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SELF-SIMILAR AUTOMORPHISMS OF A FREE GROUP OF COUNTABLE RANK

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Abstract

We investigate self-similar automorphisms of a free group F of infinite countable rank, that is automorphisms for which their actions on F and F' are similar. We show properties, examples and counterexamples of self-similar automorphisms and study the subgroup generated by self-similar automorphisms.

1. Introduction and Main Results

Let F be a free group of countable rank. As usual, F' denotes the commutator subgroup of F and we define the terms of the derived series $F^{(n)}$ of F as follows $F^{(0)} = F$, $F^{(1)} = F'$, $F^{(n+1)} = [F^{(n)}, F^{(n)}]$ for $n > 0$. We denote by N the set of natural numbers $\{1, 2, 3, \ldots\}$, and by Z the ring of integers.

We investigate self-similar automorphisms of a free group F of infinite countable rank, that is automorphisms for which their actions on F and F' are similar. We show properties, examples of self-similar automorphisms and study the subgroup generated by self-similar automorphisms.

The subgroups and the structure of the automorphism group of a free group of finite rank have been studied intensively (cf. surveys [8], [10]). However, we still know little about automorphisms of free groups of infinite rank. The group $\mathrm{Aut}(F)$ of automorphisms of a free group F of countable rank is "vast". For example, it contains an isomorphic copy of the group $S(\mathsf{N})$ of all permutations on natural numbers and an isomorphic copy of the group Z_2^{N} of the infinite series with entries 0 and 1. Some subgroups of the group $\mathrm{Aut}(F)$ are described in [4] and [7]. Many properties of $\mathrm{Aut}(F)$ can be found in [1], [2], [12], [13].

Throughout this paper if ξ is an automorphism of F then $\xi' = \xi|_{F'}$ denotes the restriction of ξ on F'. It is clear that ξ' is an automorphism of F'. Generally $\xi^{(n)}$ will denote the restriction of ξ on $F^{(n)}$.

Definition 1.1. We say that ξ is *self-similar* (or that ξ is *similar* to ξ') if there exists an isomorphism $\alpha : F \to F'$, such that the following diagram

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commutes ($\xi\alpha = \alpha\xi'$):

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & F' \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\xi'} \\ F & \xrightarrow{\alpha} & F' \end{array}$$

We denote by $\mathscr{M}$ the set of all self-similar automorphisms of F.

In terms of the language of operator groups (cf. [11]), α is an isomorphism from the operator group (F, ξ) to (F', ξ').

DEFINITION 1.2. Let X be a fixed basis of F. An automorphism $\tau \in \mathrm{Aut}(F)$ is an elementary simultaneous Nielsen automorphism if it satisfies one of the following conditions:

(1) τ permutes the set X.

(2) τ inverts some elements of X and acts trivially on the rest of elements of X.

(3) There exists the subset U of X such that $u^\tau = uv$ or vu for $u \in U$ and some v in $X \setminus U$ and $v^\tau = v$ for every $v \in X \setminus U$.

The notion of elementary simultaneous Nielsen automorphisms was introduced by Cohen in [4]. Bogopolski and Singhof in [1] call automorphisms of type (1) and (2) *monomial automorphisms*. The set of monomial automorphisms is a subgroup of $\mathrm{Aut}(F)$. We use the notation $\mathscr{E}_X$ for the subgroup generated by elementary simultaneous Nielsen automorphisms, and $\mathscr{E}$ for the subgroup generated by all $\mathscr{E}_X$, where X is a basis of F. The conjecture of D. Solitar states that $\mathscr{E}_X$ coincides with the subgroup of bounded automorphisms (see [4]).

We fix a basis (i.e. a free generator set) X of F and introduce the class $\mathscr{P}_X$ of automorphisms of F. An automorphism ξ belongs to $\mathscr{P}_X$ if there exist three pairwise disjoint, countable subsets $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$, $C = \{c_1, c_2, \ldots\}$, such that $X = A \cup B \cup C$ and for every $n \in \mathsf{N}$ $a_n^\xi = b_n$, $b_n^\xi = a_n$ and $c_n^\xi = c_n^{-1}$. Clearly, all automorphisms in $\mathscr{P}_X$ have order two and any two of them are conjugate. Shortly, we write that ξ acts on the basis $X = A \cup B \cup C$ as follows: $A \leftrightarrow B$, $C \to C^{-1}$. Let $\mathscr{P}$ be the subgroup generated by the union of all $\mathscr{P}_X$, where X is a basis of the group F.

Let $\mathscr{S}_X$ consist of automorphisms permuting the basis X. Clearly, $\mathscr{S}_X$ is a subgroup of $\mathscr{E}_X$ and $\mathscr{S}_X$ is isomorphic to the group $S(\mathsf{N})$ of all permutations of natural numbers. We use the symbol $\mathscr{S}$ for the subgroup generated by all $\mathscr{S}_X$, where X is a basis of F.

We need one more set of automorphisms, $\mathscr{L}_X$. An automorphism α belongs to $\mathscr{L}_X$ if it inverts some elements from X, and does not change remaining elements. It is clear that $\mathscr{L}_X$ is isomorphic to the group Z_2^{N} of all infinite series with coordinates 0 or 1. As previously $\mathscr{L}$ is the subgroup generated by the union of all $\mathscr{L}_X$, where X is a basis of the group F.

The intersection $\mathscr{L}_X \cap \mathscr{S}_X$ is trivial. The product $\mathscr{L}_X\mathscr{S}_X$ is the subgroup of $\mathrm{Aut}(F)$ and in fact it is the subgroup of monomial automorphisms on the set X. In the group $\mathscr{L}_X\mathscr{S}_X$ the subgroup $\mathscr{L}_X$ is normal, while $\mathscr{S}_X$ is not normal. So, the product $\mathscr{L}_X\mathscr{S}_X$ is a semidirect product $\mathscr{L}_X \rtimes \mathscr{S}_X$ of $\mathscr{L}_X$ by $\mathscr{S}_X$. If we look closely we will see that this subgroup is isomorphic to the wreath product $Z_2 \wr S(\mathsf{N})$ (cf. [5]). Elements of this wreath product have a form $(\sigma, (\varepsilon_1, \varepsilon_2, \ldots))$, where $\sigma \in S(\mathsf{N})$ and $((\varepsilon_1, \varepsilon_2, \ldots) \in Z_2^{\mathsf{N}}$. Multiplication is given by

$$(\sigma, (\varepsilon_1, \varepsilon_2, \ldots)) \cdot (\delta, (\epsilon_1, \epsilon_2, \ldots)) = (\sigma\delta, (\varepsilon_{1^\delta}, \varepsilon_{2^\delta}, \ldots) + (\epsilon_1, \epsilon_2, \ldots)),$$

and $(\sigma, (\varepsilon_1, \varepsilon_2, \ldots))$ is associated with automorphism $\alpha \in \mathscr{L}_X\mathscr{S}_X$ acting on $X = \{x_1, x_2, \ldots\}$ as follows: $x_i^\alpha = x_{i^\sigma}^{(-1)^{\varepsilon_{i^\sigma}}}$.

The aim of this paper is to prove the following theorem.

THEOREM 1.3.

(i) *Every automorphism ξ from $\mathscr{P}_X$ is self-similar, so $\mathscr{P} \subseteq \langle \mathscr{M} \rangle$.*

(ii) $\langle \mathscr{P}_X \rangle = \mathscr{L}_X\mathscr{S}_X$, $\mathscr{L}_X \cap \mathscr{S}_X = 1$, *so $\langle \mathscr{P}_X \rangle$ consists of all monomial automorphisms on the set X and is isomorphic to the wreath product $Z_2 \wr S(\mathsf{N})$.*

(iii) $\mathscr{E} = \mathscr{P} < \langle \mathscr{M} \rangle \lhd \mathrm{Aut}(F)$.

REMARK 1.4. It can be deduced from ([3], Theorem C) and Theorem 1.3 (iii) that if $\langle \mathscr{M} \rangle \neq \mathrm{Aut}(F)$ then the index of $\langle \mathscr{M} \rangle$ in $\mathrm{Aut}(F)$ equals $2^{\aleph_0}$.

This work is inspired by research on automorphisms permuting generators in free groups of finite rank (cf. [9], [15]). The following example shows the relationship between self-similar automorphisms and automorphisms permuting generators in the free group of rank two.

EXAMPLE 1.5. Let F_2 be a free group of rank 2, freely generated by x and y and let σ be the automorphism of F_2 permuting x and y. Then the commutator subgroup $F = F_2'$ of $F_2 = \langle x, y \rangle$ is a free group of infinite rank. It can be deduced from [10] (4.3) that $F = F_2'$ is freely generated by the set $X = A \cup B \cup C$, where $A = \{[x^c, y^d] : c > d,\ c, d \in \mathsf{Z}\}$, $B = \{[y^c, x^d] : c > d,\ c, d \in \mathsf{Z}\}$, $C = \{[x^c, y^c] : c \in \mathsf{Z}\}$. The automorphism $\xi = \sigma|_F$, which is the restriction of σ to $F = F_2'$, acts on this basis according to the schema $A \leftrightarrow B$, $C \to C^{-1}$. So, ξ belongs to $\mathscr{P}_X$. Then by Theorem 1.3 (i), ξ belongs to $\mathscr{M}$, that is ξ is similar to $\xi' \in \mathrm{Aut}(F') = \mathrm{Aut}(F_2'')$.

2. Properties of the set $\mathcal{M}$

PROPOSITION 2.1. *Let α be an isomorphism from F to F'. Then*

(i) *For every $w \in F$ we have $w^\alpha \in F''$ if and only if $w \in F'$. Generally $w^\alpha \in F^{(n)}$ for $n \geq 1$ if and only if $w \in F^{(n-1)}$.*

(ii) *α has no nontrivial fixed points.*

(iii) *The restriction $\alpha' = \alpha|_{F'}$ is an isomorphism between F' and F''.*

(iv) *The mapping $\beta : F/F' \to F'/F''$, given by $(wF')^\beta = w^\alpha F''$, is the isomorphism of free abelian groups.*

PROOF. (i) is clear.

(ii) Let $g \neq 1$ be a fixed point of α and let n be such a number that $g \in F^{(n-1)}$ but $g \notin F^{(n)}$. Then by (i) we have $g = g^\alpha \in F^{(n)}$, which is a contradiction.

(iii) It follows from (i) that α' is a bijection from F' onto F'', so it is an isomorphism.

(iv) First, we show that $wF' = uF'$ if and only if $w^\alpha F'' = u^\alpha F''$. Indeed $wF' = uF'$ if and only if $wu^{-1} \in F'$, then by (i) it is equivalent to $(wu^{-1})^\alpha \in F''$, and so to $w^\alpha F'' = u^\alpha F''$. Thus, β is correctly defined and is an injection. Since α is a surjection, β is also a surjection. So β is an isomorphism.

An automorphism ξ induces the automorphism $\bar{\xi}$ of the free abelian group F/F' and similarly ξ' induces the automorphism $\bar{\xi}'$ of F'/F''. Moreover, it follows from Proposition 2.1 (iv) that the following diagram:

$$\begin{array}{ccc} F/F' & \xrightarrow{\ \beta\ } & F'/F'' \\ \bar{\xi}\downarrow & & \downarrow\bar{\xi}' \\ F/F' & \xrightarrow{\ \beta\ } & F'/F'' \end{array}$$

commutes, that is $\bar{\xi}\beta = \beta\bar{\xi}'$ or equivalently:

$$\beta^{-1}\bar{\xi}\beta = \bar{\xi}' \tag{2.1}$$

Now we show examples of automorphisms not belonging to $\mathcal{M}$.

EXAMPLE 2.2. Let ξ be the automorphism of F which inverts every element of basis $\{x_1, x_2, x_3, \ldots\}$, that is for $i = 1, 2, 3, \ldots$, $x_i^\xi = x_i^{-1}$. Then groups with operators (F, ξ) and (F', ξ') are not isomorphic. So, ξ does not belong to $\mathcal{M}$.

PROOF. It is clear that ξ induces the automorphism $\bar{\xi}$ of the free abelian group F/F', which sends every element into its inverse. But $\bar{\xi}'$ does not, since for example $[x_1^{-1}, x_2^{-1}] \neq [x_1, x_2]^{-1} \bmod F''$ and the equation (2.1) does not hold.

EXAMPLE 2.3. An inner automorphism $i_g(w) = g^{-1}wg$ belongs to $\mathcal{M}$ only for $g = 1$.

PROOF. Let $\alpha : F \to F'$ be an isomorphism, for which $i_g\alpha = \alpha i'_g$. Thus for every $w \in F$ we have $(g^{-1}wg)^\alpha = g^{-1}w^\alpha g$. From this we get $gg^{-\alpha}w^\alpha = w^\alpha gg^{-\alpha}$. Since α is an isomorphism, $gg^{-\alpha}$ lies in the center of F', so $g^\alpha = g$. By Proposition 2.1 (ii) $g = 1$.

PROPOSITION 2.4. *Let ξ be a self-similar automorphism of F. Then*

(i) *ξ^{-1} also belongs to $\mathcal{M}$.*

(ii) *If ζ is an automorphism of F, conjugate to ξ then ζ belongs to $\mathcal{M}$.*

(iii) *$\mathcal{M}$ is not a subgroup.*

(iv) *For every natural number n the automorphism ξ is similar to $\xi^{(n)}$, where $\xi^{(n)} = \xi|_{F^{(n)}}$ is the restriction of ξ to $F^{(n)}$. If $\alpha : F \to F'$ is an isomorphism such that $\xi\alpha = \alpha\xi'$ then $\alpha^n : F \to F^{(n)}$ is an isomorphism such that $\xi\alpha^n = \alpha^n\xi^{(n)}$.*

PROOF. (i) If $\alpha : F \to F'$ is an isomorphism such that $\xi\alpha = \alpha\xi'$ then $\alpha^{-1}\xi\alpha = \xi'$ and $\alpha^{-1}\xi^{-1}\alpha = \xi'^{-1}$. Since ξ'^{-1} is the restriction of ξ^{-1} to F', the statement follows.

(ii) If $\zeta = \beta^{-1}\xi\beta$ then $\zeta' = \beta'^{-1}\xi'\beta'$. If $\xi = \alpha\xi'\alpha^{-1}$ then

$$\begin{aligned}\zeta = \beta^{-1}\alpha\xi'\alpha^{-1}\beta &= \beta^{-1}\alpha\beta'\beta'^{-1}\xi'\beta'\beta'^{-1}\alpha^{-1}\beta\\ &= (\beta'^{-1}\alpha^{-1}\beta)^{-1}\zeta'\beta'^{-1}\alpha^{-1}\beta,\end{aligned}$$

and since $\beta'^{-1}\alpha^{-1}\beta$ is an isomorphism mapping F onto F', ζ is self-similar.

(iii) We shall show in Section 3 that the automorphism inverting all elements of the basis X is the product of automorphisms from $\mathcal{P}_X$ and we shall show in Section 5 that the subgroup generated by $\mathcal{P}_X$ is contained in the subgroup generated by $\mathcal{M}$. But as it is shown in Example 2.2 the automorphism inverting all elements of a fixed basis does not belong to $\mathcal{M}$. So $\mathcal{M}$ is not a subgroup.

(iv) It follows from Proposition 2.1 (i) that α^n is an isomorphism that maps F onto $F^{(n)}$. The equality $\xi\alpha^n = \alpha^n\xi^{(n)}$ can be proved by induction on n.

3. Proof of Theorem 1.3 (ii)

We recall that α belongs to $\mathcal{P}_X$ if there are three countable, pairwise disjoint subsets $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$, $C = \{c_1, c_2, \ldots\}$, such that $X = A \cup B \cup C$ and for every natural n we have $a_n^\alpha = b_n$, $b_n^\alpha = a_n$, $c_n^\alpha = c_n^{-1}$. We use the short notation $\alpha : A \leftrightarrow B, C \to C^{-1}$.

PROOF OF THEOREM 1.3 (ii). Our aim is to prove that $\langle \mathscr{P}_X \rangle = \mathscr{L}_X \mathscr{S}_X$. Every automorphism from the set $\langle \mathscr{P}_X \rangle$ belongs to $\mathscr{L}_X \mathscr{S}_X$. So it remains to prove that $\mathscr{L}_X \mathscr{S}_X$ is contained in $\langle \mathscr{P}_X \rangle$.

First we show that $\mathscr{L}_X \subseteq \langle \mathscr{P}_X \rangle$.

We split X into three infinite, pairwise disjoint subsets $X = A \cup B \cup C$. Let α, β, γ be automorphisms acting on these sets as follows: $\alpha : A \leftrightarrow B, C \to C^{-1}$, $\beta : A \leftrightarrow C, B \to B^{-1}$, $\gamma : B \leftrightarrow C, A \to A^{-1}$. The automorphisms α, β, γ belong to $\mathscr{P}_X$ and $\alpha\beta\alpha\gamma$ is the automorphism acting identically on A and inverting elements in $B \cup C$.

So we have proved that every automorphism δ for which there is a partition of the set $X = X_1 \cup X_2$ into two infinite, disjoint subsets such that δ acts trivially on X_1 and inverts all elements in X_2, belongs to $\langle \mathscr{P}_X \rangle$. We define such automorphisms by giving these two sets and show now that the set of these automorphisms generates $\mathscr{L}_X$.

For every automorphism η in $\mathscr{L}_X$ there exist subsets U and V (not necessarily infinite, and even one of them can be empty), such that η acts trivially on U and inverts elements from V. If U and V are infinite then η is among generators for $X_1 = U$ and $X_2 = V$.

If U is finite (or empty) then V must be infinite. We partition V into two infinite, disjoint subsets $V = V_1 \cup V_2$. Then η is a composition of two generators. For the first one $X_1 = U \cup V_1$, $X_2 = V_2$, and for the other $X_1 = U \cup V_2$, $X_2 = V_1$.

If V is finite then U is infinite. Let V_1 and V_2 be infinite, disjoint subsets such that $X = V_1 \cup V_2$ and $V \subseteq V_2$. Then η is the product of two generators. For the first one $X_1 = V_2$, $X_2 = V_1$, and for the second $X_1 = V_2 \setminus V$, $X_2 = V \cup V_1$. Since V is finite, in the second case both sets are also infinite.

Now we prove that $\mathscr{S}_X \subseteq \langle \mathscr{P}_X \rangle$.

By [5] (Lemma 8.1A, p. 256) every permutation is a product of two involutions. So, it suffices to prove that every permutation of order two belongs to $\langle \mathscr{P}_X \rangle$.

We have shown above that every automorphism α for which there exist three infinite, pairwise disjoint subsets $A = \{a_1, a_2, \ldots\}$, $B = \{b_1, b_2, \ldots\}$, $C = \{c_1, c_2, \ldots\}$ such that $a_n^\alpha = b_n$, $b_n^\alpha = a_n$, $c_n^\alpha = c_n$ for every natural n, belongs to $\langle \mathscr{P}_X \rangle$. As previously, we use the notation $\alpha : A \leftrightarrow B, C \to C$. To define α it is enough to indicate sets A, B and C. Throughout this proof we call such automorphisms generators.

Further part of the proof is similar to the one above. Let β be any involution in $\mathscr{S}_X$ for which there exist three subsets U, V, W (not necessarily infinite), such that $\eta : U \leftrightarrow V, W \to W$. If U, V, W are infinite then η is among the generators.

If U, V are infinite and W is finite then we partition U into two disjoint subsets $U = U_1 \cup U_2$. Thus, we get the partition of the set V: $V = U^\eta = (U_1 \cup U_2)^\eta = U_1^\eta \cup U_2^\eta = V_1 \cup V_2$. Then η is the composition of two generators. The first one is defined by $A = U_1$, $B = V_1$, $C = U_2 \cup V_2 \cup W$, and the second one is defined by $A = U_2$, $B = V_2$, $C = U_1 \cup V_1 \cup W$.

If U, V are finite then W is infinite. We partition W into three infinite, disjoint subsets $W = U_1 \cup V_1 \cup W_1$. Then η is a product of two generators. The first is defined by sets $A = U_1 \cup U$, $B = V_1 \cup V$, $C = W_1$ and the second is defined by $A = U_1$, $B = V_1$, $C = U \cup V \cup W_1$.

Since $\mathscr{L}_X \subseteq \langle \mathscr{P}_X \rangle$ and $\mathscr{S}_X \subseteq \langle \mathscr{P}_X \rangle$ we have that $\mathscr{L}_X \mathscr{S}_X$ is contained in $\langle \mathscr{P}_X \rangle$.

4. Proof of Theorem 1.3 (iii)

LEMMA 4.1. $\mathscr{E}_X = \langle \mathscr{P}_{X_\tau}, \tau \in \mathscr{E}_X \rangle$.

PROOF. Let τ belong to $\mathscr{E}_X$. We partition X into three pairwise disjoint subsets $X = U \cup V \cup W$. Subset W is non "active", that is if $w \in W$ then $w^\tau = w$. Also for $v \in V$ we have $v^\tau = v$ but elements from this set act on elements from U. Let $V = \{v_1, v_2, v_3, \ldots\}$. We partition U into subsets $U = U_1 \cup U_2 \cup \ldots$. If $u \in U_i$ then $u^\tau = uv_i$. Let U_i consist of elements $u_{i1}, u_{i2}, \ldots$ for $i = 1, 2, \ldots$. Now we define a new basis Y which is the union $U' \cup W$, where $U' = U_1' \cup U_2' \cup \ldots$ and $U_i = \{u_{i1}, u_{i1}v_i, u_{i2}u_{i1}^{-1}, u_{i3}u_{i1}^{-1}, \ldots\}$. One can check that Y is a free generator set of F and that the automorphism σ defined by mapping $X \to Y$ is bounded in X (and in Y), so σ belongs to $\mathscr{E}_X$. Now we define an automorphism η. The automorphism η changes u_{i1} and $u_{i1}v_i$ for $i = 1, 2, \ldots$ and acts identically on other elements. By Theorem 1.3 (ii) η belongs to $\mathscr{P}_Y$. How does η act on elements of the basis X? Let us calculate $v_i^\eta = (u_{i1}^{-1}u_{i1}v_i)^\eta = (u_{i1}v_i)^{-1}u_{i1} = v_i^{-1}$, $u_{i1}^\eta = u_{i1}v_i$, and for $k > 1$ $u_{ik}^\eta = [(u_{ik}u_{i1}^{-1})(u_{i1})]^\eta = u_{ik}u_{i1}^{-1}u_{i1}v_i = u_{ik}v_i$. Thus $\tau = \vartheta\eta$, where ϑ inverts all v_i and acts trivially on other elements. Since $\vartheta \in \mathscr{L}_X$ and by Theorem 1.3 (ii) $\mathscr{L}_X \subseteq \mathscr{P}_X$ we get $\tau \in \langle \mathscr{P}_X, \mathscr{P}_Y \rangle$. For other variants of the mapping τ the reasoning is analogous. This completes the proof.

PROOF OF THEOREM 1.3 (iii). By Theorem 1.3 (i) (which will be proved in the following section) and Lemma 4.1 we have $\mathscr{E} = \mathscr{P} < \langle \mathscr{M} \rangle$ so the statement is true. By Proposition 2.4 (ii), $\langle \mathscr{M} \rangle$ is normal in $\mathrm{Aut}(F)$.

5. Proof of Theorem 1.3 (i)

Now we are ready to prove the point (i) of Theorem 1.3. Let ξ belong to $\mathscr{P}_X$. So, X is a union $A \cup B \cup C$ of three infinite, pairwise disjoint subsets $A = \{a_1, a_2, a_3, \ldots\}$, $B = \{b_1, b_2, b_3, \ldots\}$, $C = \{c_1, c_2, c_3, \ldots\}$ and ξ acts on this basis as follows: $a_i^\xi = b_i$, $b_i^\xi = a_i$, $c_i^\xi = c_i^{-1}$, for all $i \in \mathsf{N}$. To prove that

ξ is self-similar we have to show that F' has a basis of the form $\mathfrak{a} \cup \mathfrak{b} \cup \mathfrak{c}$ in F', on which ξ' acts in similar way as ξ on X, that is $\xi' : \mathfrak{a} \leftrightarrow \mathfrak{b}, \mathfrak{c} \to \mathfrak{c}^{-1}$. We call such a basis $\mathscr{P}$-basis. It is clear that $\xi^2 = \mathrm{id}$.

A similar basis can be constructed by using Dyer-Scott Theorem (see [6], Theorem 3) but this theorem does not imply that all sets $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are infinite.

We use the following order in the set of nontrivial powers of generators $\{a_i^{k_i} : k_i \in \mathsf{Z} \setminus \{0\}, i \in \mathsf{N}\} \cup \{b_i^{l_i} : l_i \in \mathsf{Z} \setminus \{0\}, i \in \mathsf{N}\} \cup \{c_i^{m_i} : m_i \in \mathsf{Z} \setminus \{0\}, i \in \mathsf{N}\}$:

$$(5.1) \qquad a_1^{k_1} < b_1^{l_1} < c_1^{m_1} < \cdots < a_i^{k_i} < b_i^{l_i} < c_i^{m_i} < a_{i+1}^{k_{i+1}} < b_{i+1}^{l_{i+1}} < c_{i+1}^{m_{i+1}} < \cdots,$$

and if $k < l$ then $a_i^k < a_i^l$, $b_i^k < b_i^l$ and $c_i^k < c_i^l$ for every $i \in \mathsf{N}$. It can be deduced from [14] that F' is freely generated by the set of all commutators of the form $[y^k, z^l]^{x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}}$ such that $y, z, x_1, \ldots, x_k \in A \cup B \cup C$, $y < z$, $y < x_1 \le x_2 \le \cdots \le x_k$ and $z \notin \{x_1, x_2, \ldots, x_k\}$, and $k, l, d_1, \ldots, d_k$ are integers. Let us denote this basis of F' by $\mathfrak{Y}$. This basis is not $\mathscr{P}$-basis. We have to reconstruct $\mathfrak{Y}$ to get the proper one.

We use the common, possibly trivial, symbols $\alpha_i, \beta_i, \gamma_i$ for elements of the subgroup $\langle a_i \rangle, \langle b_i \rangle, \langle c_i \rangle$, respectively. We use the symbols μ_i or μ_i' for elements of the set $\{\alpha_i, \beta_i, \gamma_i\}$. Then the basis $\mathfrak{Y}$ consists of commutators of the form $[\mu_i, \mu_j']^{\mu_{i_1} \cdots \mu_{i_k}}$, where $i \le j$, $i_1 \le i_2 \le \cdots \le i_k$ and $\mu_j' \notin \{\mu_{i_1}, \ldots, \mu_{i_k}\}$.

We split the basis $\mathfrak{Y}$ into three disjoint subsets:

$$\mathfrak{Y} = T \cup Q \cup P$$

where

$$\begin{aligned} T &= \{[\alpha_i, \beta_i]^h\}, \\ Q &= \{[\alpha_i, \gamma_i]^{\beta_i h}, [\alpha_i, \mu_j]^{\beta_i h}, i < j\}, \\ P &= \mathfrak{Y} \setminus (T \cup Q), \end{aligned}$$

where h is an ordered word in the alphabet $A \cup B \cup C$. We say that the word h in the alphabet $A \cup B \cup C$ is *ordered* if $h = x_1^{d_1} x_2^{d_2} \ldots x_k^{d_k}$, $x_1, \ldots, x_k \in A \cup B \cup C$, $d_1, \ldots, d_k$ are integers and $x_1 < x_2 < \cdots < x_k$. Let $\mathscr{H}$ denote a set of all ordered words. Ordered words appear in exponents of commutators of the basis $\mathfrak{Y}$.

LEMMA 5.1. *For every word $w \in F$ there exists a unique ordered word $\bar{w}$ and an element $t \in F'$, such that $w = \bar{w} t$.*

PROOF. We can change the letters modulo F', so every word can be uniquely ordered modulo F'.

We say that the ordered word $\bar{w}$ is the ordered image of w if there exists $t \in F'$, such that $w = \bar{w}t$. If $w \in F$ and $t \in F'$ then $\overline{wt} = \bar{w}$. It is clear that if h is an ordered word then $\bar{h} = h$.

Our plan is to change the basis consequently:

$$\mathfrak{Y} = T \cup Q \cup P \to \mathfrak{Y}' = T' \cup Q \cup P \to \mathfrak{Y}'' = T' \cup Q' \cup P \to \mathfrak{Y}''' = T' \cup Q' \cup P'$$

where $\mathfrak{Y}, \mathfrak{Y}', \mathfrak{Y}'', \mathfrak{Y}'''$ are bases of F' and sets T', Q', P' are parts of the new $\mathcal{P}$-basis. So the last set $\mathfrak{Y}'''$ is a $\mathcal{P}$-basis of F'.

In every step we use elementary simultaneous Nielsen transformations (see Section 1). These transformations are invertible and hence change any basis of F into a new basis. We call these transformations, for short, Nielsen transformations.

By the length $|w|$ of a word $w \in F$ we mean its length in the alphabet $A \cup B \cup C$.

Let us partition the set $\mathcal{H}$ of all ordered words into three disjoint subsets $\mathcal{H}_<$, $\mathcal{H}_>$ and $\mathcal{H}_=$, where:

$$\mathcal{H}_< = \{h : \ h < \overline{h^\xi}\}, \mathcal{H}_> = \{h : \ h > \overline{h^\xi}\}, \mathcal{H}_= = \{h : \ h = \overline{h^\xi}\},$$

where $<$ is the lexicographical order in $\mathcal{H}$ induced by the order (5.1) and $\overline{h^\xi}$ is the order image of h^ξ.

LEMMA 5.2.

(i) *If $w \in F$ then $\overline{\overline{w}^\xi} = \overline{w^\xi}$.*

(ii) *$h \in \mathcal{H}_<$ if and only if $\overline{h^\xi} \in \mathcal{H}_>$.*

(iii) *If $h \in \mathcal{H}_=$ then $h^\xi = ht$, where $t \in F'$ and $t^\xi = t^{-1}$.*

(iv) *For every $h' \in \mathcal{H}$ there exist $t \in F'$ and $h \in \mathcal{H}$ such that $h't = h^\xi$.*

(v) *Let $h \in \mathcal{H}$ and $\mu \in A \cup B \cup C$ then $\mu h = h'v$, where h' is an ordered word and v is a product of elements from $\mathfrak{Y}$ or their inverses, for which words in exponents are shorter than h.*

PROOF. (i) If $w \in F$ then there exists $t \in F'$, such that $w = \bar{w}t$, hence $\overline{w^\xi} = \overline{(\overline{w}t)^\xi} = \overline{\overline{w}^\xi t^\xi} = \overline{\overline{w}^\xi}$, since $t^\xi \in F'$.

(ii) If $h \in \mathcal{H}_<$ then $h < \overline{h^\xi}$, and by (i):

$$\overline{h^\xi} > h = \overline{(\overline{h^\xi})^\xi}$$

so $\overline{h^\xi} \in \mathcal{H}_>$. The converse is clear.

(iii) Let $h \in \mathcal{H}_=$ then $h = \overline{h^\xi}$. There exists $t \in F'$, such that $h^\xi = \overline{h^\xi}t = ht$ and

$$h = (h^\xi)^\xi = (ht)^\xi = htt^\xi,$$

so $tt^{\xi} = 1$.

(iv) There exist $t' \in F'$ and $h \in \mathscr{H}$, such that $(h')^{\xi} = ht'$, so $h' = h^{\xi}(t')^{\xi}$ and

$$h^{\xi} = h't, \text{ for } t = (t')^{-\xi}.$$

(v) If μh is ordered then $h' = \mu h$ and $v = 1$. If μh is not ordered then $h = h_1\mu^d h_2$, where h_1 contains all symbols less then μ, h_2 all symbols greater then μ and d is an integer (possible that $d = 0$). Then

$$\mu h = \mu h_1 \mu^d h_2 = h_1\mu[\mu, h_1]\mu^d h_2 = h_1\mu^{d+1}h_2[\mu, h_1]^{\mu^d h_2}$$

and $h' := h_1\mu^{d+1}h_2$ is ordered, $v = [\mu, h_1]^{\mu^d h_2} = [\mu^{d+1}, h_1]^{h_2}[\mu^d, h_1]^{-h_2}$. If $h_1 = \mu_1 \dots \mu_k$ then for every integer n:

$$\begin{aligned} v &= [\mu^n, h_1]^{h_2} = [\mu^n, \mu_1 \dots \mu_k]^{h_2} \\ &= [\mu^n, \mu_1]^{h_2}[\mu^n, \mu_2]^{\mu_1 h_2} \dots [\mu^n, \mu_k]^{\mu_1 \dots \mu_{k-1} h_2} \end{aligned}$$

and all words in exponents are shorter than h.

5.1. The subset T

Let us remind that

$$T = \{[\alpha_i, \beta_i]^h : h \in \mathscr{H} \text{ and } h \text{ begins with a symbol greater then } \beta_i, i \geq 1\}$$

We denote by $\mathfrak{T}$ the subgroup generated by T.

LEMMA 5.3. *Let $h \in \mathscr{H}$. Then there exists $t \in \mathfrak{T}$ such that $h^{\xi} = \overline{h^{\xi}}t$ and if $t = \prod[\alpha_i, \beta_i]^{h_i}$ then every h_i is shorter than h.*

PROOF. If h is ordered, then

$$h = (\alpha_1\beta_1\gamma_1)(\alpha_2\beta_2\gamma_2)\dots(\alpha_k\beta_k\gamma_k),$$

hence

$$\begin{aligned} h^{\xi} &= (\alpha_1^{\xi}\beta_1^{\xi}\gamma_1^{\xi})(\alpha_2^{\xi}\beta_2^{\xi}\gamma_2^{\xi})\dots(\alpha_k^{\xi}\beta_k^{\xi}\gamma_k^{\xi}) \\ &= (\beta_1'\alpha_1'\gamma_1^{-1})(\beta_2'\alpha_2'\gamma_2^{-1})\dots(\beta_k'\alpha_k'\gamma_k^{-1}) \\ &= \alpha_1'\beta_1'[\beta_1', \alpha_1']\gamma_1^{-1}\alpha_2'\beta_2'[\beta_2', \alpha_2']\gamma_2^{-1}\dots\alpha_k'\beta_k'[\beta_k', \alpha_k']\gamma_k^{-1} \\ &= \alpha_1'\beta_1'[\beta_1', \alpha_1']\gamma_1^{-1}\alpha_2'\beta_2'[\beta_2', \alpha_2']\gamma_2^{-1}\dots\alpha_k'\beta_k'\gamma_k^{-1}[\beta_k', \alpha_k']^{\gamma_k^{-1}} \\ &= \alpha_1'\beta_1'\gamma_1^{-1}\alpha_2'\beta_2'\gamma_2^{-1} \\ &\qquad \dots\alpha_k'\beta_k'\gamma_k^{-1}[\beta_1', \alpha_1']^{\gamma_1^{-1}\alpha_2'\beta_2'\gamma_2^{-1}\dots\gamma_k^{-1}}[\beta_2', \alpha_2']^{\gamma_2^{-1}\dots\gamma_k^{-1}}\dots[\beta_k', \alpha_k']^{\gamma_k^{-1}} \end{aligned}$$

where $\alpha'_i = \beta_i^{\xi} \in A$, $\beta'_i = \alpha_i^{\xi} \in B$. Thus we have $\overline{h^{\xi}} = \alpha'_1\beta'_1\gamma_1^{-1}\alpha'_2\beta'_2\gamma_2^{-1} \dots \alpha'_k\beta'_k\gamma_k^{-1}$ and

$$t = [\beta'_1, \alpha'_1]^{\gamma_1^{-1}\alpha'_2\beta'_2\gamma_2^{-1}\dots\gamma_k^{-1}}[\beta'_2, \alpha'_2]^{\gamma_2^{-1}\alpha'_3\beta'_3\gamma_2^{-1}\dots\gamma_k^{-1}} \dots [\beta'_k, \alpha'_k]^{\gamma_k^{-1}} \in \mathfrak{T},$$

and the longest word which can appear in the exponent is $\gamma_1^{-1}\alpha'_2\beta'_2\gamma_2^{-1} \dots \gamma_k^{-1}$ and it is shorter than h.

It may happen that α_1 or β_1 is equal to 1. But then $[\beta_1, \alpha_1] = 1$ and the longest word in the exponent is equal to $\gamma_2^{-1}\alpha'_3\beta'_3\gamma_2^{-1} \dots \gamma_k^{-1}$ which also is shorter then h.

It follows from Lemma 5.3 that the subgroup $\mathfrak{T} = \langle T \rangle$ is ξ-invariant, so we change the basis T inside the subgroup $\mathfrak{T}$.

LEMMA 5.4. *The subgroup $\mathfrak{T}$ possesses a $\mathscr{P}$-basis.*

PROOF. We split T into disjoint subsets, with respect to the length of the words in the exponent:

$$T = T_0 \cup T_1 \cup T_2 \cup T_3 \cup \dots$$

where $T_k = \{[\alpha_i, \beta_i]^h : \ h \text{ has the length equal to } k, i \geq 1\}$. We show, by induction on n, that every subgroup $\langle T_0 \cup \dots \cup T_n \rangle$ has a $\mathscr{P}$-basis $A_n \cup B_n \cup C_n$. It is clear that $\langle T_0 \rangle$ has a $\mathscr{P}$-basis (this construction is similar to the one in Example 1.5). Let us assume that $\langle T_0 \cup \dots \cup T_n \rangle$ has a $\mathscr{P}$-basis. Let $w \in T_{n+1}$ then $w = [\alpha_i, \beta_i]^h$, $|h| = n + 1$. We split T_{n+1} into three disjoint subsets:

$$T_{n+1} = T_< \cup T_> \cup T_=$$

where

$$T_< = \{[\alpha_i, \beta_i]^h : \ |h| = n + 1, h \in \mathscr{H}_<\},$$
$$T_> = \{[\alpha_i, \beta_i]^h : \ |h| = n + 1, h \in \mathscr{H}_>\},$$
$$T_= = \{[\alpha_i, \beta_i]^h : \ |h| = n + 1, h \in \mathscr{H}_=\}.$$

If $h \in \mathscr{H}_<$, then by Lemma 5.3 we have $h^{\xi} = h't$, where $t \in \langle T_0 \cup \dots \cup T_n \rangle$, and by Lemma 5.2 (ii) we have $h' = \overline{h^{\xi}} \in \mathscr{H}_>$. So for $[\alpha_i, \beta_i]^h \in T_<$ we have:

$$([\alpha_i, \beta_i]^h)^{\xi} = [\beta'_i, \alpha'_i]^{h't}$$

Hence we put $[\alpha_i, \beta_i]^h \in T_<$ into A_{n+1} and we transform every element $[\alpha'_i, \beta'_i]^{h'}$ by $t \in \langle T_0 \cup \dots \cup T_n \rangle$, inverse and put the element obtained in that way into B_{n+1}. Hence we get:

$$A_{n+1} \ni [a_i^k, b_i^l]^h \xrightarrow{\xi} [b_i^k, a_i^l]^{h^{\xi}} = [a_i^l, b_i^k]^{-h't} \in B_{n+1}.$$

Let us note that above transformations are Nielsen transformations because we act on elements from T_{n+1} by elements from $\langle T_0 \cup \ldots \cup T_n \rangle$.

If $[a_i^k, b_i^l]^h \in T_=$ then $h \in \mathscr{H}_=$, and by Lemma 5.2 (iii) $h^\xi = ht$ where $t \in \mathfrak{T}$ is such that $t^\xi = t^{-1}$. We have two possibilities: $k = l$ or $k \neq l$. If $k \neq l$ then for $k < l$ we put $[a_i^k, b_i^l]^h$ into A_{n+1} and for $k > l$ we transform $[a_i^k, b_i^l]^h$ by t, inverse and we put the element obtained in that way into B_{n+1}.

If $k = l$ then we change all elements $[a_i^k, b_i^k]^h$ into $[a_i^k, b_i^k]^h t$ and we put this element into C_{n+1}. Hence:

$$C_{n+1} \ni [a_i^k, b_i^k]^h t \xrightarrow{\xi} [b_i^k, a_i^k]^{ht} t^\xi = t^{-1}[b_i^k, a_i^k]^h t t^\xi = ([a_i^k, b_i^k]^h t)^{-1} \in C_{n+1}^{-1}.$$

All transformations are Nielsen transformations, so we change T_{n+1} into a $\mathscr{P}$-basis.

We have proved that every subgroup $\langle T_0 \cup \ldots \cup T_n \rangle$ has the $\mathscr{P}$-basis $A_n \cup B_n \cup C_n$ and it is clear that:

$$A_0 \cup B_0 \cup C_0 \subset A_1 \cup B_1 \cup C_1 \subset A_2 \cup B_2 \cup C_2 \subset \ldots$$

So the subgroup $\mathfrak{T} = \langle T \rangle = \bigcup_n \langle T_0 \cup \ldots \cup T_n \rangle$ has a $\mathscr{P}$-basis $\bigcup_n (A_n \cup B_n \cup C_n)$.

We have shown in Lemma 5.4 that the basis $\mathfrak{Y} = T \cup Q \cup P$ can be changed into the basis $\mathfrak{Y}' = T' \cup Q \cup P$, where T' is a $\mathscr{P}$-basis. The next step is to change Q into a $\mathscr{P}$-basis Q'.

5.2. The subset Q

We have to change the set Q into $Q' = \mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C}$, which is a $\mathscr{P}$-basis.

We split Q into two subsets:

$$Q = Q_+ \cup Q_-$$

where

$$Q_+ = \{[\alpha_i, \mu_j]^{\beta_i h} : \ \mu_j = a_j^d \vee \mu_j = c_j^l, \text{for } l > 0\},$$
$$Q_- = \{[\alpha_i, \mu_j^\xi]^{\beta_i h} : \ \mu_j = a_j^d \vee \mu_j = c_j^l, \text{for } l > 0\}.$$

There exists a bijection between Q_+ and Q_-:

$$[\alpha_i, \mu_j]^{\beta_i h} \longleftrightarrow [\alpha_i, \mu_j^\xi]^{\beta_i h'}$$

where $h^\xi = h't$.

We put every element from Q_+ into $\mathfrak{A}$ and we replace every element $[\alpha_i, \mu_j^\xi]^{\beta_i h'}$ (such that $h^\xi = h't$) from Q_- with $[\beta_i, \mu_j^\xi]^{\alpha_i h'}$, then we transform $[\beta_i, \mu_j^\xi]^{\alpha_i h'}$ by t, then invert and put the element obtained in that way into

$\mathfrak{B}$. So, now it is enough to prove that using Nielsen transformations we can change every element $[\alpha_i, \mu_j^\xi]^{\beta_i h} \in Q_-$ into $[\beta_i, \mu_j^\xi]^{\alpha_i h}$.

LEMMA 5.5. *Every element* $[\alpha_i, \mu_j^\xi]^{\beta_i h} \in Q_-$ *can be replaced by* $[\beta_i, \mu_j^\xi]^{\alpha_i h}$, *using Nielsen transformations.*

PROOF. We use an induction on the length of the word h. Let $h = 1$. We use the commutator identity:

$$[a, c]^b = [b, a][b, c]^a[a, c][a, b]^c[c, b]$$

and get:

$$[\alpha_i, \mu_j^\xi]^{\beta_i} = \underline{[\beta_i, \alpha_i]}[\beta_i, \mu_j^\xi]^{\alpha_i}\underline{[\alpha_i, \mu_j^\xi][\alpha_i, \beta_i]^{\mu_j^\xi}[\mu_j^\xi, \beta_i]}$$

Underlined elements belong to $\mathfrak{Y}' \setminus Q$, so using Nielsen transformation we can remove them getting $[\beta_i, \mu_j^\xi]^{\alpha_i}$. Let now $|h| > 1$ then:

$$[\alpha_i, \mu_j^\xi]^{\beta_i h} = \underline{[\beta_i, \alpha_i]^h}[\beta_i, \mu_j^\xi]^{\alpha_i h}\underline{[\alpha_i, \mu_j^\xi]^h[\alpha_i, \beta_i]^{\mu_j^\xi h}[\mu_j^\xi, \beta_i]^h}$$

Since $[\beta_i, \alpha_i]^h \in T$ and $[\mu_j^\xi, \beta_i]^h \in P$ (so they are not in Q) we can remove them, obtaining the new element:

$$[\beta_i, \mu_j^\xi]^{\alpha_i h}\underline{[\alpha_i, \mu_j^\xi]^h[\alpha_i, \beta_i]^{\mu_j^\xi h}} \tag{5.2}$$

The word $\mu_j^\xi h$ may be not ordered, but by Lemma 5.2 (v) we have $\mu_j^\xi h = \bar{h}v$, where $\bar{h}$ is ordered and v is a product of commutators from $\mathfrak{Y}$ or their inverses, for which words in exponents are shorter than h. So we can remove $[\alpha_i, \beta_i]^{\mu_j^\xi h}$ by multiplying (5.2) by elements from $\mathfrak{Y}' \setminus Q$ and by elements from Q_- but with shorter exponents than h. Finally we can remove $[\alpha_i, \mu_j^\xi]^h$ because it belongs to P.

So we can change the basis $\mathfrak{Y}'$ into $\mathfrak{Y}'' = T' \cup Q' \cup P$, such that T', Q' are $\mathscr{P}$-bases. Finally we change the subset P.

5.3. The subset P

Let us remind that:

$$\begin{aligned} P &= \mathfrak{Y} \setminus (T \cup Q) \\ &= \{[\alpha_i, \gamma_i]^h, [\alpha_i, \mu_j]^h : h \text{ does not contain } \beta_i, i < j\} \\ &\quad \cup \{[\gamma_i, \mu_j]^h, [\beta_i, \gamma_i]^h, [\beta_i, \mu_j]^h, i < j\} \end{aligned}$$

LEMMA 5.6. *The subset P can be changed into P' which is a $\mathscr{P}$-basis.*

PROOF. We split P into two subsets:

$$P = P_1 \cup P_2$$

where $P_1 = \{[\mu, \mu_1]^h : \mu = \alpha_i \vee \mu = c_i^l, \text{for } l > 0\}$, $P_2 = \{[\mu^\xi, \mu_1^\xi]^h : \mu = \alpha_i \vee \mu = c_i^l, \text{for } l > 0\}$. We have to change P into $P' = \mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C}$. We put elements from P_1 into $\mathfrak{A}$ and transform the elements from P_2 by t, such that $h^\xi = h't$, inverse them and put the element obtained in that way into $\mathfrak{B}$.

The Lemma 5.6 finishes transformations of the basis $\mathfrak{Y}$ and we get a $\mathscr{P}$-basis $\mathfrak{Y}'''$ for ξ in F'.

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