imply that $z = \zeta$.

The map $f = (f_p, f_{p+1}) : \bar{\mathsf{U}} \to \mathsf{C}^2$ is injective, its differential vanishes at no point of U, and it carries $b\mathsf{U}$ onto the rectifiable curve $\Gamma = b\Delta$ if $\Delta = f(\mathsf{U})$.

The algebra $\mathscr{P}(\bar{\Delta})$ is not all of $A(\Delta)$: If it were, then the algebra $B_{p,p+1}$ considered above would be all of $A(\bar{\mathsf{U}})$, but, as we have seen, it is not: It contains no outer function that vanishes at 1.

The curve Γ is somewhat better than rectifiable: The open arc $\Gamma \setminus \{0\}$ is real-analytic. The parameterization for Γ that we have given is by functions of class $\mathscr{C}^1$, but the differential of this map vanishes at the point 1, so Γ is not presented as a curve of class $\mathscr{C}^1$, and, indeed, it is not such a curve because of the singularity at the origin.

We have seen that the algebra $B_{p,p+1}$ contains no outer function. The disc algebra contains an infinite sequence of linearly independent outer functions, e.g., $\{g_n\}_{n=1,\dots}$ with $g_n(z) = (z-1)^n$. The functions are, moreover, independent mod $B_{p,p+1}$: If the polynomial P given by $P(z) = \sum_{k=0}^m c_k(z-1)^k$ lies in $B_{p,p+1}$, then, as we saw above, P has the inner factor H, which is impossible as is seen by considering rates of decay along the radius $[0, 1)$.

It follows that the quotient vector space $A(\bar{\Delta})/\mathscr{P}(\bar{\Delta})$ is infinite-dimensional.

4. Differentiably generated algebras on surfaces

We now turn our attention to uniform algebras with spectra compact surfaces. Under a mild smoothness hypothesis, such algebras admit a rather simple description.

THEOREM 4.1. *If Σ is a compact surface, perhaps with boundary, of class $\mathscr{C}^1$, if $\mathfrak{A} \subset \mathscr{C}(\Sigma)$ is a uniform algebra on Σ for which Σ is the spectrum, if there is a set $\mathscr{G}$ of generators for $\mathfrak{A}$ each of which is of class $\mathscr{C}^1$, and if Γ is the Shilov boundary for $\mathfrak{A}$, then the set $\Sigma \setminus \Gamma$ admits the structure of a one-dimensional reduced complex space on which each $f \in \mathfrak{A}$ is holomorphic, and the algebra of restrictions $\mathfrak{A}|(\Sigma \setminus \Gamma)$ is dense in the algebra $\mathscr{O}(\Sigma \setminus \Gamma)$ in the sense of uniform convergence on compacta. Moreover, the set Γ contains the boundary of Σ. If $\mathscr{G}$ can be chosen as a finite set, the complex space $\Sigma \setminus \Gamma$ is biholomorphically equivalent to a subvariety of a bounded open set in C^n, n the cardinality of $\mathscr{G}$.*

In particular we have a result about algebras on discs:

COROLLARY 4.2. *If $\mathfrak{A}$ is a uniform algebra on the closed unit disc $\bar{\mathsf{U}}$ in the plane that is generated by continuously differentiable functions, and if the the spectrum of $\mathfrak{A}$ is $\bar{\mathsf{U}}$, then the Shilov boundary for $\mathfrak{A}$ contains the boundary of* U.

It is a long-open problem to show that the same conclusion can be drawn for every uniform algebra with spectrum the closed disc (omitting the hypothesis of smooth generators). This general question appears in Jarosz's list of open problems [13, p. 150, Question 7.2].[3] In fact, the problem is much older: It is mentioned in Gamelin's book [9, p. 10] and had already been posed as an open problem by Hoffman at the Tulane Symposium of 1965. See [4, p. 348].

That $b\Sigma \subset \Gamma$ implies that the surface $\Sigma \setminus \Gamma$ is a not necessarily connected surface that is open, i.e., that each component of $\Sigma \setminus \Gamma$ is noncompact and without boundary.

In the case that $\mathfrak{A}$ admits a set of generators of class $\mathscr{C}^1$ but no finite set of such generators, the space $\Sigma \setminus \Gamma$ may not have globally bounded local embedding dimension in which case it cannot be realized as a subvariety of an open set in any C^n for n a positive integer. An example is given below.

We should point out explicitly that the complex structure on $\Sigma \setminus \Gamma$ is not claimed to be the structure of a Riemann surface, i.e., a one-dimensional complex *manifold*; there may well be singular points as in the case that Σ is the closed unit disc in the complex plane, and $\mathfrak{A}$ is the algebra generated by the functions z^2 and z^3 in which case the complex structure provided by the theorem has a singularity at the origin. The subalgebra of the disc algebra generated by the functions z^2 and z^3 does not contain the function z.

COROLLARY 4.3. *The surface $\Sigma \setminus \Gamma$ is orientable.*

PROOF OF THEOREM 4.1. We deal first with the case that the chosen set $\mathscr{G}$ of generators for $\mathfrak{A}$ is finite, say $\mathscr{G} = \{g_1, \dots, g_n\}$. Define a map $\Phi : \Sigma \to \mathsf{C}^n$ by the condition that $\Phi(s) = (g_1(s), \dots, g_n(s))$ for all $s \in \Sigma$. The map Φ is a homeomorphism from Σ onto a compact surface, perhaps with boundary, S, in C^n. As $\mathscr{G}$ is a set of generators for $\mathfrak{A}$ and Σ is the spectrum of $\mathfrak{A}$, the surface S is polynomially convex. Set $B = \Phi(\Gamma)$, so that S is the polynomially convex hull of B, and B is minimal with respect to this condition. Define the algebra $A \subset \mathscr{C}(S)$ to be the algebra $\{f \circ \Phi^{-1} : f \in \mathfrak{A}\}$. As $\mathscr{G}$ is a set of generators for $\mathfrak{A}$, it follows that $A = \mathscr{P}(S)$.

Since the surface Σ is compact and of class $\mathscr{C}^1$ and the map Φ is of class $\mathscr{C}^1$, it follows that the surface S has finite area – finite two-dimensional Hausdorff

[3] I am indebted to Alexander Izzo for this reference.

measure, whence, a fortiori, the set $S \setminus B = \widehat{B} \setminus B$ has finite area. It follows from a result due, independently, to Alexander, Basner, and Sibony [19, p. 150, Th. 3.6.3] that $S \setminus B$ is a one-dimensional variety.

We now show that $B \supset bS$. Suppose, to the contrary, that there is a point $x \in bS \setminus B$. The boundary of the surface S, if nonempty, consists of a finite number of mutually disjoint simple closed curves of class $\mathscr{C}^1$, and each point in this boundary has a neighborhood in S that is homeomorphic to the closed upper half-plane in the two-dimensional Euclidean plane. Consequently, there is an arc λ in S with both end points in bS that separates the point x from B. Moreover, the arc λ can be chosen to be rectifiable. The local maximum modulus principle implies that $x \in \widehat{\lambda}$. However, rectifiable arcs are polynomially convex. We conclude that there are no points with the property we have ascribed to x.

Finally, we must establish the density assertion of the theorem, which is equivalent to the following statement.

Lemma 4.4. *The algebra $\mathscr{P}(S)$ is dense in the algebra $\mathscr{O}(S \setminus B)$ in the sense of uniform convergence on compacta.*

Proof. Set $\Omega = S \setminus B$. Let $f \in \mathscr{O}(\Omega)$, and let K be a compact subset of Ω. Let $\gamma = \gamma_1 \cup \ldots \cup \gamma_s$ be a finite system of mutually disjoint rectifiable simple closed curves in Ω_{reg}, the set of nonsingular points of Ω, that jointly bound a relatively compact domain in Ω that contains K. The polynomially convex hull $\hat{\gamma}$ is a subset of Σ, and $\hat{\gamma} \setminus \gamma$ is a one-dimensional variety, say V. The variety V is disjoint from the Shilov boundary B: If $x \in B \cap V$, then as S is a surface, possibly with boundary, and V is a one-dimensional variety, a full neighborhood of x in S has to lie in V. Because peak points for $\mathscr{P}(S)$ are dense in B, this full neighborhood contains peak points for the algebra $\mathscr{P}(S)$; such points cannot lie in $\hat{\gamma}$. Thus, $\hat{\gamma} \subset S \setminus B$.

The variety Ω contains $\hat{\gamma}$. Accordingly, there is a polynomial polyhedron P such that $P \cap B = \emptyset$ and $P \supset \hat{\gamma}$. There is a function F holomorphic on P that agrees with f on $W \cap P$. The function F is uniformly approximable on compacta in P by polynomials by virtue of the Oka-Weil approximation theorem. In particular, the function f can be approximated uniformly on the set K by polynomials and therefore, a fortiori, by elements of $\mathscr{P}(S)$.

Theorem 4.1 is proved in the case that $\mathfrak{A}$ admits a finite set of smooth generators.

The treatment of the case of algebras that are not finitely generated is more involved. To begin with, it depends in an essential way on the theory of *maximum modulus algebras*.

Definition 4.5. A *maximum modulus algebra on X with projection p over Ω* is a quadruple (A, X, Ω, p) in which X is a locally compact space, A is an algebra of continuous $\mathbb{C}$-valued functions on X that separates points on X, Ω is an open set in $\mathbb{C}$, and p is an element of A that effects a proper map from X to Ω. It is required further that for every disc Δ contained with its closure in Ω and centered at ζ and for every $g \in A$, $\max_{x \in p^{-1}(\zeta)} |g(x)| \leq \max_{x \in p^{-1}(b\Delta)} |g(x)|$.

Recall that a map $f : \mathfrak{X} \to \mathfrak{Y}$ is *proper* if for every compact set $K \subset \mathfrak{Y}$, the preimage $f^{-1}(K)$ is a compact subset of $\mathfrak{X}$.

For our purposes the most important example of a maximum modulus algebra is given as follows. Let X_o be the spectrum of the uniform algebra A_o, and let Γ be the Shilov boundary for A_o. Let $f \in A_o$, let Ω be a component of $f(X_o) \setminus f(\Gamma)$, let $X = f^{-1}(\Omega)$, and let A be the algebra of restrictions $g|X$, $g \in A_o$. Then $f : f^{-1}(\Omega) \to \Omega$ is proper, and the quadruple $(A, X, \Omega, f|X)$ is a maximum modulus algebra on X with projection f over Ω, as follows from the local maximum modulus principle.

The theory of maximum modulus algebras is treated in detail in the book [2] of Alexander and Wermer.

A fundamental result in the theory of maximum modulus algebras exhibits one-dimensional analytic structure in the space X. Precisely, there is the following result:

THEOREM 4.6. *Let (A, X, Ω, f) be a maximum modulus algebra with projection f over Ω. Assume that for some integer n there exists a Borel set $E \subset \Omega$ of positive logarithmic capacity such that for every $\zeta \in E$, the cardinality of the fiber $f^{-1}(\zeta)$ is not more than n. Then*

(i) *for every $\xi \in \Omega$ the cardinality of the fiber $f^{-1}(\xi)$ is bounded by n, and*

(ii) *there exists a discrete subset S of Ω such that $f^{-1}(\Omega \setminus S)$ admits the structure of a Riemann surface on which every element in A is holomorphic.*

For this theorem we refer to [2, p. 76, Th. 11.8.].

We shall make essential use of an inequality from geometric measure theory:

THEOREM 4.7. *If X and Y are metric spaces and $f : X \to Y$ is a Lipschitzian map, if $A \subset X$, if $0 \leq k < \infty$, and if $0 \leq m < \infty$, then*

$$\int_Y^* \Lambda^k(A \cap f^{-1}(y))\, d\Lambda^m(y) \leq C(f, k, m)\Lambda^{k+m}(A). \tag{4}$$

In this statement $\int^*$ is the upper integral, and Λ^α is α-dimensional Hausdorff measure. Also, $C(f, k, m)$ is a constant that depends on the dimensions k and m and on the Lipschitz constant of the map f.

This result, with an additional hypothesis on Y, which will be satisfied in our application, is given in [8, p. 188, Th. 2.10.25.]. Subsequent to the publication of [8] it was shown in [6] that the supplementary hypotheses imposed on Y is unnecessary.

We now consider our uniform algebra $\mathfrak{A}$ on the surface Σ with Shilov boundary Γ. Set $\Omega = \Sigma \setminus \Gamma$.

Let $x \in \Omega$. Choose a domain D that is a relatively compact subset of Ω with $x \in D$ and with $bD = \gamma$, a finite union of mutually disjoint simple closed curves of class $\mathscr{C}^1$. There are finitely many elements $f_1, \dots, f_n \in \mathfrak{A}$ of the form $f_j = g_j - \alpha_j$ with each g_j in the set $\mathscr{G}$ of generators and therefore of class $\mathscr{C}^1$ and with $\alpha_j \in \mathbb{C}$ such that if $F = (f_1, \dots, f_n) : \Sigma \to \mathbb{C}^n$ then $F(x) = 0$ and $F(x) \notin F(\gamma)$. As $F(\gamma)$ is a union of smooth curves, $\Lambda^2(F(\gamma)) = 0$, which implies that $F(\gamma)$ is rationally convex. Accordingly there is a polynomial P such that $P(F(x)) = 0$ but $0 \notin P(F(\gamma))$. The composition $g = P \circ F$ is an element of $\mathfrak{A}$.

Denote by $\mathfrak{A}_\gamma$ the uniform closure of the algebra $\mathfrak{A}|\gamma$ of restrictions to γ of the elements of the algebra $\mathfrak{A}$. Its spectrum spec $\mathfrak{A}_\gamma$ is in a natural way the set $\mathfrak{A}$-hull γ given by

$$\mathfrak{A}\text{-hull}\,\gamma = \Big\{x \in \Sigma : \text{for all } f \in \mathfrak{A}, |f(x)| \le \sup_{y \in \gamma} |f(y)|\Big\}. \tag{5}$$

This hull, which is a certain compact subset of Σ, contains D by the local maximum principle.

Introduce the notation that $\Omega_\gamma = \operatorname{spec} \mathfrak{A}_\gamma \setminus g^{-1}(\gamma)$, a certain open, perhaps not connected, subset of spec $\mathfrak{A}$ that contains x. Put $W = g(\Omega_\gamma)$, which is a union of certain components of $\mathbb{C} \setminus g(\gamma)$. The quadruple $(\mathfrak{A}_\gamma|\Omega_\gamma, \Omega_\gamma, W, g|\Omega_\gamma)$ is a maximum modulus algebra. The set W contains the origin, i.e., the point $g(x)$.

The smoothness of g implies that the fibers $\Omega_\gamma \cap g^{-1}(\zeta)$, $\zeta \in W$ are generally finite: Apply the inequality (4) to the map $g : \Sigma \to \mathbb{C}$. The surface Σ has finite area, and g, as a function of class $\mathscr{C}^1$, satisfies a Lipschitz condition. Accordingly, the set Ω_γ has finite area, and we find that

$$\int_W^* \Lambda^0(\Omega_\gamma \cap g^{-1}(\zeta))\, d\Lambda^2(\zeta) \le \text{const. } \Lambda^2(\Omega_\gamma) < \infty,$$

whence there is an integer n such that for some set S_n of positive area and, a fortiori, positive capacity, in W, the fibers $\Omega_\gamma \cap g^{-1}(\zeta)$, $\zeta \in S_n$, have exactly n elements.

The principal result on maximum modulus algebras quoted above, Theorem 4.6, implies the existence of a discrete subset S of W such that $\Omega_\gamma \setminus g^{-1}(S)$ has the structure of a Riemann surface on which each element of $\mathfrak{A}$ is holomorphic. Moreover, for every $\zeta \in W$, the fiber $\Omega_\gamma \cap g^{-1}(\zeta)$ has cardinality not more than n.

To proceed, note that the set $\tilde{S} = g^{-1}(S)$ is a discrete subset of Ω_γ: If not, let $x \in \Omega_\gamma$ be a limit point of $\tilde{S}$. As each of the fibers $\Omega_\gamma \cap g^{-1}(\zeta)$ for $\zeta \in W$ has cardinality at most n, the point $g(x) \in W$ is necessarily a limit point of S, a contradiction of the discreteness of S as a subset of W.

We need now a simple function-theoretic lemma:

LEMMA 4.8. *Let Δ be an open disc in a surface of class $\mathscr{C}^1$, let $x \in \Delta$, and let $\Delta \setminus \{x\}$ have the structure of a Riemann surface. Let $\mathfrak{B} \subset \mathscr{C}(\Delta)$ be a point-separating algebra of bounded continuous functions each of which is holomorphic on $\Delta \setminus \{x\}$ with respect to the given conformal structure on the latter set. Then there exists a unique conformal structure on Δ with respect to which the identity map $\Delta \setminus \{x\} \hookrightarrow \Delta$ is holomorphic and such that the elements of $\mathfrak{B}$ are holomorphic on Δ with respect to the extended conformal structure.*

Perhaps it is well to be clear about the notion of conformal structure. By the given conformal structure on Δ we understand a collection $\mathscr{K} = \{(U_\iota, \psi_\iota)\}_{\iota \in I}$ of open subsets U_ι of $\Delta \setminus \{x\}$ and homeomorphisms ψ_ι from U_ι to open sets V_ι in the complex plane such that for all $\iota, \kappa \in I$, the map $\psi_\iota \circ \psi_\kappa^{-1}$ is holomorphic on its set of definition, which, if not empty, is some open set in the plane. The U_ι are supposed to constitute a cover for $\Delta \setminus \{x\}$. That $f \in \mathfrak{B}$ is holomorphic with respect to this structure means that each of the compositions $f \circ \psi_\iota^{-1}$ is holomorphic where it is defined.

To extend the conformal structure through x is to find a corresponding family $\mathscr{K}_\Delta$ on the entire disc Δ with $\mathscr{K}_\Delta \supset \mathscr{K}$.

PROOF OF LEMMA 4.8. The uniformization theorem implies that the domain $\Delta \setminus \{x\}$ is conformally equivalent to a doubly connected domain in the plane. Thus, there is a conformal[4] map $\chi : D \to \Delta \setminus \{x\}$ from a doubly connected domain $D \subset \mathbb{C}$ onto $\Delta \setminus \{x\}$. As a doubly connected domain in the plane, we can suppose D to be one of three canonical domains:

(i) $D = D_\infty = \mathbb{C} \setminus \{0\}$, or

(ii) $D = D_r = \{\zeta \in \mathbb{C} : r < |\zeta| < 1/r\}$ for a unique $r \in (0, 1)$, or

(iii) $D = D_0 = \{\zeta \in \mathbb{C} : 0 < |z| < 1\}$.

We exclude case (i) because if $D = D_\infty$, then as $\mathfrak{B}$ contains bounded, nonconstant functions f, for which $f \circ \chi$ is nonconstant bounded holomorphic function on $\mathbb{C} \setminus \{0\}$, we have a contradiction.

In case (ii), we can suppose that the cluster set of χ at $C_r = \{\zeta \in \mathbb{C} : |\zeta| = r\}$ is the set $\{x\}$. Then for a nonconstant $f \in \mathfrak{B}$, we have that the cluster set of the

[4] We understand conformal maps to be injective.

nonconstant holomorphic function $f \circ \chi$ on D_r at C_r is the set $\{f(x)\}$. This is impossible.

Consequently, the domain D must be the domain D_0. In this case the conformal map χ extends to a continuous map, also denoted by χ, from the unit disc U to Δ. The extended χ is a homeomorphism from U onto Δ. Moreover, for each function $f \in \mathfrak{B}$, the function $f \circ \chi$ is holomorphic on U, as follows from the Riemann removable singularity theorem.

Thus, if we add to the atlas $\mathscr{K}$ defining the complex structure on $\Delta \setminus \{x\}$ the pair (Δ, ψ) with $\psi = \chi^{-1}$, we obtain on Δ the structure of a Riemann surface on which each $f \in \mathfrak{B}$ is holomorphic. Moreover, the identity map from $\Delta \setminus \{x\}$ to Δ is holomorphic.

The lemma is proved.

We apply the lemma just proved at each point of the discrete subset $\tilde{E}$ of Ω_γ considered above to find that the entire open subset Ω_γ admits the structure of a Riemann surface on which the elements of the algebra $\mathfrak{A}$ are holomorphic. In particular, the neighborhood D of the point x with which we began this discussion has the structure of a Riemann surface on which all elements of $\mathfrak{A}$ are holomorphic. It follows that the whole surface $\Omega = \Sigma \setminus \Gamma$ has the structure of a Riemann surface on which all the elements of the algebra $\mathfrak{A}$ are holomorphic. This surface may not be connected.

We do not know, nor is it to be expected, that in general $\mathfrak{A}$ is dense in $\mathscr{O}(\Omega)$ in the sense of uniform convergence on compacta. To obtain the density assertion of Theorem 4.1 we must, in general, alter the complex structure on Ω to obtain a one-dimensional complex space, which, typically, will not be nonsingular.

For this construction we need a lemma:

Lemma 4.9. *The Riemann surface Ω is convex with respect to the algebra $\mathfrak{A}$.*

The assertion is that if K is a compact subset of Ω, then with the $\mathfrak{A}$-hull of K defined as in equation (5), the set $\Omega \cap (\mathfrak{A}\text{-hull } K)$ is compact.

Proof. Fix a compact set K in Ω. The maximum modulus theorem implies that $\mathfrak{A}$-hull K can contain no peak point for the algebra $\mathfrak{A}$. To prove the lemma, it suffices to show that $\mathfrak{A}$-hull K is disjoint from Γ. Suppose this is false so that there is a point $z \in (\mathfrak{A}\text{-hull } K) \cap \Gamma$. The analysis we have given above implies that the set $W = \mathfrak{A}\text{-hull } K \setminus K$ has the structure of a Riemann surface on which each element of $\mathfrak{A}$ is holomorphic. The point z lies in W and so must contain a neighborhood of the point x in Σ. As peak points for $\mathfrak{A}$ are dense in Γ, this neighborhood contains a peak point for $\mathfrak{A}$, and we have a contradiction.

The lemma is proved.

Our situation now is the following: The set Ω is known to be a Riemann surface on which each element of $\mathfrak{A}$ is holomorphic. Moreover, the algebra $\mathfrak{A}$ separates points on Ω, and Ω is convex with respect to $\mathfrak{A}$. Denote by $\mathfrak{B}$ the closure of the algebra of restrictions $\mathfrak{A}|\Omega$ in $\mathscr{C}(\Omega)$ so that $\mathfrak{B}$ is a closed subalgebra of $\mathscr{O}(\Omega)$. (We are taking $\mathscr{C}(\Omega)$ and $\mathscr{O}(\Omega)$ to have the topology of uniform convergence on compacta.) The spectrum of $\mathfrak{B}$[5] is in a natural way Ω in that the topology on Ω is that induced by $\mathfrak{B}$ and each character φ of the algebra $\mathfrak{B}$ is of the form $\varphi(g) = g(x)$ for some necessarily unique point $x \in \Omega$. The latter point is clear: The functional φ acts on the dense subalgebra $\mathfrak{A}$ of $\mathfrak{B}$ by evaluation at some point x of Σ. Moreover, the functional φ is continuous, so there is an inequality $|f(x)| = |\varphi(f)| \leq \|f\|_K$ for some compact subset K of Ω. It follows that the point x must lie in Ω.

To conclude, we need only invoke a theorem of Rossi [18, p. 147, Th. 6.8] – see also the paper [17] – which implies the existence of a complex structure on Ω with respect to which the algebra $\mathfrak{B}$ is the algebra of *all* holomorphic functions.[6]

Theorem 4.1 is proved.

EXAMPLE 4.10. We now construct the promised example to show that the complex space of Theorem 4.1 may not be biholomorphically equivalent to a subvariety of an open subset in C^n for any positive integer n. This kind of example is familiar.

Again let $\{x_n\}_{n=1,\ldots}$ be the sequence defined by $x_n = 1 - e^{-n^2}$. We have already noted, there are many functions in $A^\infty(\bar{\mathsf{U}})$ that vanish to order n at x_n for all $n = 1, \ldots$. Denote by $\mathfrak{A}_o$ the algebra of all functions $f \in A^\infty(\bar{\mathsf{U}})$ whose derivatives f' vanish to order n at x_n. Let $\mathfrak{A}$ be the uniform closure of $\mathfrak{A}_o$. The uniform algebra $\mathfrak{A}$ has $\bar{\mathsf{U}}$ as its spectrum, i.e., it separates points on $\bar{\mathsf{U}}$, and each nonzero C-algebra homomorphism from $\mathfrak{A}$ to C is of the form $g \mapsto g(z)$ for some fixed $z \in \bar{\mathsf{U}}$.

In proving this, we will use the fact that the multiplicative linear functionals, or characters, on the algebra $A^\infty(\bar{\mathsf{U}})$ are point evaluations at points of the closed unit disc $\bar{\mathsf{U}}$. Let φ be a character on $\mathfrak{A}$. Let I be the ideal of $A^\infty(\bar{\mathsf{U}})$ consisting of those functions f in $\mathfrak{A}_o$ that vanish at each point x_n, and let $\mathfrak{I}$ be the closure of I in $\mathfrak{A}$. There are two cases to consider: First, it may be that φ does not

[5] The algebra $\mathfrak{B}$ is a Fréchet algebra with identity but not a Banach algebra. By its spectrum we understand the set of its continuous characters, i.e., the set of continuous nonzeero C-algebra homomorphisms $\varphi : \mathfrak{B} \to \mathsf{C}$, endowed with the weak topology induced by the algebra $\mathfrak{B}$.

[6] The theorem of Rossi just cited is in a context much more general than ours in that it treats algebras of holomorphic functions on complex spaces of arbitrary dimension. One could rewrite its proof in the particular case of algebras on one-dimensional spaces and obtain thereby a considerably simpler demonstration of our Theorem 4.1 Doing so would require a few pages and does not seem appropriate in the present setting.

annihilate the ideal $\mathfrak{I}$, whence it does not annihilate the ideal I in $A^\infty(\bar{\mathsf{U}})$. If $g \in I$ satisfies $\varphi(g) \neq 0$, then we can consider, for any $f \in A^\infty(\bar{\mathsf{U}})$, the quantity $\varphi(fg)/\varphi(g)$. In fact this quantity is independent of the choice of g, subject only to the condition that $\varphi(g) \neq 0$: If $g, h \in I$ and neither $\varphi(g)$ nor $\varphi(h)$ vanishes, then for each $f \in A^\infty(\bar{\mathsf{U}})$,

$$\begin{aligned}\varphi(fg)/\varphi(g) &= (\varphi(fg)\varphi(h))/(\varphi(g)\varphi(h)) \\ &= \varphi(fgh)/\varphi(gh) = \varphi(fh)/\varphi(h).\end{aligned} \tag{6}$$

Thus, if we fix a $g \in I$ that is not annihilated by φ and define the functional $\tilde{\varphi}$ on $A^\infty(\bar{\mathsf{U}})$ by $\tilde{\varphi}(f) = \varphi(fg)/\varphi(g)$, we have a well-defined functional. It is linear and multiplicative, and it satisfies $\tilde{\varphi}(1) = 1$. Thus, there is a point $z_\varphi \in \bar{\mathsf{U}}$ such that for all $f \in A^\infty(\bar{\mathsf{U}})$, $\varphi(f) = z_\varphi$. The functional $\tilde{\varphi}$ agrees on the algebra $\mathfrak{A}_o$ with φ, for φ is multiplicative on $\mathfrak{A}_o$. Thus, φ acts on $\mathfrak{A}$ as evaluation at z_φ. If φ annihilates the ideal $\mathfrak{I}$, then it annihilates I, and so gives rise to a character φ^* on the quotient algebra $A^\infty(\bar{\mathsf{U}})/I$. The characters on this quotient algebra are of the form $[f] \mapsto f(y)$ for some y in the set $\{1, x_1, x_2, \ldots\}$. (Here $[f]$ denotes the residue class of f in the quotient algebra $A^\infty(\bar{\mathsf{U}})/I$.) It follows that φ acts on $\mathfrak{A}$ as evaluation at the point y. As $\mathfrak{A}$ separates points on $\bar{\mathsf{U}}$, we have that spec $\mathfrak{A} = \bar{\mathsf{U}}$.

The algebra $\mathfrak{A}$ is thus an algebra of the sort considered in Theorem 4.1.

The complex space associated with this example by Theorem 4.1 is the open unit disc U with the structure sheaf $\mathscr{F}$ defined by the condition that for each open subset V in U the space of sections $\mathscr{F}(V)$ is the space of all ordinary holomorphic functions f on V that satisfy $f^{(j)}(x_n) = 0$ in the range $1 \leq j \leq n$ for all those n for which $x_n \in V$. We denote this space by $\mathsf{U}_{\mathscr{F}}$. This is a reduced one-dimensional space and as such, it can be mapped holomorphically and homeomorphically onto a one-dimensional subvariety of C^3. See [11, p. 224, Th. VII.C.10].

There is, however, no *biholomorphic* embedding of $\mathsf{U}_{\mathscr{F}}$ as a subvariety of C^m for any positive integer m. This is so because the local embedding dimension (or tangential dimension) of $\mathsf{U}_{\mathscr{F}}$ at the point x_n is $n+1$, whence the local embedding dimension is not globally bounded. Recall that if $\mathfrak{X}$ is a complex space and $\mathfrak{x} \in \mathfrak{X}$, then the local embedding dimension of $\mathfrak{X}$ at $\mathfrak{x}$ is the dimension of the complex vector space $\mathfrak{m}_{\mathfrak{x}}/\mathfrak{m}_{\mathfrak{x}}^2$ in which $\mathfrak{m}_{\mathfrak{x}}$ is the maximal ideal in the local ring of germs of functions holomorphic at $\mathfrak{x}$. In the case of $\mathsf{U}_{\mathscr{F}}$, the maximal ideal $\mathfrak{m}_{x_n}$ is the ring of germs of functions with power series expansion about x_n of the form $\sum_{k=n+1}^\infty \alpha_{k+1}(z - x_n)^k$. Consequently, the dimension of $\mathfrak{m}_{x_n}/\mathfrak{m}_{x_n}^2$ has dimension $n+1$.

Thus, the complex space $\mathsf{U}_{\mathscr{F}}$ is not biholomorphically equivalent to a subvariety of any C^m.

5. Further questions

Certain questions suggested by the results above seem worthy of further investigation.

1. Find the correct analogue of Theorem 4.1 without the condition that the algebra admit a set of differentiable generators. The essence of the problem is found already in the disc case: If $\mathfrak{A}$ is a uniform algebra with the closed unit disc $\bar{\mathsf{U}}$ as its spectrum and with Shilov boundary contained in the circle $b\mathsf{U}$, is there a complex structure on U with respect to which the elements of $\mathfrak{A}$ are holomorphic?

2. If $\mathscr{M}$ is a compact manifold of dimension $n > 2$ and of class $\mathscr{C}^1$, if $\mathfrak{A}$ is a uniform algebra on $\mathscr{M}$ generated by functions of class $\mathscr{C}^1$ and with $\mathscr{M}$ as its spectrum, and if Γ is the Shilov boundary for $\mathfrak{A}$, does the set $\mathscr{M} \setminus \Gamma$, if not empty, exhibit some residual one-dimensional analytic structure? Specifically, is there any one-dimensional complex manifold in $\mathscr{M} \setminus \Gamma$ on which the elements of $\mathfrak{A}$ are holomorphic? Erlend Fornæss Wold has recently informed me that he, in collaboration with Alexander Izzo and Håken Samuelsson, has settled this question.

3. Show that if Σ is a one-dimensional subvariety of a bounded open set in C^n with $b\Sigma$ a finite union of mutually disjoint rectifiable simple closed curves, then the spectrum of the Banach algebra $A(\bar{\Sigma})$ is $\bar{\Sigma}$. In the case that Σ is a relatively compact subset of a larger ambient Riemann surface, the result is known and due to Arens [3].

The following remarks throw some light on this question but do not settle it.

We know that the set $\bar{\Sigma}$ is rationally convex so that spec $\mathscr{R}(\bar{\Sigma})$ is naturally identified with $\bar{\Sigma}$.

The inclusion $\mathscr{R}(\bar{\Sigma}) \hookrightarrow A(\bar{\Sigma})$ induces a map $\rho : \operatorname{spec} A(\bar{\Sigma}) \to \operatorname{spec} \mathscr{R}(\bar{\Sigma}) = \bar{\Sigma}$. This map is surjective; it is merely the restriction map that takes $\varphi \in \operatorname{spec} A(\bar{\Sigma})$ to $\varphi|\mathscr{R}(\bar{\Sigma}) \in \operatorname{spec} \mathscr{R}(\bar{\Sigma})$. What must be verified is that ρ is injective, i.e., that two elements of spec $A(\bar{\Sigma})$ that agree on $\mathscr{R}(\bar{\Sigma})$ coincide.

We first note that the map is injective over $\bar{\Sigma} \setminus \Gamma = \Sigma \setminus \Gamma$. That is, we show that if $\varphi \in \operatorname{spec} A(\bar{\Sigma})$, if $x \in \Sigma \setminus \Gamma$, and if for each $g \in \mathscr{R}(\bar{\Sigma})$, $\varphi(g) = g(x)$, then for each $f \in A(\bar{\Sigma})$, $\varphi(f) = f(x)$. Equivalently, we show that if $f \in A(\bar{\Sigma})$ vanishes at x, then $\varphi(f) = 0$. Suppose $\varphi \in \operatorname{spec} A(\bar{\Sigma})$ and that for all $f \in \mathscr{R}(\bar{\Sigma})$, $\varphi(f) = f(x)$ for the fixed point $x \in \Sigma \setminus \Gamma$. The rational convexity of Γ implies the existence of a function $g \in \mathscr{R}(\bar{\Sigma})$ that vanishes at x and that does not vanish at any point of the boundary Γ. Thus, the zero locus $g^{-1}(0)$ is a finite subset of Σ. If $f \in A(\bar{\Sigma})$ vanishes at x, then the *Nullstellensatz* yields the existence of a function h holomorphic on a neighborhood of x in Σ such that for some positive integer μ we have the equality $f^\mu = gh$ near x. As $h = f^\mu/g$ near x, we see that h is naturally defined as a holomorphic function on of $(\Sigma \setminus g^{-1}(0)) \cup \{x\}$ and that it has continuous boundary values along Γ. Let $r \in \mathscr{R}(\bar{\Sigma})$ satisfy $r(x) = 1$ and $r(y) = 0$ for all y other than x at which g vanishes. Then for a sufficiently large positive integer ν we have that $r^\nu f^\mu = ghr^\nu$ with the function hr^ν an element of $A(\bar{\Sigma})$. Thus $\varphi(f^\mu) = \varphi(r^\nu f^\mu) = \varphi(g)\varphi(hr^\nu) = 0$ whence $\varphi(f) = 0$. Accordingly for all $f \in A(\bar{\Sigma})$, $\varphi(f) = f(x)$.

Next, ρ is injective over the set of peak points for $\mathscr{R}(\bar{\Sigma})$. To see this, let be a peak point for $\mathscr{R}(\bar{\Sigma})$, and let $h_o \in \mathscr{R}(\bar{\Sigma})$ peak at x. Suppose $\varphi \in \operatorname{spec} A(\bar{\Sigma})$ to satisfy $\rho\varphi = x$. There is a probability measure μ on Γ such that for all $f \in A(\bar{\Sigma})$, $\varphi(f) = \int f\, d\mu$. For every $f \in A(\bar{\Sigma})$ we have

$$\varphi(f) = \varphi(f)\varphi(h_o^k) = \int f h_o^k \, d\mu \xrightarrow{k\to\infty} f(x)\mu(\{x\}),$$

so the kernel φ contains the maximal ideal of $A(\bar{\Sigma})$ that consists of the functions in $A(\bar{\Sigma})$ that vanish at the point x. Thus φ is found to be evaluation at x, and ρ is seen to be injective over the peak point x.

It remains to show that ρ is injective over points of the Shilov boundary for $\mathscr{R}(\bar{\Sigma})$ that are not peak points for $\mathscr{R}(\bar{\Sigma})$. Such points do exist in general as we have seen in examples given above.

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