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THE ALGEBRA OF SEMIGROUPS OF SETS

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Abstract

We study the algebra of semigroups of sets (i.e. families of sets closed under finite unions) and its applications. For each $n > 1$ we produce two finite nested families of pairwise different semigroups of sets consisting of subsets of R^n without the Baire property.

1. Introduction

An interesting extension of the family $\mathscr{M}$ of all meager subsets of the real line R, as well as the family $\mathscr{O}$ of all open subsets of R, in the family $\mathscr{P}(\mathsf{R})$ of all subsets of R is the family $\mathscr{B}_p$ of all sets possessing the Baire property. The property is a classical notion which is related to the thesis of R. Baire. Recall that $B \in \mathscr{B}_p$ if there are an $O \in \mathscr{O}$ and an $M \in \mathscr{M}$ such that $B = O \bigtriangleup M$.

It is well known that the family $\mathscr{B}_p$ is a σ-algebra of sets invariant under homeomorphisms of the real line R, and the complement $\mathscr{B}_p^C = \mathscr{P}(\mathsf{R}) \setminus \mathscr{B}_p$ of $\mathscr{B}_p$ in $\mathscr{P}(\mathsf{R})$ is not empty (for example, each Vitali set S of R ([7]) is an element of $\mathscr{B}_p^C$). Moreover, there are elements of $\mathscr{B}_p^C$ with a natural algebraic structure (see [4] for subgroups of the additive group R, which are elements of $\mathscr{B}_p^C$).

In [2] Chatyrko and Nyagahakwa looked for subfamilies of the family $\mathscr{B}_p^C$ which have some algebraic structures. They proved that the family $\mathscr{V}_1$ of all finite unions of Vitali sets of R and its extension $\mathscr{V}_2$ which elements are all sets of the type $A \bigtriangleup B$, where $A \in \mathscr{V}_1$ and $B \in \mathscr{M}$, are semigroups of sets (i.e. families of sets closed under finite unions) invariant under translations of the real line R and consisting of zero-dimensional subsets of $\mathscr{B}_p^C$. Furthermore, Chatyrko and Nyagahakwa extended the result to the Euclidean spaces R^n, where n is any positive integer.

In this paper we pay attention to the algebra of semigroups of sets. We look at the behavior of semigroups of sets under several operations. Then we suggest some applications. First, we show that the results from [2] can be obtained by the use of the theory. Moreover, we can suggest many different semigroups of sets in $\mathscr{B}_p^C$. After that for each $n > 1$ we produce two finite nested families of

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pairwise different semigroups of sets consisting of subsets of R^n without the Baire property.

2. Auxiliary notions

Recall that a non-empty set $\mathscr{S}$ is called *a semigroup* if there is an operation $\alpha : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ such that $\alpha(\alpha(s_1, s_2), s_3) = \alpha(s_1, \alpha(s_2, s_3))$. The semigroup $\mathscr{S}$ is called *abelian* if $\alpha(s_1, s_2) = \alpha(s_2, s_1)$.

Let X be a set and $\mathscr{P}(X)$ be the family of all subsets of X. In the paper we will be interested in subsets $\mathscr{S}$ of $\mathscr{P}(X)$ such that for each $A, B \in \mathscr{S}$ we have $A \cup B \in \mathscr{S}$. It is evident that such a family of sets is an abelian semigroup with respect to the operation of union of sets (in brief, *a semigroup of sets*).

Let $\mathscr{A} \subset \mathscr{P}(X)$. Put $\mathscr{S}_{\mathscr{A}} = \{\cup_{i \le n} A_i : A_i \in \mathscr{A}, n \in \mathsf{N}\}$. Note that $\mathscr{S}_{\mathscr{A}}$ is a semigroup of sets. Recall that a set $\mathscr{I} \subset \mathscr{P}(X)$ is called *an ideal of sets* if $\mathscr{I}$ is a semigroup of sets and if $A \in \mathscr{I}$ and $B \subset A$ then $B \in \mathscr{I}$. Put $\mathscr{I}_{\mathscr{A}} = \{B \in \mathscr{P}(X) : \text{there is } A \in \mathscr{S}_{\mathscr{A}} \text{ such that } B \subset A\}$. Note that $\mathscr{I}_{\mathscr{A}}$ is an ideal of sets.

For $x \in \mathsf{R}$ denote by T_x the translation of R by x, i.e. $T_x(y) = y + x$ for each $y \in \mathsf{R}$. If A is a subset of R and $x \in \mathsf{R}$, we denote $T_x(A)$ by A_x.

The equivalence relation E on R is defined as follows. For $x, y \in \mathsf{R}$, let xEy iff $x - y \in \mathsf{Q}$, where Q is the set of rational numbers. Let us denote its equivalence classes by E_α, $\alpha \in I$. It is evident that $|I| = c$ (continuum), and for each $\alpha \in I$ and each $x \in E_\alpha$, $E_\alpha = \mathsf{Q}_x$. Let us also note that every equivalence class E_α is dense in R. Recall ([7]) that *a Vitali set of* R is any subset S of R such that $|S \cap E_\alpha| = 1$ for each $\alpha \in I$, and each Vitali set neither possess the Baire property in R nor it is measurable in the sense of Lebesgue.

For other notions and notations we refer to [3] and [6].

3. Semigroups of sets and ideals of sets

Let $\mathscr{A}, \mathscr{B} \subset \mathscr{P}(X)$. Put $\mathscr{A} \cup \mathscr{B} = \{A \cup B : A \in \mathscr{A}, B \in \mathscr{B}\}$, $\mathscr{A} \triangle \mathscr{B} = \{A \triangle B : A \in \mathscr{A}, B \in \mathscr{B}\}$ and $\mathscr{A} * \mathscr{B} = \{(A \setminus B_1) \cup B_2 : A \in \mathscr{A}; B_1, B_2 \in \mathscr{B}\}$. However, $\mathscr{A} \cap \mathscr{B}$ denotes the intersection of $\mathscr{A}, \mathscr{B}$, i.e. the family of common elements of $\mathscr{A}, \mathscr{B}$.

It is evident that $\mathscr{A} \cup \mathscr{B} = \mathscr{B} \cup \mathscr{A}$ and $\mathscr{A} \triangle \mathscr{B} = \mathscr{B} \triangle \mathscr{A}$. Since $A \cup B = (A \setminus B) \cup B = (B \setminus A) \cup A$, we have $\mathscr{A} \cup \mathscr{B} \subset \mathscr{A} * \mathscr{B}$ and $\mathscr{A} \cup \mathscr{B} \subset \mathscr{B} * \mathscr{A}$. Moreover, if $\mathscr{A}, \mathscr{B}$ are both semigroups of sets or both ideals of sets then the family $\mathscr{A} \cup \mathscr{B}$ is of the same type.

On the other hand as we will see in the following examples in general for given semigroups of sets $\mathscr{A}, \mathscr{B}$ the families $\mathscr{A} \triangle \mathscr{B}$, $\mathscr{A} * \mathscr{B}$, $\mathscr{B} * \mathscr{A}$ do not need to be semigroups of sets and none of the statements $\mathscr{A} \triangle \mathscr{B} \subseteq \mathscr{A} \cup \mathscr{B}$, $\mathscr{A} \triangle \mathscr{B} \supseteq \mathscr{A} \cup \mathscr{B}$, $\mathscr{A} \triangle \mathscr{B} \subseteq \mathscr{A} * \mathscr{B}$, $\mathscr{A} \triangle \mathscr{B} \supseteq \mathscr{A} * \mathscr{B}$, $\mathscr{A} * \mathscr{B} \subseteq \mathscr{B} * \mathscr{A}$

needs to hold. Moreover, one of the families $\mathscr{A} * \mathscr{B}$, $\mathscr{B} * \mathscr{A}$ can be a semigroup of sets while the other is not.

EXAMPLE 3.1. Let $|X| \geq 2$ and A be a non-empty proper subset of X. Put $B = X \setminus A$, $\mathscr{A} = \{A, X\}$ and $\mathscr{B} = \{B, X\}$. Note that $\mathscr{A} = \mathscr{S}_{\mathscr{A}}$, $\mathscr{B} = \mathscr{S}_{\mathscr{B}}$ and the families $\mathscr{A} \cup \mathscr{B} = \{X\}$, $\mathscr{A} \bigtriangleup \mathscr{B} = \{\emptyset, A, B, X\}$, $\mathscr{A} * \mathscr{B} = \{B, X\}$, $\mathscr{B} * \mathscr{A} = \{A, X\}$ are semigroups of sets. Moreover, none of the following inclusions $\mathscr{A} \bigtriangleup \mathscr{B} \subseteq \mathscr{A} \cup \mathscr{B}$, $\mathscr{A} \bigtriangleup \mathscr{B} \subseteq \mathscr{A} * \mathscr{B}$, $\mathscr{A} * \mathscr{B} \subseteq \mathscr{B} * \mathscr{A}$ and $\mathscr{B} * \mathscr{A} \subseteq \mathscr{A} * \mathscr{B}$ holds.

EXAMPLE 3.2. Let $X = \{1, 2, 3, 4\}$, $A_1 = \{1, 3\}$, $A_2 = \{2, 4\}$, $B_1 = \{1, 2\}$, $B_2 = \{3, 4\}$, $C = \{1, 4\}$, $D = \{2, 3\}$, $\mathscr{A} = \{\emptyset, A_1, A_2\}$ and $\mathscr{B} = \{\emptyset, B_1, B_2\}$. Note that $\mathscr{S}_{\mathscr{A}} = \{\emptyset, A_1, A_2, X\}$ and $\mathscr{S}_{\mathscr{B}} = \{\emptyset, B_1, B_2, X\}$. Moreover, we have $\mathscr{S}_{\mathscr{A}} \cup \mathscr{S}_{\mathscr{B}} = \{\emptyset, A_1, A_2, B_1, B_2, \{1\}^-, \{2\}^-, \{3\}^-, \{4\}^-, X\}$ (here Y^- denotes the complement of a set Y in the set X), $\mathscr{S}_{\mathscr{A}} \bigtriangleup \mathscr{S}_{\mathscr{B}} = \{\emptyset, A_1, A_2, B_1, B_2, C, D, X\}$ and $\mathscr{S}_{\mathscr{A}} * \mathscr{S}_{\mathscr{B}} = \mathscr{S}_{\mathscr{B}} * \mathscr{S}_{\mathscr{A}} = \mathscr{P}(X) \setminus \{C, D\}$. It is easy to see that the inclusions $\mathscr{S}_{\mathscr{A}} * \mathscr{S}_{\mathscr{B}} \subseteq \mathscr{S}_{\mathscr{A}} \bigtriangleup \mathscr{S}_{\mathscr{B}}$ and $\mathscr{S}_{\mathscr{A}} \cup \mathscr{S}_{\mathscr{B}} \subseteq \mathscr{S}_{\mathscr{A}} \bigtriangleup \mathscr{S}_{\mathscr{B}}$ do not hold. We note also that none of the families $\mathscr{S}_{\mathscr{A}} \bigtriangleup \mathscr{S}_{\mathscr{B}}$, $\mathscr{S}_{\mathscr{A}} * \mathscr{S}_{\mathscr{B}}$ and $\mathscr{S}_{\mathscr{B}} * \mathscr{S}_{\mathscr{A}}$ is a semigroup of sets. In fact, $A_1, D \in \mathscr{S}_{\mathscr{A}} \bigtriangleup \mathscr{S}_{\mathscr{B}}$ but $A_1 \cup D = 4^- \notin \mathscr{S}_{\mathscr{A}} \bigtriangleup \mathscr{S}_{\mathscr{B}}$, and $\{1\}, \{4\} \in \mathscr{S}_{\mathscr{A}} * \mathscr{S}_{\mathscr{B}}$ but $\{1\} \cup \{4\} = C \notin \mathscr{S}_{\mathscr{A}} * \mathscr{S}_{\mathscr{B}}$.

EXAMPLE 3.3. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A_1 = \{1, 2, 4, 5, 7, 8\}$, $A_2 = \{2, 3, 5, 6, 8, 9\}$, $B_1 = \{1, 2, 3, 4, 5, 6\}$, $B_2 = \{4, 5, 6, 7, 8, 9\}$, $\mathscr{A} = \{A_1, A_2\}$, $\mathscr{B} = \{\emptyset, B_1, B_2\}$. Note that $\mathscr{S}_{\mathscr{A}} = \{A_1, A_2, X\}$ and $\mathscr{S}_{\mathscr{B}} = \{\emptyset, B_1, B_2, X\}$. First we will show that the family $\mathscr{S}_{\mathscr{A}} * \mathscr{S}_{\mathscr{B}}$ is not a semigroup of sets. It is enough to prove that the set $C = ((A_1 \setminus B_1) \cup \emptyset) \cup ((A_2 \setminus B_2) \cup \emptyset) \notin \mathscr{S}_{\mathscr{A}} * \mathscr{S}_{\mathscr{B}}$. Note that $C = (A_1 \setminus B_1) \cup (A_2 \setminus B_2) = \{2, 3, 7, 8\}$. Assume that $C \in \mathscr{S}_{\mathscr{A}} * \mathscr{S}_{\mathscr{B}}$. Thus $C = (S_1 \setminus S_2) \cup S_3$ for some $S_1 \in \mathscr{S}_{\mathscr{A}}$ and $S_2, S_3 \in \mathscr{S}_{\mathscr{B}}$. Since $|C| = 4$, we have $S_3 = \emptyset$. Let $S_1 = A_1$. Then $|S_1 \setminus S_2|$ is either 2 (if S_2 is B_1 or B_2), 0 (if $S_2 = X$) or 6 (if $S_2 = \emptyset$). We have a contradiction. If $S_1 = A_2$, we also have a contradiction by a similar argument as above. Assume now that $S_1 = X$. Then $|S_1 \setminus S_2|$ is either 3 (if S_2 is B_1 or B_2), 0 (if $S_2 = X$) or 9 (if $S_2 = \emptyset$). We have again a contradiction that proves the statement.

Further note that $\mathscr{S}_{\mathscr{B}} * \mathscr{S}_{\mathscr{A}} = \{A_1, A_2, \{1\}^-, \{3\}^-, \{7\}^-, \{9\}^-, X\} = \mathscr{S}_{\mathscr{A}} \cup \mathscr{S}_{\mathscr{B}}$. Hence, the family $\mathscr{S}_{\mathscr{B}} * \mathscr{S}_{\mathscr{A}}$ is a semigroup of sets.

PROPOSITION 3.4. *Let $\mathscr{S}$ be a semigroup of sets and $\mathscr{I}$ be an ideal of sets. Then the family $\mathscr{S} * \mathscr{I}$ is a semigroup of sets.*

PROOF. In fact, let $S_i \in \mathscr{S}$ and $I'_i, I''_i \in \mathscr{I}$, $i = 1, 2$. Proceed as follows: $U = ((S_1 \setminus I'_1) \cup I''_1) \cup ((S_2 \setminus I'_2) \cup I''_2) = (S_1 \setminus I'_1) \cup (S_2 \setminus I'_2) \cup (I''_1 \cup I''_2)$. Put $I_2 = I''_1 \cup I''_2$ and continue: $U = ((S_1 \cap I'^-_1) \cup (S_2 \cap I'^-_2))^{--} \cup I_2 = ((S_1 \cap I'^-_1)^- \cap (S_2 \cap I'^-_2)^-)^- \cup I_2 = ((S_1^- \cup I'_1) \cap (S_2^- \cup I'_2))^- \cup I_2 = ((S_1^- \cap S_2^-) \cup (S_1^- \cap I'_2) \cup$

$(S_2^- \cap I_1') \cup (I_1' \cap I_2'))^- \cup I_2$. Put $I_1 = (S_1^- \cap I_2') \cup (S_2^- \cap I_1') \cup (I_1' \cap I_2')$ and note that $U = ((S_1^- \cap S_2^-)^- \cap I_1^-) \cup I_2 = ((S_1 \cup S_2) \cap I_1^-) \cup I_2 = ((S_1 \cup S_2) \setminus I_1) \cup I_2$. It is easy to see that $S_1 \cup S_2 \in \mathscr{S}$ and $I_1, I_2 \in \mathscr{I}$. Hence, $U \in \mathscr{S} * \mathscr{I}$.

Let (X, τ) be a topological space and $\mathscr{M}_{(X,\tau)}$ be a family of meager subsets of (X, τ). It is easy to see that the family τ is a semigroup of sets and $\mathscr{M}_{(X,\tau)}$ is an ideal of sets (in fact, σ-ideal of sets). The family $\mathscr{B}_{(X,\tau)}$ of sets with the Baire property is defined as the family $\tau \bigtriangleup \mathscr{M}_{(X,\tau)}$. It is well known that $\tau \bigtriangleup \mathscr{M}_{(X,\tau)} = \tau * \mathscr{M}_{(X,\tau)}$. In fact, this equality is a particular case of the following general statement.

PROPOSITION 3.5. *Let $\mathscr{S}$ be a semigroup of sets and $\mathscr{I}$ be an ideal of sets. Then*

(a) $\mathscr{S} * \mathscr{I} = \mathscr{S} \bigtriangleup \mathscr{I} \supset \mathscr{S} \cup \mathscr{I} = \mathscr{I} * \mathscr{S} \supset \mathscr{S}$;

(b) $(\mathscr{S} * \mathscr{I}) * \mathscr{I} = \mathscr{S} * \mathscr{I}$, $\mathscr{I} * (\mathscr{I} * \mathscr{S}) = \mathscr{I} * \mathscr{S}$.

PROOF. (a) Note that for any set $S \in \mathscr{S}$ and for any set $I \in \mathscr{I}$ we have $S \bigtriangleup I = (S \setminus I) \cup (I \setminus S) \in \mathscr{S} * \mathscr{I}$, $S \cup I = S \bigtriangleup (I \setminus S) \in \mathscr{S} \bigtriangleup \mathscr{I}$, $S \cup I = (I \setminus S) \cup S \in \mathscr{I} * \mathscr{S}$ and $S = S \cup \emptyset \in \mathscr{S} \cup \mathscr{I}$. Thus, $\mathscr{S} * \mathscr{I} \supset \mathscr{S} \bigtriangleup \mathscr{I} \supset \mathscr{S} \cup \mathscr{I} \supset \mathscr{S}$ and $\mathscr{I} * \mathscr{S} \supset \mathscr{S} \cup \mathscr{I}$. Observe also that for any sets $S_1, S_2 \in \mathscr{S}$ and any sets $I_1, I_2 \in \mathscr{I}$ we have $(S_1 \setminus I_1) \cup I_2 = S_1 \bigtriangleup I \in \mathscr{S} \bigtriangleup \mathscr{I}$, where $I = ((I_1 \cap S_1) \setminus I_2) \cup (I_2 \setminus S_1)$ and $(I_1 \setminus S_1) \cup S_2 \in \mathscr{S} \cup \mathscr{I}$. Thereby, $\mathscr{S} * \mathscr{I} \subset \mathscr{S} \bigtriangleup \mathscr{I}$ and $\mathscr{I} * \mathscr{S} \subset \mathscr{S} \cup \mathscr{I}$.

(b) Let $S \in \mathscr{S}$ and $I_1, I_2, I_3, I_4 \in \mathscr{I}$. Observe that $(((S \setminus I_1) \cup I_2) \setminus I_3) \cup I_4 = (S \setminus (I_1 \cup I_3)) \cup ((I_2 \setminus I_3) \cup I_4) \in \mathscr{S} * \mathscr{I}$. Hence, $(\mathscr{S} * \mathscr{I}) * \mathscr{I} \subset \mathscr{S} * \mathscr{I}$. The opposite inclusion is evident.

Let $I_1, I_2, I_3 \in \mathscr{I}$ and $S_1, S_2, S_3, S_4 \in \mathscr{S}$. Note that $(I_1 \setminus ((I_2 \setminus S_1) \cup S_2)) \cup ((I_3 \setminus S_3) \cup S_4) = ((I_1 \setminus ((I_2 \setminus S_1) \cup S_2)) \cup (I_3 \setminus S_3)) \cup S_4 = I \cup S_4 \in \mathscr{I} * \mathscr{S}$, where $I = (I_1 \setminus ((I_2 \setminus S_1) \cup S_2)) \cup (I_3 \setminus S_3)$. Hence, $\mathscr{I} * (\mathscr{I} * \mathscr{S}) \subset \mathscr{I} * \mathscr{S}$. The opposite inclusion is evident.

COROLLARY 3.6. *Let $\mathscr{S}$ be a semigroup of sets and $\mathscr{I}$ be an ideal of sets. Then*

(a) *the families $\mathscr{S} \bigtriangleup \mathscr{I}$, $\mathscr{I} * \mathscr{S}$ are semigroups of sets;*

(b) $(\mathscr{I} * \mathscr{S}) * \mathscr{I} = \mathscr{I} * (\mathscr{S} * \mathscr{I}) = \mathscr{S} * \mathscr{I}$.

PROOF. We will show only (b). Note that

(1) $\mathscr{S} * \mathscr{I} = (\mathscr{S} * \mathscr{I}) * \mathscr{I} \supset (\mathscr{I} * \mathscr{S}) * \mathscr{I} \supset \mathscr{S} * \mathscr{I}$;

(2) $\mathscr{S} * \mathscr{I} = (\mathscr{S} * \mathscr{I}) * \mathscr{I} \supset \mathscr{I} * (\mathscr{S} * \mathscr{I}) \supset \mathscr{S} * \mathscr{I}$.

The following statement is evident.

COROLLARY 3.7. *Let $\mathscr{I}_1, \mathscr{I}_2$ be ideals of sets. Then the family $\mathscr{I}_1 * \mathscr{I}_2$ is an ideal of sets. Moreover, $\mathscr{I}_1 * \mathscr{I}_2 = \mathscr{I}_2 * \mathscr{I}_1 = \mathscr{I}_1 \triangle \mathscr{I}_2 = \mathscr{I}_1 \cup \mathscr{I}_2$.*

EXAMPLE 3.8. Let $X = \{1, 2\}$, $A = X$, $B = \{1\}$, $C = \{2\}$, $\mathscr{A} = \{A\}$, $\mathscr{B} = \{B\}$. Note that $\mathscr{S}_{\mathscr{A}} = \{A\}$, $\mathscr{I}_{\mathscr{B}} = \{\emptyset, B\}$, $\mathscr{S}_{\mathscr{A}} * \mathscr{I}_{\mathscr{B}} = \{A, C\}$ and $\mathscr{I}_{\mathscr{B}} * \mathscr{S}_{\mathscr{A}} = \{A\}$. Thus, in general, none of the following statements is valid: $\mathscr{S} * \mathscr{I} = \mathscr{I} * \mathscr{S}$, $\mathscr{S} * \mathscr{I} \supset \mathscr{I}$, the family $\mathscr{S} * \mathscr{I}$ is an ideal of sets or $\mathscr{I} * \mathscr{S}$ is an ideal of sets, even if $\mathscr{S}$ is a semigroup of sets and $\mathscr{I}$ is an ideal of sets.

The next statement is useful in the search of pairs of semigroups without common elements.

PROPOSITION 3.9 (See [2, Proposition 3.1]). *Let $\mathscr{I}$ be an ideal of sets and $\mathscr{A}, \mathscr{B} \subset \mathscr{P}(X)$ such that*

(a) $\mathscr{A} \cap \mathscr{I} = \emptyset$;

(b) *for each element $U \in \mathscr{S}_{\mathscr{A}}$ and each non-empty element $B \in \mathscr{B}$ there is an element $A \in \mathscr{A}$ such that $A \subset B \setminus U$.*

Then

(1) *for each element $I \in \mathscr{I}$, each element $U \in \mathscr{S}_{\mathscr{A}}$ and each non-empty element $B \in \mathscr{B}$ we have $(U \cup I)^- \cap B \neq \emptyset$;*

(2) *for each elements $I_1, I_2 \in \mathscr{I}$, each element $U \in \mathscr{S}_{\mathscr{A}}$ and each non-empty element $B \in \mathscr{B}$ we have $(U \cup I_1)^- \cap (B \setminus I_2) \neq \emptyset$;*

(3) *for each elements $I_1, I_2, I_3, I_4 \in \mathscr{I}$, each element $U \in \mathscr{S}_{\mathscr{A}}$ and each element $V \in \mathscr{S}_{\mathscr{B}}$ we have $(U \setminus I_1) \cup I_2 \neq (V \setminus I_3) \cup I_4$. i.e. $(\mathscr{S}_{\mathscr{A}} * \mathscr{I}) \cap (\mathscr{S}_{\mathscr{B}} * \mathscr{I}) = \emptyset$.*

PROOF. Our proof is very close to the proof of [2, Proposition 3.1].

(1) Assume that $U \cup I \supset B$ for some non-empty element $B \in \mathscr{B}$. By (b) there is $A \in \mathscr{A}$ such that $A \subset B \setminus U$. Note that $A \subset (U \cup I) \setminus U \subset I$. But this contradicts (a).

(2) Assume that $U \cup I_1 \supset (B \setminus I_2)$ for some non-empty element $B \in \mathscr{B}$ and some element $I_2 \in \mathscr{I}$. Note that $U \cup (I_1 \cup I_2) = (U \cup I_1) \cup I_2 \supset (B \setminus I_2) \cup I_2 \supset B$. But this contradicts (1).

(3) Assume that $(U \setminus I_1) \cup I_2 = (V \setminus I_3) \cup I_4$ for some elements $U \in \mathscr{S}_{\mathscr{A}}$, $V \in \mathscr{S}_{\mathscr{B}}$ and $I_3, I_4 \in \mathscr{I}$. If $V = \emptyset$, then $(U \setminus I_1) \cup I_2 = I_4$ and so $U \subset I_1 \cup I_4$. But this contradicts (a). Hence $V \neq \emptyset$. Note that there is a non-empty element $B \in \mathscr{B}$ such that $B \subset V$. Further observe that $U \cup I_2 \supset (U \setminus I_1) \cup I_2 = (V \setminus I_3) \cup I_4 \supset B \setminus I_3$. But this contradicts (2).

EXAMPLE 3.10 ([2]).

(a) The family $\mathscr{V}$ of all Vitali sets of R as $\mathscr{A}$, the family $\mathscr{O}$ of all open sets of R as $\mathscr{B}$ and the family $\mathscr{M}$ of all meager sets of R as $\mathscr{I}$ satisfy the conditions of Proposition 3.3. Note that $\mathcal{S}_{\mathcal{V}} = \mathcal{V}_1$, $\mathcal{S}_{\mathcal{O}} = \mathcal{O}$, $\mathcal{V}_1 * \mathcal{M} = \mathcal{V}_2$ and $\mathcal{O} * \mathcal{M} = \mathcal{B}_p$ (the notations are from the Introduction). Hence, $\mathcal{V}_2 \cap \mathcal{B}_p = \emptyset$.

(b) Consider the Euclidean space R^n for some $n > 1$. A Vitali set of R^n is any set $\overline{S} = \prod_{j=1}^n S(j)$, where $S(j)$ is a Vitali set of R for each $j = 1, \ldots, n$. The family $\mathcal{V}^n$ of all Vitali sets of R^n as $\mathcal{A}$, the family $\mathcal{O}^n$ of all open sets of R^n as $\mathcal{B}$ and the family $\mathcal{M}^n$ of all meager sets of R^n as $\mathcal{I}$ satisfy the conditions of Proposition 3.3. Let $\mathcal{V}_1^n$ be the family of all finite unions of Vitali sets of R^n, $\mathcal{V}_2^n = \mathcal{V}_1^n * \mathcal{M}^n$ and $\mathcal{B}_p^n$ be the family of all sets of R^n with the Baire property. Note that $\mathcal{S}_{\mathcal{V}^n} = \mathcal{V}_1^n$, $\mathcal{S}_{\mathcal{O}^n} = \mathcal{O}^n$, $\mathcal{B}_p^n = \mathcal{O}^n * \mathcal{M}^n$ and $\mathcal{V}_2^n \cap \mathcal{B}_p^n = \emptyset$.

There is even a generalization of the result for the products $\mathsf{R}^n \times \mathsf{R}_S^m$, where R_S is the Sorgenfrey line (see [3] for the definition).

4. Applications

In [2, Theorem 3.2] one can find the following statements about the families $\mathcal{V}^n, \mathcal{V}_1^n, \mathcal{V}_2^n$, where $n \geq 1$.

(i) $\mathcal{V}^n \subset \mathcal{V}_1^n \subset \mathcal{V}_2^n \subset (B_p^n)^C$.

(ii) For each $U \in \mathcal{V}_1^n$, $\dim U = 0$, and for each $W \in \mathcal{V}_2^n$, $\dim W \leq n - 1$.

(iii) The families $\mathcal{V}^n, \mathcal{V}_1^n, \mathcal{V}_2^n$ are invariant under translations of R^n.

(iv) The families $\mathcal{V}_1^n, \mathcal{V}_2^n$ are semigroups of sets.

4.1. Two nested families of semigroups of sets

It follows easily from Corollary 3.1 and Proposition 3.2 that the family $\mathcal{M}^n * \mathcal{V}_1^n$ is another semigroup of sets invariant under translations of R^n such that $\mathcal{V}_1^n \subset \mathcal{M}^n * \mathcal{V}_1^n \subset \mathcal{V}_2^n$. The following statement extends the variety of semigroups of sets without the Baire property based on the family $\mathcal{V}_1^n$.

THEOREM 4.1. *Let $n > 1$. Then there are two finite families $\{\mathcal{L}^{n,k}\}_{k=0}^{n-1}$, $\{\mathcal{R}^{n,k}\}_{k=0}^{n-1}$ of pairwise distinct semigroups of sets invariant under translations of the Euclidean space R^n such that*

(a) *for each $0 \leq k \leq n-2$ we have $\mathcal{L}^{n,k} \subset \mathcal{L}^{n,k+1}$ and $\mathcal{R}^{n,k} \subset \mathcal{R}^{n,k+1}$,*

(b) *for each $L \in \mathcal{L}^{n,k}$ and $R \in \mathcal{R}^{n,k}$ we have $\dim L \leq k$ and $\dim R \leq k$ and there are $L_0 \in \mathcal{L}^{n,k}$ and $R_0 \in \mathcal{R}^{n,k}$ such that $\dim L_0 = \dim R_0 = k$, where $0 \leq k \leq n-1$,*

(c) *for each $0 \leq k \leq n-1$ we have $\mathcal{L}^{n,k} \subset \mathcal{R}^{n,k}$ but $\mathcal{R}^{n,k-1}$ does not contain $\mathcal{L}^{n,k}$,*

(d) *$\mathcal{R}^{n,n-1} \subset \mathcal{V}_2^n$ but $\mathcal{R}^{n,n-1} \neq \mathcal{V}_2^n$, $\mathcal{L}^{n,n-1} \subset \mathcal{M}^n * \mathcal{V}_1^n$ but $\mathcal{L}^{n,n-1} \neq \mathcal{M}^n * \mathcal{V}_1^n$ and $\mathcal{M}^n * \mathcal{V}_1^n$ does not contain $\mathcal{R}^{n,0}$, $\mathcal{V}_1^n \subset \mathcal{L}^{n,0}$ but $\mathcal{V}_1^n \neq \mathcal{L}^{n,0}$.*

PROOF. For each $0 \leq k < n$ let us consider the family $\mathscr{F}_k$ of all closed k-dimensional subsets of R^n. Note that every family $\mathscr{F}_k$ is a semigroup of sets, and the inclusion $\mathscr{I}_{\mathscr{F}_k} \subset \mathscr{I}_{\mathscr{F}_{k+1}}$ holds for each $0 \leq k \leq n-2$. Since every element of $\mathscr{I}_{\mathscr{F}_{n-1}}$ is nowhere dense in the Euclidean space R^n we have $\mathscr{I}_{\mathscr{F}_{n-1}} \subset \mathscr{M}^n$. For each $0 \leq k < n$ put $\mathscr{R}^{n,k} = \mathscr{V}_1^n * \mathscr{I}_{\mathscr{F}_k}$ and $\mathscr{L}^{n,k} = \mathscr{I}_{\mathscr{F}_k} * \mathscr{V}_1^n$. The point (a) is evident. It follows from Proposition 3.1 and Corollary 3.1 that the families $\mathscr{R}^{n,k}$, $\mathscr{L}^{n,k}$ are semigroups of sets for each $0 \leq k < n$. It is also clear that the families $\mathscr{R}^{n,k}$, $\mathscr{L}^{n,k}$ consist of sets which are invariant under translations of R^n and which have dimension dim $\leq k$. Since for each Vitali set S of R^n the union $S \cup I^k = (I^k \setminus S) \cup S = (S \setminus I^k) \cup I^k$, where I^k is any subset of R^n homeomorphic to the k-dimensional cube $[0,1]^k$, belongs to both families $\mathscr{L}^{n,k}$, $\mathscr{R}^{n,k}$ and $\dim(S \cup I^k) = k$, we have (b). Note that Proposition 3.2 implies the inclusion of (c), and (b) implies that $\mathscr{L}^{n,k-1} \neq \mathscr{L}^{n,k}$, $\mathscr{R}^{n,k-1} \neq \mathscr{R}^{n,k}$ and that the family $\mathscr{R}^{n,k-1}$ cannot contain the family $\mathscr{L}^{n,k}$. On the other hand for each Vitali set S of R^n the difference $S \setminus \{p\}$, where $p \in S$, cannot belong to the family $\mathscr{M}^n * \mathscr{V}_1^n$ but it belongs to the family $\mathscr{R}^{n,0}$. Hence, $\mathscr{L}^{n,k} \neq \mathscr{R}^{n,l}$ for each $0 \leq k, l < n-1$. Note that $\mathscr{R}^{n,n-1} \subset \mathscr{V}_2^n$, $\mathscr{L}^{n,n-1} \subset \mathscr{M}^n * \mathscr{V}_1^n$ and $\mathscr{V}_1^n \subset \mathscr{L}^{n,0}$. In order to finish the proof of (d) let us recall (see [2, Lemma 3.4]) that for each element $U \in \mathscr{V}_1^n$ there are elements $V_1, \dots, V_n \in \mathscr{V}_1$ such that $U \subset \prod_{i=1}^n V_i$. This easily implies that no element of $\mathscr{V}_1^n$ can contain a countable subset of R^n consisting of points with rational coordinates. Thus the set $\mathsf{C}^n \cup S = (\mathsf{C}^n \setminus S) \cup S \in \mathscr{L}^{n,0}$, where C is the standard Cantor set of $[0,1]$ and S is any Vitali set of R^n, is not an element of $\mathscr{V}_1^n$, and the set $\mathsf{Q}^n \cup S = (\mathsf{Q}^n \setminus S) \cup S \in \mathscr{M}^n * \mathscr{V}_1^n$, where Q is the set of all rational numbers of R and S is any Vitali set of R^n, is no element of $\mathscr{R}^{n,n-1}$. This completes the proof of (d).

4.2. Supersemigroups based on the Vitali sets

Let Q be a countable dense subgroup of the additive group of the real numbers. One can consider the Vitali construction ([7]) with the group Q instead of the group Q of rational numbers (cf. [4]). The analogue of a Vitali set with respect to the group Q we will call *a Vitali Q-selector of* R. One can introduce in the same way as above a Vitali Q-selector of R^n, $n \geq 1$ and the corresponding families $\mathscr{V}^n(Q)$, $\mathscr{V}_1^n(Q)$, $\mathscr{M}^n * \mathscr{V}_1^n(Q)$, $V_2^n(Q)$, $\mathscr{L}^{n,k}(Q)$, $\mathscr{R}^{n,k}(Q)$, where $0 \leq k < n$. Note that similar statements as in part 4.1 are valid for the families.

Let $\mathscr{F}$ be the family of all countable dense subgroups of the additive group of the real numbers.

Set $\mathscr{V}^{\sup} = \{V : V \in \mathscr{V}^1(Q), Q \in \mathscr{F}\}$, $\mathscr{V}_1^{\sup} = \mathscr{S}_{\mathscr{V}^{\sup}}$ and $\mathscr{V}_2^{\sup} = \mathscr{V}_1^{\sup} * \mathscr{M}$.

It is easy to see that

(i) for each $Q \in \mathscr{F}$ we have $\mathscr{V}_2^1(Q) \subset \mathscr{V}_2^{\sup}$.

(ii) $\mathscr{V}_1^{\sup}, \mathscr{V}_2^{\sup}$ are semigroups of sets invariant under translations of R.

(One can even show that for each $Q \in \mathscr{F}$ we have $\mathscr{V}^1(Q) \subseteq \mathscr{V}^{\sup}$ but $\mathscr{V}^1(Q) \neq \mathscr{V}^{\sup}$, resp. $\mathscr{V}_1^1(Q) \subseteq \mathscr{V}_1^{\sup}$ but $\mathscr{V}_1^1(Q) \neq \mathscr{V}_1^{\sup}$. We do not know if $\mathscr{V}_2^1(Q) \neq \mathscr{V}_2^{\sup}$ for each $Q \in \mathscr{F}$.)

We will call the family $\mathscr{V}_1^{\sup}$ *the supersemigroup of sets based on the Vitali sets*.

LEMMA 4.2. *For any set $U \in \mathscr{V}_1^{\sup}$ and any non-empty open set O of* R *there is a set $V \in \mathscr{V}^{\sup}$ such that $V \subset O \setminus U$.*

PROOF. Let $U = \cup_{i=1}^n V_i$, where $V_i \in \mathscr{V}^1(Q_i)$ and $Q_i \in \mathscr{F}$. Note that the statement is valid when $Q_1 = \cdots = Q_n$ (see [2, Lemma 3.1]). Now we will consider the general case. Put $Q = \sum_{i=1}^n Q_i = \left\{\sum_{i=1}^n q_i : q_i \in Q_i\right\}$ and note that $Q \in \mathscr{F}$.

CLAIM 4.3. *For each $x \in \mathsf{R}$ we have $|Q_x \cap (O \setminus U)| \geq 1$.*
(In fact, $|Q_x \cap (O \setminus U)| = \aleph_0$.)

PROOF. For $n = 1$ the statement evidently holds ([2, Lemma 3.1]).

Let $n \geq 2$. Let O_i, $i \leq n$, be non-empty open sets of R such that $x + O_1 + \cdots + O_n = \{x + x_1 + \cdots + x_n : x_i \in O_i, i \leq n\} \subset O$. For each $i \leq n$ choose $n+1$ different points $q_i(j)$, $j \leq n+1$, of $O_i \cap Q_i$.

Let now $Q_i = \{q_i^j : j \geq 1\}$, $i = 1 \leq n$, and $q_i^j = q_i(j)$, $i \leq n$; $j \leq n+1$. Observe that for each $i \leq n$ and each $j_1, \ldots, \widehat{j_i}, \ldots, j_n$ (the notation $\widehat{a}$ means that a is not there) the set $\{x + q_1^{j_1} + \cdots q_i^k + \cdots + q_n^{j_n} : k \geq 1\}$ consists of countably many different points (a coset of Q_i) and only one of them belongs to V_i.

Consider now an n-dimensional digital box $B = \{(j_1, \ldots, j_n) : j_i \leq n+1, i \leq n\}$. Note that $|B| = (n+1)^n$ and call the elements of B by cells. Put in each cell $(j_1, \ldots, j_n)$ of B the sum $x + q_1^{j_1} + \cdots + q_n^{j_n}$.

Fix $i \leq n$ and observe that each interval $I(j_1, \ldots, \widehat{j_i}, \ldots, j_n) = \{(j_1, \ldots, k, \ldots, j_n) : k \leq n+1\}$ of cells contains at most one element of V_i. So the whole box B contains at most $(n+1)^{n-1}$ elements of V_i. Summarizing we have at most $n(n+1)^{n-1}$ elements of U in the box B. Since $(n+1)^n > n(n+1)^{n-1}$ for $n \geq 2$, there are points p in B which are not elements of U. But such p must be elements of the set $Q_x \cap O$ by our choice. The claim is proved.

Let us finish the proof of the lemma. For each equivalence class Q_x choose a point from the set $Q_x \cap (O \setminus U)$. The set of such points is a Vitali Q-selector V of R such that $V \subset O \setminus U$.

THEOREM 4.4.
(a) $\mathscr{V}_2^{\sup} \subset \mathscr{B}_p^C$.
(b) *for each* $A \in \mathscr{V}_2^{\sup}$ *we have* $\dim A = 0$.
(c) *for each* $Q \in \mathscr{F}$ *we have* $\mathscr{V}_2^1(Q) \subset \mathscr{V}_2^{\sup}$.
(d) $\mathscr{V}_2^{\sup}$ *is a semigroup of sets invariant under translations of* R.

PROOF. (a) and (b) follow Lemma 4.1 and Proposition 3.3. (c) and (d) were observed in (i) and (ii) of this section.

REMARK 4.5.
(a) Considering different ideals of sets in the real line R (the ideal of finite sets, the ideal of countable sets, the ideal of closed discrete sets, the ideal of nowhere dense sets, etc) we can produce many different semigroups of sets in $\mathscr{B}_p^C$ by the use of the operation $*$ and the semigroups $\mathscr{V}_1^1(Q)$, $Q \in \mathscr{F}$, and $\mathscr{V}_1^{\sup}$.
(b) Let us note that one can define supersemigroups of sets based on the Vitali sets in R^n, $n \geq 2$, by a similar argument as above.

4.3. A nonmeasurable case

In [5] Kharazishvili proved that each element U of the family $\mathscr{V}_1$ is nonmeasurable in the Lebesgue sense. Let $\mathscr{N}$ be the family of all measurable sets in the Lebesgue sense on the real line R and $\mathscr{N}_0 \subset \mathscr{N}$ be the family of all sets of the Lebesgue measure zero. Recall that the family $\mathscr{N}_0$ is an ideal of sets (in fact, a σ-ideal). It follows from Propositions 3.1 and 3.2 that the families $\mathscr{V}_1$, $\mathscr{N}_0 * \mathscr{V}_1$ and $\mathscr{V}_1 * \mathscr{N}_0$ are three different semigroups of sets invariant under translations of R and $\mathscr{V}_1 \subset \mathscr{N}_0 * \mathscr{V}_1 \subset \mathscr{V}_1 * \mathscr{N}_0$. We have the following generalization of Kharazishvili's result.

PROPOSITION 4.6. *Each element of the family* $\mathscr{V}_1 * \mathscr{N}_0$ *is nonmeasurable in the Lebesgue sense.*

PROOF. In fact, let $A \in \mathscr{V}_1 * \mathscr{N}_0$ and assume that $A \in \mathscr{N}$. By Proposition 3.2 there are an $U \in \mathscr{V}_1$ and an $N \in \mathscr{N}_0$ such that $A = U \triangle N$. It is known that if A_1, A_2 are sets such that $A_1 \in \mathscr{N}$ and the set $A_1 \triangle A_2$ is of the Lebesgue measure zero then the set A_2 must belong to the family $\mathscr{N}$ (see [1]). But $A \triangle U = (U \triangle N) \triangle U = N$, hence $U \in \mathscr{N}$. This is a contradiction with [5]. So $A \notin \mathscr{N}$.

QUESTION 4.7. Is each element U of the family $\mathscr{V}_1^{\sup}$ nonmeasurable in the Lebesgue sense?

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REFERENCES

1. Capiński, M., and Kopp, E., *Measure, Integral and Probability*. Second edition. Springer Undergraduate Mathematics Series. Springer-Verlag, London, 2004.
2. Chatyrko, V. A., and Nyagahakwa, V., *On the families of sets without the Baire property generated by Vitali ses*, p-Adic Numbers Ultrametric Anal. Appl. 3 (2011), no. 2, 100–107.
3. Engelking, R., *General Topology*, Heldermann, Berlin 1989.
4. Kharazishvili, A. B., *Nonmeasurable Sets and Functions*, Elsevier, Amsterdam 2004.
5. Kharazishvili, A. B., *Measurability properties of Vitali sets*, Amer. Math. Monthly 118 (2011), 693–703
6. Kuratowski, K., *Topology I*, Academic Press, London 1966.
7. Vitali, G., *Sul problema della mesura dei gruppi di punti di una retta*, Bologna 1905.

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