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UNIVALENCE CRITERIA AND LIPSCHITZ-TYPE SPACES ON PLURIHARMONIC MAPPINGS

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Abstract

In this paper, we investigate some properties of pluriharmonic mappings defined in the unit ball. First, we discuss the relationship between the univalence of pluriharmonic mappings and linearly connected domains, and then we study Lipschitz-type spaces for pluriharmonic mappings.

1. Introduction and main results

Let C^n denote the complex Euclidean n-space. For $z = (z_1, \dots, z_n) \in \mathsf{C}^n$, the conjugate of z, denoted by $\overline{z}$, is defined by $\overline{z} = (\overline{z}_1, \dots, \overline{z}_n)$. For z and $w = (w_1, \dots, w_n) \in \mathsf{C}^n$, the standard *Hermitian scalar product* on C^n and the *Euclidean norm* of z are given by $\langle z, w\rangle := \sum_{k=1}^n z_k\overline{w}_k$ and $\|z\| := \langle z, z\rangle^{1/2}$, respectively. For $a \in \mathsf{C}^n$,

$$\mathsf{B}^n(a, r) = \{z \in \mathsf{C}^n : \|z - a\| < r\}$$

is the (open) ball of radius r with center a. Also, we let $\mathsf{B}^n(r) := \mathsf{B}^n(0, r)$ and use B^n to denote the unit ball $\mathsf{B}^n(1)$, and $\mathsf{D} = \mathsf{B}^1$. We can interpret C^n as the real $2n$-space R^{2n} so that a ball in C^n is also a ball in R^{2n}. We use the following standard notations. For $a \in \mathsf{R}^n$, we may let

$$\mathsf{B}^n_{\mathsf{R}}(a, r) = \{x \in \mathsf{R}^n : \|x - a\| < r\}$$

so that $\mathsf{B}^n_{\mathsf{R}}(r) := \mathsf{B}^n_{\mathsf{R}}(0, r)$ and $\mathsf{B}^n_{\mathsf{R}} = \mathsf{B}^n_{\mathsf{R}}(1)$ denotes the open unit ball in R^n centered at the origin.

For a complex-valued and differentiable function f from B^n into C, we introduce (see for instance [4], [5], [6])

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right) \qquad \text{and} \qquad \overline{\nabla} f = \left(\frac{\partial f}{\partial \overline{z}_1}, \dots, \frac{\partial f}{\partial \overline{z}_n}\right).$$

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Throughout, $\mathscr{H}(\mathsf{B}^n, \mathsf{C}^n)$ denotes the set of all continuously differentiable mappings f from B^n into C^n with $f = (f_1, \ldots, f_n)$ and $f_j(z) = u_j(z) + i v_j(z)$ $(1 \leq j \leq n)$, where u_j and v_j are real-valued functions from B^n into R. For $f \in \mathscr{H}(\mathsf{B}^n, \mathsf{C}^n)$, the *real Jacobian matrix* of f is given by

$$J_f = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial y_2} & \cdots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_n} \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial y_1} & \frac{\partial v_n}{\partial x_2} & \frac{\partial v_n}{\partial y_2} & \cdots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_n} \end{pmatrix}.$$

Moreover, for each $f = (f_1, \ldots, f_n) \in \mathscr{H}(\mathsf{B}^n, \mathsf{C}^n)$, denote by

$$Df = \big(\nabla f_1, \ldots, \nabla f_n\big)^T$$

the matrix formed by the complex gradients $\nabla f_1, \ldots, \nabla f_n$, and let

$$\overline{D} f = \big(\overline{\nabla} f_1, \ldots, \overline{\nabla} f_n\big)^T,$$

where T means the matrix transpose.

For an $n \times n$ complex matrix A, we introduce the *operator norm*

$$\|A\| = \sup_{z \neq 0} \frac{\|Az\|}{\|z\|} = \max\left\{\|A\theta\| : \theta \in \partial\mathsf{B}^n\right\}.$$

We use $L(\mathsf{C}^n, \mathsf{C}^m)$ to denote the space of continuous *linear operators* from C^n into C^m with the operator norm, and let I_n be the *identity operator* in $L(\mathsf{C}^n, \mathsf{C}^n)$.

A continuous complex-valued function f defined on a domain $G \subset \mathsf{C}^n$ is said to be *pluriharmonic* if for each fixed $z \in G$ and $\theta \in \partial\mathsf{B}^n$, the function $f(z + \theta\zeta)$ is harmonic in $\{\zeta : \|\zeta\| < d_G(z)\}$, where $d_G(z)$ denotes the distance from z to the boundary ∂G of G. It follows from [15, Theorem 4.4.9] that if G is simply connected, then a real-valued function u defined on G is pluriharmonic if and only if u is the real part of a holomorphic function on G. Clearly, a mapping $f : \mathsf{B}^n \to \mathsf{C}$ is pluriharmonic if and only if f has a representation $f = h + \overline{g}$, where g and h are holomorphic. We refer to [2], [4], [11], [12] for more details on pluriharmonic mappings.

A *vector-valued mapping* $f = (f_1, \ldots, f_n) \in \mathscr{H}(\mathsf{B}^n, \mathsf{C}^n)$ is said to be pluriharmonic, if each component f_j $(1 \leq j \leq n)$ is a pluriharmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}$. We denote by $\mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ the set of all *vector-valued pluriharmonic mappings* from $\mathbb{B}^n$ into $\mathbb{C}^n$. Let $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$, where h and g are holomorphic in $\mathbb{B}^n$. Then the real Jacobian determinant of f can be rewritten in the following form

$$\det J_f = \det \begin{pmatrix} Dh & \overline{Dg} \\ Dg & \overline{Dh} \end{pmatrix}$$

and if h is locally biholomorphic, then the determinant of J_f can be written as follows

$$\det J_f = |\det Dh|^2 \det\bigl(I_n - Dg[Dh]^{-1}\overline{Dg[Dh]^{-1}}\bigr).$$

Recall that the determinant of the Jacobian J_f of a *planar harmonic mapping* $f = h + \overline{g}$ is given by

$$\det J_f = |f_z|^2 - |f_{\overline{z}}|^2 := |h'|^2 - |g'|^2,$$

and so, f is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$; or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property that $|\omega(z)| < 1$ in $\mathbb{D}$ (see [10], [13]). For $f = h + \overline{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$, the condition $\|Dg[Dh]^{-1}\| < 1$ is sufficient for $\det J_f$ to be positive and hence for f to be sense-preserving, which is a natural generalization of one-variable condition $|g'(z)| < |h'(z)|$ (cf. [11]).

A domain $D \subset \mathbb{C}^n$ is said to be *M-linearly connected* if there exists a positive constant $M < \infty$ such that any two points $w_1, w_2 \in D$ are joined by a path $\gamma \subset D$ with

$$\ell(\gamma) \leq M\|w_1 - w_2\|,$$

where $\ell(\gamma) = \inf\bigl\{\int_\gamma \|dz\| : \gamma \subset D\bigr\}$. It is not difficult to verify that a 1-linearly connected domain is convex. For extensive discussions on linearly connected domains, see [1], [3], [8], [14]. In [8], Chuaqui and Hernández discussed the relationship between the linear connectivity of the images $\mathbb{D}$ under the planar harmonic mappings $f = h + \overline{g}$ and under their corresponding holomorphic counterparts h, where h and g are holomorphic in $\mathbb{D}$. We generalize the corresponding results of [8] to the following forms.

THEOREM 1.1. *Let $f = h + \overline{g}$ be a univalent pluriharmonic mapping from $\mathbb{B}^n$ into $\mathbb{C}^n$, where h is locally biholomorphic and g is holomorphic in $\mathbb{B}^n$. If $\Omega = f(\mathbb{B}^n)$ is M-linearly connected and*

$$\|Dg(z)[Dh(z)]^{-1}\| < \frac{1}{M+1} \qquad \textit{for} \quad z \in \mathbb{B}^n,$$

then h is biholomorphic. If furthermore

$$\|Dg(z)[Dh(z)]^{-1}\| \leq C < \frac{1}{M+1} \qquad \textit{for} \quad z \in \mathsf{B}^n$$

and for some positive constant C, then $h(\mathsf{B}^n)$ is M'-linearly connected, where $M' = \frac{M}{1-C(1+M)}$.

We remark that Theorem 1.1 is a generalization of [8, Theorem 2].

THEOREM 1.2. *Let $f = h + \overline{g}$ be a univalent pluriharmonic mapping from B^n into C^n, where h is locally biholomorphic and g is holomorphic in B^n. If $\Omega = f(\mathsf{B}^n)$ is M-linearly connected and*

$$\|Dg(z)[Dh(z)]^{-1}\| < \frac{1}{2M+1} \qquad \textit{for} \quad z \in \mathsf{B}^n,$$

then $F = h + \overline{g}A$ is univalent for every $A \in L(\mathsf{C}^n, \mathsf{C}^n)$ with $\|A\| = 1$.

We remark that if $n = 1$, then Theorem 1.2 coincides with [8, Theorem 3].
The following result is similar to [11, Theorem 6].

THEOREM 1.3. *Let $f = h + \overline{g} \in \mathscr{PH}(\mathsf{B}^n, \mathsf{C}^n)$, where h is biholomorphic and g is holomorphic in B^n. If $h(\mathsf{B}^n)$ is M-linearly connected and*

$$\|Dg(z)[Dh(z)]^{-1}\| \leq C < \frac{1}{M}$$

for some positive constant C, then $F = h + \overline{g}A$ is univalent and $F(\mathsf{B}^n)$ is M'-linearly connected, where $M' = \frac{(1+C)M}{1-MC}$ for every $A \in L(\mathsf{C}^n, \mathsf{C}^n)$ with $\|A\| = 1$.

For $r \in (0, 1)$, a univalent mapping $f = h + \overline{g} \in \mathscr{PH}(\mathsf{B}^n, \mathsf{C}^n)$ with

$$\|Dg[Dh]^{-1}\| < 1$$

is called *fully convex* if it maps every ball $\overline{\mathsf{B}^n(r)}$ onto a convex domain, where h is locally biholomorphic and g is holomorphic in B^n (cf. [7]). The following result is analogous to [9, Corollary 5.8].

THEOREM 1.4. *Let $f = h + \overline{g} \in \mathscr{PH}(\mathsf{B}^n, \mathsf{C}^n)$ be fully convex, where $Dg(0) = 0$. Then for every $r \in (0, 1)$ and any two points $z_1,\ z_2 \in \overline{\mathsf{B}^n(r)}$,*

$$\|h(z_2) - h(z_1)\| \leq \frac{1}{1-r}\|f(z_2) - f(z_1)\|.$$

The proofs of Theorems 1.1, 1.2, 1.3 and 1.4 will be presented in Section 2.

2. The univalence criteria and Lipschitz-type spaces on pluriharmonic mappings

The following lemma plays a key role in the proof of Theorems 1.1, 1.2 and 1.4.

LEMMA 2.1. *Let A be an $n \times n$ complex matrix with $\|A\| < 1$. Then $I_n \pm A$ are nonsingular matrixes and*

$$\|(I_n \pm A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

Lemma 2.1 may be referred to as the Neumann expansion theorem, and so the proof is omitted here.

PROOF OF THEOREM 1.1. Suppose that there are two distinct points $z_1,\ z_2 \in \mathsf{B}^n$ such that $h(z_1) = h(z_2)$. Then

$$f(z_2) - f(z_1) = \overline{g(z_2)} - \overline{g(z_1)}, \qquad \text{i.e.} \quad \overline{w}_2 - \overline{w}_1 = G(w_2) - G(w_1),$$

where $G = g \circ f^{-1}$ and $w = f(z)$. It follows from the inverse mapping theorem and Lemma 2.1 that f^{-1} is differentiable. Differentiation of the equation

$$f^{-1}(f(z)) = z$$

yields the following relations

$$\begin{aligned} Df^{-1}Dh + \overline{D}f^{-1}Dg &= I_n, \\ Df^{-1}\overline{Dg} + \overline{D}f^{-1}\overline{Dh} &= 0, \end{aligned}$$

which give

$$\begin{aligned} Df^{-1} &= [Dh]^{-1}\big(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\big)^{-1}, \\ \overline{D}f^{-1} &= -[Dh]^{-1}\big(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\big)^{-1}\overline{Dg}[\overline{Dh}]^{-1}. \end{aligned} \tag{1}$$

By (1) and Lemma 2.1, we get

$$\begin{aligned} &\|DgDf^{-1}\| + \|Dg\overline{D}f^{-1}\| \\ &\quad = \big\|Dg[Dh]^{-1}\big(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\big)^{-1}\big\| \\ &\qquad + \big\|Dg[Dh]^{-1}\big(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\big)^{-1}\overline{Dg}[\overline{Dh}]^{-1}\big\| \end{aligned}$$

$$
\begin{aligned}
&\le \left\|\left(I_n - \overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right)^{-1}\right\| \|Dg[Dh]^{-1}\| \times (1 + \|Dg[Dh]^{-1}\|)\\
&\le \frac{\|Dg[Dh]^{-1}\|(1 + \|Dg[Dh]^{-1}\|)}{1 - \left\|\overline{Dg}[\overline{Dh}]^{-1}Dg[Dh]^{-1}\right\|}\\
&\le \frac{\|Dg[Dh]^{-1}\|(1 + \|Dg[Dh]^{-1}\|)}{1 - \|Dg[Dh]^{-1}\|^2}\\
&\le \frac{\|Dg[Dh]^{-1}\|}{1 - \|Dg[Dh]^{-1}\|}\\
&< \frac{1}{M}.
\end{aligned}
$$

Let $w = f(z)$ and $\gamma \subset \Omega$ be a curve joining $w_1,\ w_2$ with $l(\gamma) \le M\|w_1 - w_2\|$. Then, using the last inequality, we see that

$$
\begin{aligned}
\|\overline{w}_2 - \overline{w}_1\| &= \|G(w_2) - G(w_1)\|\\
&\le \int_\gamma \left(\|DG(w)\| + \|\overline{D}G(w)\|\right) \|dw\|\\
&= \int_\gamma \left(\|Dg(z)Df^{-1}(w)\| + \|Dg(z)\overline{D}f^{-1}(w)\|\right) \|dw\|\\
&\le \int_\gamma \frac{\|Dg(z)[Dh(z)]^{-1}\|}{1 - \|Dg(z)[Dh(z)]^{-1}\|} \|dw\|\\
&< \frac{1}{M} l(\gamma)\\
&\le \|w_2 - w_1\|,
\end{aligned}
$$

which is a contradiction. Therefore, h is biholomorphic. The proof of the first part of the theorem is complete.

Now, we prove the second part of the theorem. Define

$$
H(w) = h(f^{-1}(w)) = w - \overline{G(w)},
$$

where $G = g \circ f^{-1}$ and $w = f(z)$. Let $\gamma \subset \Omega$ be a curve joining w_1, w_2 with $l(\gamma) \le M\|w_1 - w_2\|$. By the chain rule, we have

$$
DH = I_n - \overline{Dg}\,\overline{\overline{D}f^{-1}}
$$

and

$$
\overline{D}H = -\overline{Dg}\,\overline{Df^{-1}}.
$$

Then

$$\begin{aligned}\|DH\| + \|\overline{D}H\| &= \left\|I_n - \overline{Dg}\,\overline{\overline{D}f^{-1}}\right\| + \left\|\overline{Dg}\,\overline{Df^{-1}}\right\| \\ &\le 1 + \left\|\overline{Dg}\,\overline{\overline{D}f^{-1}}\right\| + \left\|\overline{Dg}\,\overline{Df^{-1}}\right\| \\ &\le 1 + \frac{\|Dg[Dh]^{-1}\|}{1 - \|Dg[Dh]^{-1}\|} \\ &= \frac{1}{1 - \|Dg[Dh]^{-1}\|} \\ &\le \frac{1}{1-C},\end{aligned}$$

which gives

$$\begin{aligned}l(H(\gamma)) &\le \int_\gamma \left(\|DH(w)\| + \|\overline{D}H(w)\|\right)\|dw\| \\ &\le \frac{l(\gamma)}{1-C} \le \frac{M}{1-C}\|w_2 - w_1\|.\end{aligned} \tag{2}$$

On the other hand,

$$\begin{aligned}\|H(w_2) - H(w_1)\| &= \|w_2 - \overline{G(w_2)} - (w_1 - \overline{G(w_1)})\| \\ &\ge \|w_2 - w_1\| - \|G(w_2) - G(w_1)\| \\ &\ge \|w_2 - w_1\| - \int_\gamma \left(\|DG(w)\| + \|\overline{D}G(w)\|\right)\|dw\| \\ &\ge \|w_2 - w_1\| - \frac{C}{1-C} l(\gamma) \\ &\ge \frac{1 - C(1+M)}{1-C}\|w_2 - w_1\| \\ &\ge (1 - C(1+M))\frac{l\big(H(\gamma)\big)}{M},\end{aligned}$$

by (2).

Hence

$$l(H(\gamma)) \le \frac{M}{1 - C(1+M)}\|H(w_2) - H(w_1)\|.$$

The proof of the theorem is complete.

PROOF OF THEOREM 1.2. We rewrite $F = h + \overline{g}A = f + \overline{g}(A - I_n)$. Suppose that there are two distinct points $z_1, z_2 \in \mathsf{B}^n$ such that $F(z_1) = F(z_2)$. Then

$$f(z_2) - f(z_1) = \left(\overline{g(z_2)} - \overline{g(z_1)}\right)(A - I_n),$$

that is,

$$\overline{w}_2 - \overline{w}_1 = (G(w_2) - G(w_1))(\overline{A} - I_n),$$

where $G = g \circ f^{-1}$ and $w = f(z)$. By calculations, we have

$$\begin{aligned}\|w_2 - w_1\| &= \|(G(w_2) - G(w_1))(\overline{A} - I_n)\| \\ &\le \|G(w_2) - G(w_1)\| \|\overline{A} - I_n\| \\ &\le (1 + \|A\|)\|G(w_2) - G(w_1)\| \\ &= 2\|G(w_2) - G(w_1)\|,\end{aligned}$$

which implies

$$\|G(w_2) - G(w_1)\| \ge \frac{\|w_2 - w_1\|}{2}. \tag{3}$$

On the other hand, by using arguments similar to those in the proof of Theorem 1.1, we have

$$\begin{aligned}\|DG(w)\| + \|\overline{D}G(w)\| &\le \|Dg(z)Df^{-1}(w)\| + \|Dg(z)\overline{D}f^{-1}(w)\| \\ &\le \frac{\|Dg(z)[Dh(z)]^{-1}\|}{1 - \|Dg(z)[Dh(z)]^{-1}\|} < \frac{1}{2M},\end{aligned}$$

by assumption.

Let $\gamma \subset \Omega$ be a curve joining w_1, w_2 with $l(\gamma) \le M\|w_1 - w_2\|$. Then

$$\begin{aligned}\|G(w_2) - G(w_1)\| &\le \int_\gamma \big(\|DG(w)\| + \|\overline{D}G(w)\|\big)\, \|dw\| \\ &< \frac{l(\gamma)}{2M} \le \frac{\|w_2 - w_1\|}{2}\end{aligned}$$

which is a contradiction to (3). Therefore, F is univalent. The proof of the theorem is complete.

PROOF OF THEOREM 1.3. First, we prove $F(\mathsf{B}^n)$ is linearly connected. Define $\Omega = h(\mathsf{B}^n)$ and

$$H(w) = w + \overline{g(h^{-1}(w))}A$$

for $w \in \Omega$. For any two distinct points $w_1,\ w_2 \in \Omega$, by hypothesis, there is a curve $\gamma \subset \Omega$ joining w_1 and w_2 such that $l(\gamma) \le M\|w_1 - w_2\|$. Also, we let

$\Gamma = H(\gamma)$. Then we find that

$$\begin{aligned} l(\Gamma) &= \int_\Gamma \|dH(w)\| \\ &\le \int_\gamma \left(\|DH(w)\| + \|\overline{D}H(w)\|\right)\|dw\| \\ &\le \int_\gamma \left(\|I_n\| + \|\overline{Dg(z)[Dh(z)]^{-1}}\|\,\|A\|\right)\|dw\| \\ &\le (1+C)M\|w_2 - w_1\|. \qquad (4) \end{aligned}$$

On the other hand, the definition of H gives

$$\begin{aligned} \|H(w_2) - H(w_1)\| &\ge \|w_2 - w_1\| - \|g(h^{-1}(w_2)) - g(h^{-1}(w_1))\| \qquad (5) \\ &\ge \|w_2 - w_1\| - \int_\gamma \|Dg(z)[Dh(z)]^{-1}\|\,\|dw\| \\ &\ge \|w_2 - w_1\|(1 - MC), \end{aligned}$$

and therefore, (4) gives

$$l(\Gamma) \le M'\|H(w_2) - H(w_1)\|,$$

where $M' = \frac{(1+C)M}{1-MC}$.

Finally, we show the univalency of $F = h + \overline{g}A$ for every $A \in L(\mathsf{C}^n, \mathsf{C}^n)$. Suppose that F fails to be univalent. Then there are two distinct points $w_1,\ w_2$ such that $H(w_1) = H(w_2)$ which is impossible, by (5). The proof of the theorem is complete.

PROOF OF THEOREM 1.4. Since $\Omega = f(\overline{\mathsf{B}^n(r)})$ is convex, for any two points $z_1,\ z_2 \in \overline{\mathsf{B}^n(r)}$ and $t \in [0, 1]$, we have

$$\varphi(t) = (f(z_2) - f(z_1))t + f(z_1) \in \Omega,$$

where $f = (f_1, \ldots, f_n)$. Let $\gamma = f^{-1} \circ \varphi$. For any fixed $\theta \in \partial\mathsf{B}^n$, let $A_\theta = Dg[Dh]^{-1}\theta$. By Schwarz's lemma, for $z \in \mathsf{B}^n(r)$, $\|A_\theta(z)\| < \|z\|$ if $r \in (0, 1)$. The arbitrariness of $\theta \in \partial\mathsf{B}^n$ gives

$$\|Dg(z)[Dh(z)]^{-1}\| < r \qquad (6)$$

for $z \in \mathsf{B}^n(r)$. By (1), (6) and Lemma 2.1, we have

$$
\begin{aligned}
&\|h(z_2) - h(z_1)\| \\
&= \left\| \int_\gamma Dh(z)\, dz \right\| = \left\| \int_0^1 Dh(\gamma(t)) \frac{d}{dt}\gamma(t)\, dt \right\| \\
&= \left\| \int_0^1 Dh(\gamma(t)) \big[Df^{-1}(\varphi(t)) D\varphi(t) + \overline{D} f^{-1}(\varphi(t)) \overline{D\varphi(t)} \,\big]\, dt \right\| \\
&\leq \int_0^1 \big(\|Dh(\gamma(t)) Df^{-1}(\varphi(t))\| + \|Dh(\gamma(t)) \overline{D} f^{-1}(\varphi(t))\| \big) \|D\varphi(t)\|\, dt \\
&\leq \|f(z_2) - f(z_1)\| \int_0^1 \big(1 + \|Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\| \big) \\
&\qquad \times \left\| I_n - \overline{Dg(\gamma(t))} \big[\overline{Dh(\gamma(t))} \big]^{-1} Dg(\gamma(t)) [Dh(\gamma(t))]^{-1} \right\| dt \\
&\leq \int_0^1 \frac{1 + \|Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\|}{1 - \left\| \overline{Dg(\gamma(t))} [\overline{Dh(\gamma(t))}]^{-1} Dg(\gamma(t)) [Dh(\gamma(t))]^{-1} \right\|}\, dt \\
&\qquad \times \|f(z_2) - f(z_1)\| \\
&\leq \|f(z_2) - f(z_1)\| \int_0^1 \frac{1}{1 - \|Dg(\gamma(t))[Dh(\gamma(t))]^{-1}\|}\, dt \\
&\leq \frac{1}{1-r} \|f(z_2) - f(z_1)\|,
\end{aligned}
$$

where $D\varphi(t) = \begin{pmatrix} \phi_1(z_2 - z_1) & 0 & \cdots & 0 \\ 0 & \phi_2(z_2 - z_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n(z_2 - z_1) \end{pmatrix}$ is a diagonal matrix, where for convenience we write $\phi_n(z_2 - z_1) = f_n(z_2) - f_n(z_1)$. The proof of this theorem is complete.

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