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# ORLICZ REGULARITY FOR NON-DIVERGENCE PARABOLIC SYSTEMS WITH PARTIALLY VMO COEFFICIENTS

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## Abstract

This work treats the interior Orlicz regularity for strong solutions of a class of non-divergence parabolic systems with coefficients just measurable in time and VMO in the spatial variables.

## 1. Introduction

Let us consider the following parabolic systems

$$u_t^\alpha + \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} a_{\alpha\beta}^{ij}(x,t) u_{x_i x_j}^\beta = f^\alpha \tag{1}$$

in some domain $\Omega_T \subset \mathrm{R}^{n+1}$, where $\alpha, \beta = 1, \ldots, N$, $i, j = 1, 2, \ldots, n$. In this paper, the summation is understood for repeated indices. There were many works on the $W^{2,p}$ regularity for (1), that is, local or global $L^p$ estimates for the second order derivatives of strong solutions of (1). Let us mention some of them. In the scalar case ($N = 1$), when the coefficients belong to $C^0(\overline{\Omega}_T)$, Ladyzhenskaya in [13] showed that a solution of (1) actually belongs to $W_p^{1,2}(\Omega_T)$ ($2 < p < \infty$) by Fourier multiplier theory; when the coefficients are discontinuous but belong to *VMO*, Bramanti and Cerutti in [2] obtained a similar result by using Coifman-Rochberg-Weiss commutator theorem. The approach in [2] was further used in the study of Morrey regularity for non-divergence parabolic problems with discontinuous coefficients, see [14], [15], [16], [17], [18] and references therein. When the coefficients are just measurable in time and *VMO* in spatial variables, solvability of (1) in Sobolev spaces was investigated by Krylov in [11] and [12]. Later, the results in [11] and [12] were extended to parabolic systems ($N \neq 1$) in [3].

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In this paper, we are interested in the Orlicz regularity problem, more accurately, for any Young function $\phi \in \triangle_2 \cap \nabla_2$ and $Q' \Subset \Omega_T$, if $u = (u^1, \dots, u^N)$ is a strong solution of (1) in $L^\phi(\Omega_T; \mathsf{R}^N)$ and $f^\alpha \in L^\phi(\Omega_T)$, whether $|D^2u|$ still belongs to $L^\phi$, at least locally? The main result of this paper will give an affirmative answer to this problem and show that the results in [12] and [3] are still valid in the setting of general Orlicz spaces. Unlike in [2], [12] and [3], the approach used here is inspired by Wang [19] which is based on the weak compactness, a version of Vitali's covering lemma and maximal functions. We remark that the method in [19] has been widely used to deal with the $L^p$ or Orlicz regularity in Reifenberg flat domains for divergence elliptic or parabolic systems, see [8], [9], [4], [5], [6] and references therein.

This paper is organized as follows: in Section 2 we introduce the notations and state precisely the assumptions and the main result of this paper. In Section 3 we first prove some approximation results, and then deduce some local estimates on the Hardy-Littlewood maximal function of $|D^2u|^2$. The last Section is devoted to proving the regularity in Orlicz spaces for strong solutions of (1).

### *1.1. Notations and definitions*

Let $\Omega$ be an open bounded subset of $\mathsf{R}^n$ and set

$$\Omega_T = \Omega \times (0, T]$$

for some fixed time $T > 0$.

Denote the open ball in $\mathsf{R}^n$ of radius $r$ centered at $x$ by $B_r(x)$, and define the parabolic cylinder by

$$Q_r(x,t) = B_r(x) \times (t, t + r^2], \qquad r \in (0, \infty)$$

with its boundary by

$$\partial Q_r(x,t) = B_r(x) \times \{t = T\} \cup \partial B_r(x) \times [0, T].$$

We also use the centered parabolic cylinder

$$C_r(x,t) = B_r(x) \times \left(t - \frac{r^2}{2}, t + \frac{r^2}{2}\right]$$

and adopt the convention of writing $Q_r$ instead of $Q_r(x,t)$, when the "center" $(x,t)$ is not important or is clear from the context.

In order to simplify notation, henceforth we will write $z$ for $(x,t)$, $|D^2u|$ for $|u_{xx}|^2+|u_t|^2$, and write

$$u_t+\sum_{\beta=1}^{N}\sum_{i,j=1}^{n}a^{ij}_{\alpha\beta}(x,t)u_{x_ix_j}=\mathbf{F}$$

for the system (1). Denote the Lebesgue measure of $\Omega_T$ by $|\Omega_T|$ and set

$$\|u\|_{L^p(\Omega_T;\mathbb{R}^N)}=\||u|\|_{L^p(\Omega_T)},\qquad \text{with}\quad |u|=\left(\sum_{\alpha=1}^{N}|u^\alpha|^2\right)^{1/2},$$

$$\|D^2u\|_{L^p(\Omega_T;\mathbb{R}^N)}=\||D^2u|\|_{L^p(\Omega_T)},\quad \text{with}\quad |D^2u|=\left(\sum_{\alpha=1}^{N}(|D^2u^\alpha|^2)\right)^{1/2},$$

$$\|\mathbf{F}\|_{L^p(\Omega_T;\mathbb{R}^N)}=\||\mathbf{F}|\|_{L^p(\Omega_T)},\qquad \text{with}\quad |\mathbf{F}|=\left(\sum_{\alpha=1}^{N}(|f^\alpha|^2)\right)^{1/2}$$

and

$$\|u_x\|_{L^p(\Omega_T;\mathbb{R}^N)}=\||u_x|\|_{L^p(\Omega_T)},\qquad \text{with}\quad |u_x|=\left(\sum_{\alpha=1}^{N}\sum_{i=1}^{n}(|u^\alpha_{x_i}|^2)\right)^{1/2}.$$

DEFINITION 1 (*$VMO_x$* and weakly $(\delta,R)$-vanishing). Denote

$$\operatorname{osc}(a,Q_r(x,t))=r^{-2}|B_r|^{-2}\int_t^{t+r^2}\int_{y,z\in B_r(x)}|a(y,s)-a(z,s)|\,dy\,dz\,ds,$$

$$a_R^{\sharp(x)}=\sup_{(t,x)\in\Omega_T}\sup_{r\le R}\operatorname{osc}(a,Q_r(t,x)).$$

We say that $a$ is weakly $(\delta,R)$-vanishing, if $\sup_{r<R}a_r^{\sharp(x)}\le\delta^2$; We say that $a\in VMO_x$, if

$$\lim_{R\to0}a_R^{\sharp(x)}=0.$$

The function $a_R^{\sharp(x)}$ is called the local $VMO_x$ modulus of $a$.

DEFINITION 2 (Sobolev space). Let $1\le p\le\infty$. A function $u$ is said to belong to the Sobolev space $W_p^{1,2}(\Omega_T)$, if $|u|$, $|u_x|$, $|D^2u|\in L^p(\Omega_T)$, and we set

$$\|u\|_{W_p^{1,2}(\Omega_T)}:=\|u\|_{L^p(\Omega_T)}+\|D^2u\|_{L^p(\Omega_T)}+\|u_x\|_{L^p(\Omega_T)}.$$

By $\mathring{W}_p^{1,2}(\Omega_T)$, we mean the subspace of $W_p^{1,2}(\Omega_T)$ consisting of functions $u(z)$ vanishing near the parabolic boundary $\partial\Omega_T$.

DEFINITION 3. If $u \in W_2^{1,1}(\Omega_T; \mathsf{R}^N)$ and satisfies

$$\int_{\Omega_T} u_t \varphi \, dz - \int_{\Omega_T} a_{\alpha\beta}^{ij}(z) u_{x_i} \varphi_{x_j} \, dz = \int_{\Omega_T} \mathbf{F} \varphi \, dz,$$

for every $\varphi \in \mathring{W}_2^{1,1}(\Omega_T; \mathsf{R}^N)$, then $u$ is called a weak solution of

$$u_t + \left(a_{\alpha\beta}^{ij}(z) u_{x_i}\right)_{x_j} = \mathbf{F}.$$

DEFINITION 4. We say that $u \in W_2^{1,2}(\Omega_T; \mathsf{R}^N)$ is a strong solution of (1), if there are sequences of smooth vector functions $\{u_n\}, \{f_n\}$ such that $u_n \to u$, $f_n \to f$ in $L^2(\Omega_T; \mathsf{R}^N)$ and

$$\left(u_n^\alpha\right)_t + \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} a_{\alpha\beta}^{ij}(x,t) \left(u_n^\beta\right)_{x_i x_j} = f_n^\alpha$$

for each $n$.

### *1.2. Orlicz spaces*

DEFINITION 5. A nonnegative real-valued function $\phi$ is said to be a Young function if $\phi$ is increasing, convex and satisfies

$$\phi(0) = 0; \quad \phi(\infty) = \lim_{t \to \infty} \phi(t) = \infty; \quad \lim_{t \to 0^+} \frac{\phi(t)}{t} = \lim_{t \to \infty} \frac{t}{\phi(t)} = 0.$$

DEFINITION 6. For a given Young function $\phi$ and a bounded domain $\Omega_T \subset \mathsf{R}^{n+1}$, the Orlicz class $K^\phi(\Omega_T)$ is the set of all measurable functions $f : \Omega_T \to \mathsf{R}^1$ satisfying

$$\int_{\Omega_T} \phi(|f(z)|) \, dz < \infty.$$

The Orlicz space $L^\phi(\Omega_T)$ is defined to be the linear hull of $K^\phi(\Omega_T)$, that is, the smallest linear space (under pointwise addition and scalar multiplication) containing $K^\phi(\Omega_T)$.

DEFINITION 7. We say that a Young function $\phi$ satisfies the $\triangle_2$-condition, denoted by $\phi \in \triangle_2$, if for some number $\alpha > 0$ and for all $t > 0$, $\phi(2t) \leq \alpha\phi(t)$; A Young function $\phi$ is said to satisfy the $\bigtriangledown_2$-condition, denoted by $\phi \in \bigtriangledown_2$, if for some number $\beta > 1$ and for all $t > 0$, $2\beta\phi(t) \leq \phi(\beta t)$.

REMARK 8. We will write a Young function $\phi \in \triangle_2 \cap \bigtriangledown_2$, if $\phi$ is assumed to satisfy both $\triangle_2$ and $\bigtriangledown_2$ conditions. This condition ensures that a Young function grows neither too slowly nor too fast. For any $p > 1$, the Young function $\phi(t) = t^p \in \triangle_2 \cap \nabla_2$, thus Lebesgue spaces $L^p(\Omega_T)$ are special cases of Orlicz spaces $L^\phi(\Omega_T)$.

DEFINITION 9. Given a Young function $\phi \in \triangle_2 \cap \nabla_2$, the Luxemburg norm $\|\cdot\|_{L^\phi(\Omega_T)}$ is defined by

$$\|f\|_{L^\phi(\Omega_T)} = \inf\left\{\rho > 0 : \int_{\Omega_T} \phi(|f|/\rho)\,dz \le 1\right\}.$$

With the norm $\|\cdot\|_{L^\phi(\Omega_T)}$, $(L^\phi(\Omega_T), \|\cdot\|_{L^\phi(\Omega_T)})$ is a Banach space.

LEMMA 10. *Let $\phi$ be a Young function. Then $\phi(t) \in \triangle_2 \cap \nabla_2$ if and only if there exist constants $A_2 \ge A_1 > 0$ and $\alpha_1 \ge \alpha_2 > 1$ such that for any $0 < s \le t$,*

$$A_1\left(\frac{s}{t}\right)^{\alpha_1} \le \frac{\phi(s)}{\phi(t)} \le A_2\left(\frac{s}{t}\right)^{\alpha_2}. \tag{2}$$

*Moreover, the condition* (2) *implies that for* $0 < \theta_1 \le 1 \le \theta_2 < \infty$,

$$\phi(\theta_1 t) \le A_2\theta_1^{\alpha_2}\phi(t) \qquad \text{and} \qquad \phi(\theta_2 t) \le A_1^{-1}\theta_2^{\alpha_1}\phi(t).$$

LEMMA 11 ([9]). *Given a Young function $\phi \in \triangle_2 \cap \nabla_2$, suppose $f \in L^\phi(\Omega_T)$. Then $\int_{\Omega_T} \phi(|f(z)|)\,dz$ can be rewritten as an integral of the distribution $\mu_f(\lambda) = |\{z \in \Omega_T : |f| > \lambda\}|$. That is, for any $N > 1$,*

$$\int_{\Omega_T} \phi(|f(z)|)\,dz = \sum_{k=-\infty}^{\infty} \int_{N^k}^{N^{k+1}} \mu_f(\lambda)\,d\lambda.$$

*1.3. Assumptions and main results*

ASSUMPTION (H). The coefficients $a_{\alpha\beta}^{ij}(z)$ in (1) are real valued, bounded measurable functions defined in $\Omega_T$ and satisfy the strong Legendre-Hadamard condition, that is, there exists a constant $\mu > 0$ such that

$$\mu|\xi|^2|\zeta|^2 \le a_{\alpha\beta}^{ij}(z)\xi_i\xi_j\zeta^\alpha\zeta^\beta \le \mu^{-1}|\xi|^2|\zeta|^2 \tag{3}$$

for any $\xi \in \mathsf{R}^n$, $\zeta \in \mathsf{R}^N$ and a.e. $z \in \Omega_T$. Furthermore, we assume that the coefficients belong to $VMO_x \cap L^\infty$.

Let us state the main result of this paper.

THEOREM 12. *Under the assumption* (H)*, suppose* $\phi \in \triangle_2 \cap \nabla_2$ *and* $\mathbf{F} \in L^{\phi}(\Omega_T; \mathsf{R}^N)$. *If* $u \in W_2^{1,2}(\Omega_T; \mathsf{R}^N) \cap L^{\phi}(\Omega_T)$ *is a strong solution of* (1)*, then* $|D^2u| \in L^{\phi}(Q')$*; moreover,*

$$\int_{Q'} \phi(|D^2u|^2)\,dz \le c\Big(\int_{\Omega_T} \phi(|u|^2)\,dz + \int_{\Omega_T} \phi(|\mathbf{F}|^2)\,dz\Big),$$

*where the constant c depends on* $\mu$, $\phi$, $Q'$, $\Omega_T$ *and the local* $VMO_x$ *moduli of the coefficients in* $Q'$.

Throughout this paper, denote by the letter $c$ some positive constant which may vary from line to line.

## 2. Approximation and preliminary results

### 2.1. Approximation

LEMMA 13 (Poincaré's inequality, [19]). *There exist positive constants* $r_0$ *and* $c$*, such that for any* $u \in W_2^{1,2}(\Omega_T)$, $R < r_0$,

$$\big\|u - u_{Q_R} - (\nabla u)_{Q_R} \cdot x\big\|_{L^2(Q_R)} \le cR^2\|D^2u\|_{L^2(Q_R)}. \tag{4}$$

THEOREM 14. *For any* $\varepsilon > 0$*, there is a small constant* $\delta = \delta(\varepsilon) > 0$ *such that if* $u \in W_2^{1,2}(Q_T; \mathsf{R}^N)$ *is a weak solution of* (1) *in* $Q_4 \Subset Q_T$ *with*

$$\begin{aligned} &\frac{1}{|Q_4|}\int_{Q_4} |D^2u|^2 dz \le 1,\\ &\frac{1}{|Q_4|}\int_{Q_4} \Big(|\mathbf{F}|^2 + \big|a^{ij}_{\alpha\beta} - \big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)\big|^2\Big)\,dz \le \delta^2, \end{aligned} \tag{5}$$

*where* $\big(a^{ij}_{\alpha\beta}\big)_{B_4}(t) = \frac{1}{|B_4|}\int_{B_4} a^{ij}_{\alpha\beta}(x,t)\,dx$*, then there exists a solution* $v$ *of the system*

$$v_t + \big(a^{ij}_{\alpha\beta}\big)_{B_4}(t) v_{x_i x_j}(z) = 0 \qquad in \quad Q_4 \tag{6}$$

*such that*

$$\int_{Q_4} |u - v|^2 dz \le \varepsilon^2. \tag{7}$$

PROOF. Just to simplify notations, assume that the center of $Q_r$ is the origin. We prove this conclusion by the contradiction. If not, there exist a constant

$\varepsilon_0 > 0$, and sequences $\left\{a_{\alpha\beta}^{ijk}(z)\right\}_{k=1}^{\infty}$, $\{u^k\}_{k=1}^{\infty}$, and $\{\mathbf{F}_k\}_{k=1}^{\infty}$ such that $u^k$ is a strong solution of the system

$$u_t^k + a_{\alpha\beta}^{ijk}(z) u_{x_i x_j}^k = \mathbf{F}^k \qquad \text{in} \quad Q_4 \tag{8}$$

with

$$\begin{aligned} &\frac{1}{|Q_4|}\int_{Q_4} |D^2 u^k|^2\, dz \le 1, \\ &\frac{1}{|Q_4|}\int_{Q_4} \left(|\mathbf{F}^k|^2 + \left|a_{\alpha\beta}^{ijk} - \left(a_{\alpha\beta}^{ijk}\right)_{B_4}(t)\right|^2\right) dz \le \frac{1}{k^2}, \end{aligned} \tag{9}$$

but

$$\int_{Q_4} |u^k - v^k|^2\, dz > \varepsilon_0^2, \tag{10}$$

where $v^k$ is any strong solution of the system

$$v_t + \left(a_{\alpha\beta}^{ijk}\right)_{B_4}(t) v_{x_i x_j} = 0 \qquad \text{in} \quad Q_4. \tag{11}$$

By Lemma 13,

$$\frac{1}{|Q_4|}\int_{Q_4} \left|u^k - u_{Q_4}^k - (\nabla u^k)_{Q_4}\cdot x\right|^2 dz \le \frac{c}{|Q_4|}\int_{Q_4} |D^2 u^k|^2\, dz \le c,$$

then by using the interpolation theorem, we know that $\{u^k - u_{Q_4}^k - (\nabla u^k)_{Q_4} \cdot x\}_{k=1}^{\infty}$ is bounded in $W_2^{1,2}(Q_4)$. Without loss of generality, we may assume $u_{Q_4}^k + (\nabla u^k)_{Q_4} \cdot x = 0$, and then there exists a subsequence of $\{u^k\}_{k=1}^{\infty}$ which still be denoted by $\{u^k\}_{k=1}^{\infty}$, such that for some $u_0 \in W_2^{1,2}(Q_4)$,

$$\begin{aligned} u^k &\to u_0 \qquad \text{in} \quad L^2, \\ u_{xx}^k, u_t^k &\to (u_0)_{xx}, (u_0)_t \qquad \text{weakly in} \quad L^2. \end{aligned} \tag{12}$$

Since $\left\{\left(a_{\alpha\beta}^{ijk}\right)_{B_4}(t)\right\}_{k=1}^{\infty}$ is bounded in $L^\infty(Q_4)$, and so is in $L^2(Q_4)$, there exist a subsequence which still be denoted by $\left\{\left(a_{\alpha\beta}^{ijk}\right)_{B_4}(t)\right\}_{k=1}^{\infty}$, and some function $\bar{a}_{\alpha\beta}^{ij}(t) \in L^2(Q_4)$, such that

$$\left(a_{\alpha\beta}^{ijk}\right)_{B_4}(t) \to \bar{a}_{\alpha\beta}^{ij}(t) \qquad \text{weakly in } L^2, \text{ as } k \to \infty. \tag{13}$$

Now we claim that $u_0$ itself is a solution of the system

$$(u_0)_t + \bar{a}_{\alpha\beta}^{ij}(t)(u_0)_{x_i x_j} = 0 \qquad \text{in} \quad Q_4. \tag{14}$$

For this, fix a $\varphi \in C_0^\infty(Q_4)$, then

$$\begin{aligned}(15)\quad &\int_{Q_4}\big(u_t^k + a_{\alpha\beta}^{ijk}(z)u_{x_ix_j}^k\big)\varphi\,dz\\ &=\int_{Q_4}\Big(u_t^k + \big(a_{\alpha\beta}^{ijk}(z) - \big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t)\big)u_{x_ix_j}^k + \big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t)u_{x_ix_j}^k\Big)\varphi\,dz.\end{aligned}$$

By Hölder's inequality and (9), we know

$$\begin{aligned}(16)\quad &\bigg(\int_{Q_4}\Big(a_{\alpha\beta}^{ijk}(z) - \big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t)\Big)u_{x_ix_j}^k\varphi\,dz\bigg)^2\\ &\le\bigg(\int_{Q_4}\Big(a_{\alpha\beta}^{ijk}(z) - \big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t)\Big)^2 dz\bigg)\cdot\int_{Q_4}\big(u_{x_ix_j}^k\varphi\big)^2\,dz \to 0.\end{aligned}$$

Since $\big\{\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t)\big\}_{k=1}^\infty$ is uniformly bounded in $L^\infty(Q_4)$, we see that (12) and (13) imply

$$\begin{aligned}&\int_{Q_4}\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t)u_{x_ix_j}^k\varphi\,dz - \int_{Q_4}\bar a_{\alpha\beta}^{ij}(t)(u_0)_{x_ix_j}\varphi\,dz\\ &\le\int_{Q_4}\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t)\big(u_{x_ix_j}^k - (u_0)_{x_ix_j}\big)\varphi\,dz\\ (17)\quad &\qquad+\int_{Q_4}\Big(\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t) - \bar a_{\alpha\beta}^{ij}(t)\Big)u_0\varphi_{x_ix_j}\,dz\\ &\le\int_{Q_4}(u^k - u_0)^2\,dz\int_{Q_4}\Big(\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t)\varphi_{x_ix_j}\Big)^2\,dz\\ &\qquad+\int_{Q_4}\Big(\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t) - \bar a_{\alpha\beta}^{ij}(t)\Big)u_0\varphi_{x_ix_j}\,dz \to 0,\end{aligned}$$

as $k\to\infty$, and

$$(18)\qquad \int_{Q_4}u_t^k\varphi\,dz \to \int_{Q_4}u_t^0\varphi\,dz,\qquad \text{as}\quad k\to\infty.$$

Summing up (15), (16), (17) and (18), it yields

$$(19)\qquad u_t^k + a_{\alpha\beta}^{ijk}(z)u_{x_ix_j}^k \to (u_0)_t + \bar a_{\alpha\beta}^{ij}(t)(u_0)_{x_ix_j},\qquad \text{weakly in}\quad Q_4.$$

This convergence and (9) give

$$\begin{aligned}\int_{Q_4}\big((u_0)_t + \bar a_{\alpha\beta}^{ij}(t)(u_0)_{x_ix_j}\big)^2\,dz &\le \liminf\int_{Q_4}\big(u_t^k + a_{\alpha\beta}^{ijk}(z)u_{x_ix_j}^k\big)^2\,dz\\ &= \liminf\int_{Q_4}|\mathbf{F}^k|^2\,dz = 0,\end{aligned}$$

which means $(u_0)_t + \bar{a}^{ij}_{\alpha\beta}(t)(u_0)_{x_ix_j} = 0$ a.e. in $Q_4$, and then $u_0$ is a strong solution of (14).

Noting that

$$\begin{aligned}(u_0)_t &+ \big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t)(u_0)_{x_ix_j}\\ &= (u_0)_t + \bar{a}^{ij}_{\alpha\beta}(t)(u_0)_{x_ix_j} + \Big(\big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t) - \bar{a}^{ij}_{\alpha\beta}(t)\Big)(u_0)_{x_ix_j}\\ &= \Big(\big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t) - \bar{a}^{ij}_{\alpha\beta}(t)\Big)(u_0)_{x_ix_j},\end{aligned}$$

and by using [3, Thm. 2.4], we know that the problem

$$\begin{cases} h^k_t + \big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t)h^k_{x_ix_j} = \Big(\big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t) - \bar{a}^{ij}_{\alpha\beta}(t)\Big)(u_0)_{x_ix_j},\\ h^k = 0, \qquad \text{on} \quad \partial Q_4 \end{cases} \tag{20}$$

has a unique solution $h^k$ satisfying

$$\int_{Q_4} |D^2h^k|^2\,dz \le \int_{Q_4} \Big(\Big(\big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t) - \bar{a}^{ij}_{\alpha\beta}(t)\Big)(u_0)_{x_ix_j}\Big)^2\,dz. \tag{21}$$

From [3, Lemma 3.3], it follows

$$\sup_{z\in Q_3} |(u_0)_{x_ix_j}| \le c\big(\|(u_0)_x\|_{L^2(Q_4)} + \|u_0\|_{L^2(Q_4)}\big). \tag{22}$$

Also, since $\big\{a^{ijk}_{\alpha\beta}\big\}_{k=1}^{\infty}$ is bounded and $\bar{a}^{ij}_{\alpha\beta}(t) \in L^2$, there is a positive constant $C$ such that

$$\Big\|\big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t) - \bar{a}^{ij}_{\alpha\beta}(t)\Big\|_{L^2(Q_4)} \le C. \tag{23}$$

Combining (21),(22) and (23), it obtains that $\big\{|D^2h^k|\big\}_{k=1}^{\infty}$ is bounded in $L^2(Q_4)$, hence

$$h^k \to h_0, \qquad \text{in} \quad L^2 \tag{24}$$

for some $h_0 \in L^2(Q_4)$.

Now we show $h_0 = 0$ a.e. in $Q_4$. In fact, since $h^k$ is also a weak solution of the problem

$$\begin{cases} h^k_t + \Big(\big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t)h^k_{x_i}\Big)_{x_j} = \Big(\big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t) - \bar{a}^{ij}_{\alpha\beta}(t)\Big)(u_0)_{x_ix_j},\\ h^k = 0, \qquad \text{on} \quad \partial Q^s_4, \end{cases} \tag{25}$$

where $Q^s_4 = B_4(0) \times (s, 16]$, we take $h^k$ as a test function in (25), and then

$$\int_{Q^s_4} h^k_t h^k - \big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t)h^k_{x_i}h^k_{x_j}\,dx\,dt = \int_{Q^s_4} \Big(\big(a^{ijk}_{\alpha\beta}\big)_{B_4}(t) - \bar{a}^{ij}_{\alpha\beta}(t)\Big)(u_0)_{x_ix_j}h^k\,dx\,dt,$$

hence

$$-\int_{B_4(0)} (h^k)^2(x,s)\,dx - \int_{Q_4^s} \big(a_{\alpha\beta}^{ijk}\big)_{B_4(0)}(t) h_{x_i}^k h_{x_j}^k\,dx\,dt = \int_{Q_4^s} \big(\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t) - \bar{a}_{\alpha\beta}^{ij}(t)\big)(u_0)_{x_i x_j} h^k\,dx\,dt. \tag{26}$$

By (22), we know $\big(\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t) - \bar{a}_{\alpha\beta}^{ij}(t)\big)(u_0)_{x_i x_j} \to 0$ weakly in $L^2$. Also since $h^k \to h_0$ in $L^2$, it follows that

$$\int_{Q_4^s} \big(\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t) - \bar{a}_{\alpha\beta}^{ij}(t)\big)(u_0)_{x_i x_j} h^k\,dx\,dt \to 0, \qquad \text{in} \quad L^2, \tag{27}$$

which and (3) give

$$\mu \int_{Q_4^s} |h_x^k|^2\,dx\,dt \le \int_{B_4} |h^k|^2(x,s)\,dx + \int_{Q_4^s} \big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t) h_{x_i}^k h_{x_j}^k\,dx\,dt \to 0,$$

as $k \to \infty$. Thus $(h_0)_x = 0$ a.e. in $Q_4$, which means that $h_0$ is independent of $x$ in $Q_4$. Also we have by (26) and (27) that

$$\begin{aligned}\int_{B_4} |h_0|^2(s)\,dx &= \int_{B_4} |h_0|^2(x,s)\,dx \\ &\le \liminf \int_{B_4} |h^k|^2(x,s)\,dx \\ &\le \liminf \int_{Q_4^s} \big(\big(a_{\alpha\beta}^{ijk}\big)_{B_4}(t) - \bar{a}_{\alpha\beta}^{ij}(t)\big)(u_0)_{x_i x_j} h^k\,dx\,dt \\ &= 0,\end{aligned}$$

then

$$h_0(s) = 0 \qquad \text{a.e. in} \quad (0, 16),$$

which implies $h_0(z) = 0$ a.e. in $Q_4$.

Combining (24) and (12), we have

$$\int_{Q_4} |u^k - (u_0 - h^k)|^2\,dz \le c\left(\int_{Q_4} |u^k - u_0|^2\,dz + \int_{Q_4} |h^k|^2\,dz\right) \to 0. \tag{28}$$

On the other hand, $u_0 - h^k$ is still a solution of (11), it follows from (10) that

$$\int_{Q_4} |u^k - (u_0 - h^k)|^2 dz \ge \varepsilon_0,$$

which contradicts (28).

LEMMA 15 ([7]). *Let $\psi(t)$ be a nonnegative bounded function defined on the interval $[T_0, T_1]$, where $T_1 > T_0 \geq 0$. Suppose that for any $T_0 \leq t \leq s \leq T_1$, $\psi$ satisfies*

$$\psi(t) \leq \vartheta\psi(s) + \frac{A}{(s-t)^\alpha} + B,$$

*where $\vartheta$, $A$, $B$, $\alpha$ are nonnegative constants, and $\vartheta < 1$. Then for any $T_0 \leq \rho < R \leq T_1$,*

$$\psi(\rho) \leq c_\alpha\left[\frac{A}{(R-\rho)^\alpha} + B\right],$$

*where $c_\alpha$ only depends on $\alpha$.*

LEMMA 16. *There exists a constant $N_0 > 0$, such that*

$$\sup_{z\in Q_2} |D^2 v| \leq N_0, \tag{29}$$

*where $v$ is the function in Theorem 14.*

PROOF. From [3, Lemma 3.3], we know

$$\sup_{z\in Q_2} |D^2 v| \leq c\big(\|v_x\|_{L^2(Q_3)} + \|v\|_{L^2(Q_3)}\big). \tag{30}$$

Now, we try to remove the term $\|v_x\|_{L^2(Q_3)}$. Since $v$ is a strong solution of (6) and the coefficients are independent of $x$, one sees that $v$ is a weak solution of

$$v_t + \big(\big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)v_{x_j}\big)_{x_i} = 0 \qquad \text{in} \quad Q_4. \tag{31}$$

For $2 \leq l < s \leq 3$, we choose a cutoff function $\varphi(x)$ satisfying

$$\begin{aligned} &0 < \varphi(x) \leq 1 \quad \text{in} \quad B_4, &&\varphi(x) \equiv 1 \quad \text{in} \quad B_l,\\ &\varphi(x) \equiv 0 \quad \text{in} \quad B_3\backslash B_s, &&|\varphi_x| \leq \frac{c}{s-l} \quad \text{in} \quad B_4,\end{aligned}$$

and $\eta$ with the form

$$\eta(t) = \begin{cases} \dfrac{s^2-t}{s^2-l^2} \in [l^2, s^2),\\ 1, t \in [0, l^2). \end{cases}$$

Taking $v\eta(t)\varphi^2(x)$ as a test function in (31), we have

$$\int_{Q_4} \big(v_t v\eta\varphi^2 - \big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)v_{x_j}(v\eta\varphi^2)_{x_i}\big)\, dz = 0.$$

Since

$$\begin{aligned}\int_{Q_4} v_t v\eta\varphi^2\,dz &= \int_{Q_4}\left(\frac{1}{2}v^2\eta\right)_t\varphi^2\,dz - \int_{Q_4}\frac{1}{2}v^2\eta_t\varphi^2\,dz\\ &= -\int_{B_4}\frac{1}{2}v^2\varphi^2\,dx - \int_{Q_4}\frac{1}{2}v^2\eta_t\varphi^2\,dz\end{aligned}$$

and

$$\begin{aligned}&\int_{Q_4}\big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)v_{x_j}(v\eta\varphi^2)_{x_i}\,dz\\ &\qquad= \int_{Q_4}\Big(\big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)v_{x_j}v_{x_i}\eta\varphi^2 + 2\big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)v\eta\varphi v_{x_j}\varphi_{x_i}\Big)\,dz,\end{aligned}$$

it follows

$$\begin{aligned}&\int_{Q_4}\big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)v_{x_j}v_{x_i}\eta\varphi^2\,dz + \int_{B_4}\frac{1}{2}v^2\varphi^2\,dx\\ &\qquad= -2\int_{Q_4}\big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)v\eta\varphi v_{x_j}\varphi_{x_i}dz - \int_{Q_4}\frac{1}{2}v^2\eta_t\varphi^2\,dz.\end{aligned}$$

Because of $\int_{B_4}\frac{1}{2}v^2\varphi^2dx \geq 0$, then Young's inequality, (3) and the properties of $\eta$ and $\varphi$ imply

$$\begin{aligned}\mu\int_{Q_l}|v_x|^2\,dz &\leq \int_{Q_4}\big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)v_{x_j}v_{x_i}\eta\varphi^2\,dz\\ &\leq \int_{Q_s}\frac{1}{2}|v^2\eta_t\varphi^2|\,dz + 2\int_{Q_s}\big|\big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)v\eta\varphi v_{x_j}\varphi_{x_i}\big|\,dz\\ &\leq \frac{c}{(s-l)^2}\int_{Q_s}v^2\,dz + \frac{1}{4}\int_{Q_s}|v_x|^2\,dz.\end{aligned}$$

It follows by Lemma 15 that

$$\int_{Q_2}|v_x|^2\,dz \leq c\int_{Q_3}v^2\,dz,$$

hence

$$\sup_{z\in Q_2}|D^2v| \leq c\|v\|_{L^2(Q_4)}. \tag{32}$$

From (4), we know

$$
\begin{aligned}
\sup_{z\in Q_2} |D^2 v| &\le \sup_{z\in Q_2} \big|D^2(v - u_{Q_4} - (\nabla u_{Q_4})\cdot x)\big| \\
&\le \big\| |(v - u_{Q_4} - (\nabla u_{Q_4})\cdot x)| \big\|_{L^2(Q_4)} \\
&\le c\big( \big\||v-u|\big\|_{L^2(Q_4)} + \big\|(u - u_{Q_4} - (\nabla u_{Q_4})\cdot x)\big\|_{L^2(Q_4)}\big) \\
&\le c\big( \big\||v-u|\big\|_{L^2(Q_4)} + \big\||D^2 u|\big\|_{L^2(Q_4)}\big).
\end{aligned}
$$

By (7) and (5), we have

$$
\sup_{z\in Q_2} |D^2 v| \le c(\varepsilon + 1) \le N_0
$$

for some positive constant $N_0$.

THEOREM 17. *For any $\varepsilon > 0$, there is a small $\delta = \delta(\varepsilon) > 0$ such that if $u \in W_2^{1,2}(Q_4; \mathsf{R}^N)$ is a strong solution of* (1) *in $Q_4 \subset \Omega_T$ with* (5) *holds, then there exists a strong solution $v$ of (6) such that*

$$
\frac{1}{|Q_2|}\int_{Q_2} \big|D^2(u-v)\big|^2\,dz \le \varepsilon^2.
$$

PROOF. From Theorem 14, it shows that for any $\eta > 0$, there exist a small $\delta = \delta(\eta) > 0$ and a solution $v$ of (6) in $Q_4$, such that

$$
\int_{Q_4} |u-v|^2\,dz \le \eta^2. \tag{33}
$$

Let us first note that $u - v$ is a strong solution of the system

$$
(u-v)_t + \big(a^{ij}_{\alpha\beta}(z)(u-v)_{x_i x_j}(z)\big) = \big(\mathbf{F}(z) - \big(a^{ij}_{\alpha\beta}(z) - \big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)\big)v_{x_i x_j}\big) \tag{34}
$$

in $Q_4$. Using a priori $L^2$ estimates in [3, Thm. 2.4], we have

$$
\begin{aligned}
&\big\||D^2(u-v)|\big\|_{L^2(Q_2)} \\
&\le \big\||u-v|\big\|_{L^2(Q_3)} + \big\|\big|\mathbf{F} - \big(a^{ij}_{\alpha\beta}(z) - \big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)\big)v_{x_i x_j}\big|\big\|_{L^2(Q_3)} \\
&\le \big\||u-v|\big\|_{L^2(Q_3)} + \big\||\mathbf{F}|\big\|_{L^2(Q_3)} + \big\|\big(a^{ij}_{\alpha\beta}(z) - \big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)\big)v_{x_i x_j}\big\|_{L^2(Q_3)} \\
&\le \big\||u-v|\big\|_{L^2(Q_4)} + \big\||\mathbf{F}|\big\|_{L^2(Q_4)} + \sup_{Q_3}|D^2 v|\cdot \big\|a^{ij}_{\alpha\beta}(z) - \big(a^{ij}_{\alpha\beta}\big)_{B_4}(t)\big\|_{L^2(Q_3)}.
\end{aligned}
$$

By (33), (5) and Lemma 16, we have

$$\big\| |D^2(u-v)| \big\|_{L^2(Q_2)} \le \eta + \delta + N_0\delta = \varepsilon,$$

for suitable choice of $\eta$ and $\delta$. This ends the proof.

*2.2. Local estimates of $\mathcal{M}(D^2u)(z)$*

In this subsection, we will use the parabolic maximal function defined by

$$\mathcal{M} f(z) = \sup_{z\in\Omega_T, r>0} \frac{1}{|C_r(z)\cap\Omega_T|} \int_{C_r(z)\cap\Omega_T} f(y,s)\,dy\,ds.$$

The following lemma gives a characterization of those functions $\phi \in \triangle_2 \cap \nabla_2$.

LEMMA 18 ([10]). *If $f \in L^1_{loc}(\Omega_T)$, then for every $\alpha > 0$,*

$$\big|\{z\in\Omega_T : (\mathcal{M} f)(z) > \alpha\}\big| \le \frac{c}{\alpha}\int_{\Omega_T} |f(z)|\,dz;$$

*if $\phi \in \triangle_2 \cap \nabla_2$ and $f \in L^\phi(\Omega_T)$, then $(\mathcal{M} f)(z) \in L^\phi(\Omega_T)$ and*

$$\int_{\mathsf{R}^{n+1}} \phi(\mathcal{M}(f))\,dz \le c\int_{\mathsf{R}^{n+1}} \phi(cf)\,dz,$$

*where the bound c depends only on $\phi$.*

THEOREM 19. *For any $\varepsilon > 0$ and $C_1(z') \subset Q_6 \subset \Omega_T$, there exist a positive constant $N_1$ and a small $\delta = \delta(\varepsilon) > 0$, such that if u is a strong solution of* (1) *in $\Omega_T$ with*

(35)

$$C_1(z')\cap\big\{z\in Q_6 : \mathcal{M}(|D^2u|^2)(z)\le 1\big\}\cap\big\{z\in Q_6 : \mathcal{M}(|\mathbf{F}|^2)(z)\le\delta^2\big\}\ne\emptyset$$

*and the coefficients $a^{ij}_{\alpha\beta}(z)$ being weakly $(\delta, 6)$-vanishing, then*

$$\big|C_1(z')\cap\big\{z\in Q_6 : \mathcal{M}\big(|D^2u|^2\big)(z) > N_1^2\big\}\big| < \varepsilon|C_1(z')|. \tag{36}$$

PROOF. From (35), there exists a point $z_0 \in C_1(z')$ such that for any $\rho > 0$,

$$\frac{1}{|C_\rho(z_0)|}\int_{C_\rho(z_0)} |D^2u|^2\,dz \le 1, \tag{37}$$

$$\frac{1}{|C_\rho(z_0)|}\int_{C_\rho(z_0)} |\mathbf{F}|^2\,dz \le \delta^2. \tag{38}$$

Since $C_4(z') \subset C_5(z_0)$, we derive by (38) that

$$\frac{1}{|C_4(z')|}\iint_{C_4(z')} |\mathbf{F}|^2\, dy\, ds \le \frac{|C_5(z_0)|}{|Q_4(z')|}\frac{1}{|C_5(z_0)|}\iint_{C_5(z_0)} |\mathbf{F}|^2\, dy\, ds \tag{39}$$
$$\le \left(\frac{5}{4}\right)^{n+2}\delta^2.$$

Similarly, one finds by (37) that

$$\frac{1}{|C_4(z')|}\iint_{C_4(z')} |D^2u|^2\, dy\, ds \le \left(\frac{5}{4}\right)^{n+2}. \tag{40}$$

By (39), (40) and the assumption on $a^{ij}_{\alpha\beta}(z)$ (weakly $(\delta, 6)$-vanishing), we apply Theorem 17 (with $u$ replaced by $u' = \left(\frac{4}{5}\right)^{n+2}u$ and $\mathbf{F}$ replaced by $\mathbf{F}' = \left(\frac{4}{5}\right)^{n+2}\mathbf{F}$) and obtain that for any $\eta > 0$, there exist a small $\delta(\eta) > 0$ and a strong solution $v'$ of the system

$$v'_t + \left(a^{ij}_{\alpha\beta}\right)_{B_4(x')}(t)v'_{x_ix_j} = 0 \qquad \text{in} \quad Q_4(z')$$

such that

$$\frac{1}{|C_2(z')|}\int_{C_2(z')} |D^2(u'-v')|^2\, dz \le \eta^2. \tag{41}$$

Recall that

$$\|D^2v'\|^2_{L^\infty(C_2(z'))} \le N_0^2, \tag{42}$$

we claim

$$\left\{z \in Q_6 : \mathcal{M}\left(|D^2u'|^2\right)(z) > N_1^2\right\} \cap C_1(z') \tag{43}$$
$$\subset \left\{z \in Q_6 : \mathcal{M}\left(|D^2(u'-v')|^2\right)(z) > N_0^2\right\} \cap C_1(z'),$$

where $N_1^2 = \sup\{4^{n+2}, 4N_0^2\}$.

In fact, to see this, suppose

$$z_1 \in \left\{z \in Q_6 : \mathcal{M}\left(|D^2(u'-v')|^2\right)(z) \le N_0^2\right\} \cap C_1(z'). \tag{44}$$

When $\rho \leq 1$, it follows $C_\rho(z_1) \subset C_2(z')$, and then (42) and (44) imply

$$\begin{aligned} &\frac{1}{|C_\rho(z_1)|}\int_{C_\rho(z_1)} |D^2u'|^2\,dz \\ &\leq \frac{2}{|C_\rho(z_1)|}\int_{C_\rho(z_1)} |D^2(u'-v')|^2\,dz + \frac{2}{|C_\rho(z_1)|}\int_{C_\rho(z_1)} |D^2v'|^2\,dz \\ &\leq 4N_0^2; \end{aligned} \tag{45}$$

when $\rho > 1$, we conclude $C_\rho(z_1) \subset C_{4\rho}(z_0)$, and then by (38),

$$\begin{aligned} \frac{1}{|C_\rho(z_1)|}\int_{C_\rho(z_1)} |D^2u'|^2\,dz &\leq \frac{C_{4\rho}(z_0)}{|C_\rho(z_1)|}\frac{1}{|C_{4\rho}(z_0)|}\int_{C_{4\rho}(z_0)} |D^2u'|^2\,dz \\ &\leq 4^{n+2}\frac{1}{|C_{4\rho}(z_0)|}\int_{C_{4\rho}(z_0)} |D^2u'|^2\,dz \\ &\leq 4^{n+2}. \end{aligned} \tag{46}$$

Summing up (45) and (46), it shows

$$z_1 \in \big\{z \in Q_6 : \mathcal{M}\big(|D^2u'|^2\big)(z) \leq N_1^2\big\} \cap C_1(z'). \tag{47}$$

Thus (43) follows from (44) and (47).

By (43), Lemma 18 and (41), we have

$$\begin{aligned} &\big|\big\{z \in Q_6 : \mathcal{M}\big(|D^2u'|^2\big) > N_1^2\big\} \cap C_1(z')\big| \\ &\quad\leq \big|\big\{z \in Q_6 : \mathcal{M}\big(|D^2(u'-v')|^2\big) > N_0^2\big\} \cap C_1(z')\big| \\ &\quad\leq \frac{c}{N_0^2}\int_{C_2(z')} \big(\big|D^2(u'-v')\big|^2\big)\,dz \leq c\eta^2 \leq \varepsilon|C_1(z')|, \end{aligned}$$

for suitable choice of $\eta$. This completes the proof.

With a scaling argument, we obtain the following

COROLLARY 20. *For any $\varepsilon > 0$, there exist a positive constant $N_1$ and a small $\delta = \delta(\varepsilon) > 0$ such that if $u$ is a strong solution of (1) in $\Omega_T$ with*

$$\big|\big\{z \in Q_1 : \mathcal{M}\big(|D^2u|^2\big)(z) > N_1^2\big\} \cap C_r(z')\big| \geq \varepsilon|C_r(z')|,$$

*and the coefficients being weakly $(\delta, 6)$-vanishing. Then*

$$C_r(z') \cap Q_1 \subset \big\{z \in Q_1 : \mathcal{M}\big(|D^2u|^2\big)(z) > 1\big\} \cup \big\{z \in Q_1 : \mathcal{M}\big(|f|^2\big)(z) > \delta^2\big\}.$$

## 3. Regularity in Orlicz spaces

In this section, we prove the main result of this paper.

LEMMA 21 ([19]). *Let $0 < \varepsilon < 1$, $C$ and $D$ be two measurable sets satisfying $C \subset D \subset Q_1$, $|C| < \varepsilon|Q_1|$ and the following property: for every $z \in Q_1$ with $|C \cap C_r(z)| \geq \varepsilon|C_r(z)|$, it follows $C_r(z) \cap Q_1 \subset D$. Then*

$$|C| \leq 20^{n+2}\varepsilon|D|.$$

THEOREM 22. *Suppose that $u$ is a strong solution of* (1) *in $\Omega_T$ satisfying*

$$\left|\left\{z \in Q_1 : \mathcal{M}\left(|D^2u|^2\right)(z) > N_1^2\right\}\right| < \varepsilon|Q_1|.$$

*Then for any positive integer $m$,*

$$\begin{aligned} \left|\left\{z \in Q_1 : \mathcal{M}|D^2u|^2(z) > N_1^{2(m+1)}\right\}\right| &\\ \leq \varepsilon_1\Big\{\left|\left\{z \in Q_1 : \mathcal{M}|\mathbf{F}|^2(z) > \delta^2 N_1^{2m}\right\}\right| &\\ + \left|\left\{z \in Q_1 : \mathcal{M}|D^2u|^2(z) > N_1^{2m}\right\}\right|\Big\}, & \end{aligned} \tag{48}$$

*where $\varepsilon_1 = 20^{n+2}\varepsilon$.*

PROOF. We only prove for the case $m = 0$, otherwise replace $u$ by $\frac{u}{N_1^m}$ and $\mathbf{F}$ by $\frac{\mathbf{F}}{N_1^m}$. Let

$$C = \left\{z \in Q_1 : \mathcal{M}\left(|D^2u|^2\right)(z) > N_1^2\right\}$$

and

$$D = \left\{z \in Q_1 : \mathcal{M}\left(|\mathbf{F}|^2\right)(z) > \delta^2\right\} \cup \left\{z \in Q_1 : \mathcal{M}\left(|D^2u|^2\right)(z) > 1\right\}.$$

Since $N_1 \geq 1$, $C \subset D \subset Q_1$ and $|C| < \varepsilon Q_1$, let $z \in Q_1$ such that

$$|C \cap C_r(z)| \geq \varepsilon|C_r(z)|.$$

Then by Corollary 20,

$$C_r(z) \cap Q_1 \subset D,$$

and by Lemma 21,

$$|C| \leq 20^{(n+2)}\varepsilon|D|,$$

which is the conclusion for $m = 0$.

PROOF OF THEOREM 12. For any $\varepsilon > 0$ to be chosen later, let us pick $\delta$ as in Theorem 22. Since the $a_{\alpha\beta}^{ij}$'s belong to $VMO_x(\Omega_T)$, there exists $R$ depending on $Q'$, $\Omega_T$ and $\delta$ such that the $a^{ij}_{\alpha\beta}$'s are weakly $(\delta, 4R)$-vanishing. Without loss of generality, we may assume that the $a^{ij}_{\alpha\beta}$'s are weakly $(\delta, 4)$-vanishing. By using Lemma 11 with $f = \mathcal{M}(|D^2u|^2)(z)$ and $N = N_1^2$, a computation gives

$$\begin{aligned}\int_{Q_1} \phi\big(|D^2u|^2\big)\,dz &\le c\int_{Q_1} \phi\big(\big|\mathcal{M}\big(|D^2u|^2\big)\big|\big)\,dz\\ &= \sum_{k=-\infty}^{\infty} \int_{N_1^{2k}}^{N_1^{2(k+1)}} \big|\big\{z \in Q_1 : \mathcal{M}\big(|D^2u|^2\big) > \lambda\big\}\big|\,d\phi(\lambda)\\ &\le \sum_{k=-\infty}^{\infty} \phi(N_1^{2k})\big|\big\{z \in Q_1 : \mathcal{M}\big(|D^2u|^2\big) > N_1^{2k}\big\}\big|,\end{aligned}$$

hence

$$\begin{aligned}&\int_{Q_1} \phi\big(|D^2u|^2\big)\,dz\\ &\quad\le \sum_{k=-\infty}^{M} \phi\big(N_1^{2(k+1)}\big)\big|\big\{z \in Q_1 : \mathcal{M}\big(|D^2u|^2\big)(z) > N_1^{2(k+1)}\big\}\big|\\ &\quad\le \Big(\sum_{k=M}^{\infty} + \sum_{k=-\infty}^{M-1}\Big)\phi\big(N_1^{2(k+1)}\big)\big|\big\{z \in Q_1 : \mathcal{M}\big(|D^2u|^2\big)(z) > N_1^{2(k+1)}\big\}\big|\\ &\quad= I + II.\end{aligned}$$

We take $M$ such that $N_1^{2(M+1)} = c\int_{\Omega_T}\big(|u|^2 + |\mathbf{F}|^2\big)\,dz > 1$, and have by Jensen's inequality that

$$II \le c\phi\Big(\int_{\Omega_T}\big(|u|^2 + |\mathbf{F}|^2\big)\,dz\Big) \le c\int_{\Omega_T}\big(\phi(|u|^2) + \phi(|\mathbf{F}|^2)\big)\,dz. \tag{49}$$

Now, we estimate $I$. By Theorem 22,

$$\begin{aligned}I &\le \sum_{k=M}^{\infty} \phi\big(N_1^{2(k+1)}\big)\big|\big\{z \in Q_1 : \mathcal{M}\big(|D^2u|^2\big)(z) > N_1^{2(k+1)}\big\}\big|\\ &\le \sum_{k=M}^{\infty} \phi\big(N_1^{2k+2}\big)\Big\{\varepsilon_1\big|\big\{z \in Q_1 : \mathcal{M}\big(|\mathbf{F}|^2\big)(z) > \delta^2 N_1^{2k}\big\}\big|\\ &\qquad + \varepsilon_1\big|\big\{z \in Q_1 : \mathcal{M}\big(|D^2u|^2\big)(z) > N_1^{2k}\big\}\big|\Big\}.\end{aligned}$$

Since $\phi \in \triangle_2 \cap \nabla_2$, $N_1 > 1$, we see by Lemma 10 that

$$\phi\big(N_1^{2k+2}\big) = \phi\big(N_1^{2k} \cdot N_1^2\big) \le A_1^{-1} N_1^{2\alpha_1} \phi\big(N_1^{2k}\big),$$

and by Lemma 11 and Lemma 18,

$$\begin{aligned} I &\le \varepsilon_1 N_1^{2\alpha_1} \sum_{k=M}^{\infty} \phi\big(N_1^{2k}\big) \Big\{ \varepsilon_1 \big| \big\{ z \in Q_1 : \mathcal{M}\big(|\mathbf{F}|^2\big)(z) > \delta^2 N_1^{2k} \big\} \big| \\ &\qquad + \big| \big\{ z \in Q_1 : \mathcal{M}\big(|D^2 u|^2\big)(z) > N_1^{2k} \big\} \big| \Big\} \\ &\le \varepsilon_1 N_1^{2\alpha_1} \sum_{k=M}^{\infty} \phi\bigg(\frac{\delta^2 N_1^{2k}}{\delta^2}\bigg) \Big\{ \varepsilon_1 \big| \big\{ z \in Q_1 : \mathcal{M}\big(|\mathbf{F}|^2\big)(z) > \delta^2 N_1^{2k} \big\} \big| \Big\} \\ &\qquad + \varepsilon_1 N_1^{2\alpha_1} \int_{Q_1} \phi\big((|D^2 u|)^2\big)\, dz. \end{aligned}$$

Using Lemma 10 and selecting $\varepsilon > 0$ small enough such that $N_1^{2\alpha_1} \varepsilon_1 < 1/2$, it follows

$$I \le \frac{1}{2} \int_{Q_1} \phi\big((|D^2 u|)^2\big)\, dz + \delta^{-2\alpha_2} \int_{Q_1} \phi\big(|\mathbf{F}|^2\big)\, dz. \tag{50}$$

Combining (49) and (50), we have

$$\int_{Q_1} \phi\big(|D^2 u|^2\big)\, dz \le c \bigg( \int_{\Omega_T} \phi\big(|u|^2\big)\, dz + \int_{\Omega_T} \phi\big(|\mathbf{F}|^2\big)\, dz \bigg).$$

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