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COACTIONS ON CUNTZ-PIMSNER ALGEBRAS

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Abstract

We investigate how a correspondence coaction gives rise to a coaction on the associated Cuntz-Pimsner algebra. We apply this to recover a recent result of Hao and Ng concerning Cuntz-Pimsner algebras of crossed products of correspondences by actions of amenable groups.

1. Introduction

The Cuntz-Pimsner algebra $\mathcal{O}_X$ associated to a C^*-correspondence X is a C^*-algebra whose representations encode the Cuntz-Pimsner covariant representations of X. These were introduced by Pimsner in [13], and generalize both crossed products by Z and graph algebras when the underlying graph has no sources. Further work by Katsura in [9] has expanded the class of Cuntz-Pimsner algebras to include graph algebras of arbitrary graphs, crossed products by partial automorphisms and topological graph algebras.

As in the cases of the above mentioned C^*-algebras, it is fruitful to investigate how C^*-constructions involving $\mathcal{O}_X$ can be studied in terms of corresponding constructions involving X. For example, it has been understood for some time how actions of groups on $\mathcal{O}_X$ can be studied in terms of actions on X, see [5] for example. In this paper we show how coactions of a locally compact group G on $\mathcal{O}_X$ can be studied in terms of suitable coactions of G on X.

In order to say what "suitable" should mean, we appeal to [7], where we showed that the passage from X to $\mathcal{O}_X$ is functorial for certain categories. Specifically, the target category is C^*-algebras and nondegenerate homomorphisms into multiplier algebras, and the domain category is correspondence and *Cuntz-Pimsner covariant homomorphisms* (defined in [7]). To see how this should be applied, note that a coaction of G on $\mathcal{O}_X$ is a nondegenerate homomorphism $\zeta : \mathcal{O}_X \to M(\mathcal{O}_X \otimes C^*(G))$ satisfying appropriate conditions, and similarly a coaction of G on X (as defined in [4]) is a homomorphism $\sigma : X \to M(X \otimes C^*(G))$. In order to apply the techniques from [7], we want ζ to be determined by σ. If we knew that $\mathcal{O}_X \otimes C^*(G)$ were equal to

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$\mathcal{O}_{X\otimes C^*(G)}$, the Cuntz-Pimsner algebra of the external-tensor-product correspondence, then the main result of [7] would tell us that we should require the correspondence homomorphism σ to be *Cuntz-Pimsner covariant* in the sense defined there. As it happens, due to the nonexactness of minimal C^*-tensor products, we need a slightly stronger version of Cuntz-Pimsner covariance, specifically suited for correspondence coactions. We work this out in an abstract setting toward the end of Section 2, then we use this to prove our main result concerning coactions on Cuntz-Pimsner algebras at the start of Section 3, after which we go on to develop a few tools dealing with inner coactions on correspondences.

In Section 4 we show how to recognize covariant representations of the coaction ζ on $\mathcal{O}_X$ using the coaction σ on X. In Theorem 4.4 we show that under a mild technical condition the crossed product $\mathcal{O}_X \rtimes_\zeta G$ is isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_{X\rtimes_\sigma G}$ of the crossed-product correspondence. We list in Lemma 4.6 a couple of situations in which the technical condition is guaranteed to hold. We also show that, as in the C^*-case, the crossed product of X by an inner coaction is isomorphic to the tensor product $X \otimes C_0(G)$, and that if G is amenable and acts on X then the dual coaction on the crossed product $X \rtimes G$ satisfies our stronger version of Cuntz-Pimsner covariance. For all we know the amenability hypothesis in the latter result is unnecessary, but anyway we will apply this in Section 5 to recover a recent result of Hao and Ng [5]; they show that if G acts on X then $\mathcal{O}_X \rtimes G \cong \mathcal{O}_{X\rtimes G}$, and we give a substantially different proof using the techniques of the present paper.

2. Preliminaries

We are mainly interested in correspondences over a single coefficient C^*-algebra, but occasionally we will find it convenient to allow the left and right coefficient C^*-algebras to be different. We denote an $A - B$ correspondence X by (A, X, B) and write $\varphi_A : A \to \mathcal{L}(X)$ for the left action of A on X. If $A = B$ we denote the A-correspondence X by (X, A). All correspondences will be assumed *nondegenerate* in the sense that $A \cdot X = X$.

We record here the notation and results that we will need.

The *multiplier correspondence* of a correspondence (A, X, B) is $M(X) := \mathcal{L}_B(B, X)$, which is an $M(A) - M(B)$ correspondence in a natural way. If (A, X, B) and (C, Y, D) are correspondences, a *correspondence homomorphism* $(\pi, \psi, \rho) : (A, X, B) \to (M(C), M(Y), M(D))$ comprises homomorphisms $\pi : A \to M(C)$ and $\rho : B \to M(D)$ and a linear map $\psi : X \to M(Y)$ preserving the correspondence operations. The homomorphism (π, ψ, ρ) is *nondegenerate* if $\overline{\text{span}}\{\psi(X) \cdot D\} = Y$ and both π and ρ are nondegenerate, and then there is a unique strictly continuous extension $(\overline{\pi}, \overline{\psi}, \overline{\rho}) : (M(A), M(X), M(B)) \to (M(C), M(Y), M(D))$, and also a unique nondegenerate homomorphism $\psi^{(1)} : \mathcal{K}(X) \to \mathcal{L}(Y)$ such that $(\psi^{(1)}, \psi, \rho) : (\mathcal{K}(X), X, B) \to (\mathcal{L}(Y), M(Y), M(D))$ is a nondegenerate correspondence homomorphism. The diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & M(B) \\ {\scriptstyle \varphi_A}\downarrow & & \downarrow{\scriptstyle \overline{\varphi_B}} \\ \mathcal{L}(X) & \xrightarrow[\overline{\psi^{(1)}}]{} & \mathcal{L}(Y) \end{array} \tag{2.1}$$

commutes, and $\psi^{(1)}$ is determined by

$$\psi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*.$$

If $A = B, C = D$, and $\pi = \rho$, we write $(\psi, \pi) : (X, A) \to (M(Y), M(C))$.

We refer to [7, Section 2] for an exposition of the properties of the "relative multipliers" from [3, Appendix A]. Very briefly, if (X, A) is a nondegenerate correspondence and $\kappa : C \to M(A)$ is a nondegenerate homomorphism, the C-*multipliers* of X are

$$M_C(X) := \{m \in M(X) : \kappa(C) \cdot m \cup m \cdot \kappa(C) \subset X\}.$$

The main purpose of relative multipliers is the following extension theorem [3, Proposition A.11]: let X and Y be nondegenerate correspondences over A and B, respectively, let $\kappa : C \to M(A)$ and $\sigma : D \to M(B)$ be nondegenerate homomorphisms. If there is a nondegenerate homomorphism $\lambda : C \to M(\sigma(D))$ such that

$$\pi(\kappa(c)a) = \lambda(c)\pi(a) \qquad \text{for} \quad c \in C, a \in A,$$

then for any correspondence homomorphism $(\psi, \pi) : (X, A) \to (M_D(Y), M_D(B))$ there is a unique C-strict to D-strictly continuous correspondence homomorphism $(\overline{\psi}, \overline{\pi})$ making the diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{(\psi,\pi)} & (M_D(Y), M_D(B)) \\ \big\downarrow & \overset{!}{\nearrow}{\scriptstyle (\overline{\psi},\overline{\pi})} & \\ (M_C(X), M_C(A)) & & \end{array}$$

commute.

We will also need to use the method of [7] to construct homomorphisms of Cuntz-Pimsner algebras from correspondence homomorphisms. Following Katsura [8], we define an ideal J_X of A by

$$J_X := \{a \in A : \varphi_A(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \ker \varphi_A\}.$$

For a C^*-correspondence (X, A), we denote the associated Cuntz-Pimsner algebra by $\mathcal{O}_X$, and universal covariant representation by $(k_X, k_A) : (X, A) \to \mathcal{O}_X$; see [7] for details.

We say a homomorphism $(\psi, \pi) : (X, A) \to (M(Y), M(B))$ is *Cuntz-Pimsner covariant* if

(i) $\psi(X) \subset M_B(Y)$,

(ii) $\pi : A \to M(B)$ is nondegenerate,

(iii) $\pi(J_X) \subset M(B; J_Y)$, and

(iv) the diagram

$$\begin{array}{ccc} J_X & \xrightarrow{\pi|} & M(B; J_Y) \\ {\scriptstyle \varphi_A|}\downarrow & & \downarrow{\scriptstyle \overline{\varphi_B}|} \\ \mathcal{K}(X) & \xrightarrow[\psi^{(1)}]{} & M_B(\mathcal{K}(Y)) \end{array} \tag{2.2}$$

commutes,

where, for an ideal I of a C^*-algebra A, we follow [1] by defining

$$M(A; I) = \{m \in M(A) : mA \cup Am \subset I\}.$$

By [7, Corollary 3.6], when (ψ, π) is Cuntz-Pimsner covariant there is a unique homomorphism $\mathcal{O}_{\psi,\pi}$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{(\psi,\pi)} & M_B(Y) \\ {\scriptstyle k_X}\downarrow & & \downarrow{\scriptstyle \overline{k_Y}} \\ \mathcal{O}_X & \xrightarrow[\mathcal{O}_{\psi,\pi}]{} & M_B(\mathcal{O}_Y) \end{array}$$

commute.

If G is a locally compact group and (X, A) is a correspondence we will write

$$M_{C^*(G)}(A \otimes C^*(G)) = M_{1 \otimes C^*(G)}(A \otimes C^*(G))$$
$$M_{C^*(G)}(X \otimes C^*(G)) = M_{1 \otimes C^*(G)}(X \otimes C^*(G)).$$

Recall that a *coaction* of G on a C^*-algebra A is a nondegenerate injective homomorphism $\delta : A \to M(A \otimes C^*(G))$ satisfying the *coaction identity* given by the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\delta} & M(A \otimes C^*(G)) \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \overline{\delta \otimes \mathrm{id}}} \\ M(A \otimes C^*(G)) & \xrightarrow[\overline{\mathrm{id} \otimes \delta_G}]{} & M(A \otimes C^*(G) \otimes C^*(G)), \end{array} \tag{2.3}$$

and satisfying the *coaction-nondegeneracy* condition

$$\overline{\operatorname{span}}\{\delta(A)(1 \otimes C^*(G))\} = A \otimes C^*(G).$$

REMARKS 2.1. (1) Note that, as has become customary in recent years, we have built coaction-nondegeneracy into the definition of coaction, and of course it follows that $\delta(A) \subset M_{C^*(G)}(A \otimes C^*(G))$.

(2) The coaction identity requires δ to be nondegenerate as a homomorphism, so that it extends uniquely to multipliers. However, if we know that $\delta(A) \subset M_{C^*(G)}(A \otimes C^*(G))$, then, even without knowing δ is nondegenerate, the coaction identity makes sense when the upper right and lower left corners of the commutative diagram (2.3) are replaced by $M_{C^*(G)}(A \otimes C^*(G))$.

(3) Coaction-nondegeneracy implies nondegeneracy as a homomorphism. However, an under-appreciated result of Katayama [8, Lemma 4], implies that, assuming we know δ satisfies all the other coaction axioms except for coaction-nondegeneracy, the closed span of the products $\delta(A)(1 \otimes C^*(G))$ is actually a C^*-subalgebra of $A \otimes C^*(G)$, and hence to show coaction-nondegeneracy it suffices to verify the seemingly weaker condition

(2.4) $\delta(A)(1 \otimes C^*(G))$ generates $A \otimes C^*(G)$ as a C^*-algebra.

A nondegenerate homomorphism $\mu : C_0(G) \to M(A)$ implements an *inner coaction* δ^μ on A via

$$\delta^\mu(a) = \operatorname{Ad} \overline{\mu \otimes \mathrm{id}}(w_G)(a \otimes 1),$$

where

$$w_G \in M(C_0(G) \otimes C^*(G)) = C_b(G, M(C^*(G)))$$

is the (strictly continuous) function given by the canonical embedding of G into the unitary group of $M(C^*(G))$. The *trivial coaction* $\delta^1 = \mathrm{id}_A \otimes 1$ on A is implemented by the homomorphism

$$f \mapsto f(e)1_{M(A)} \qquad \text{for} \quad f \in C_0(G).$$

A coaction (A, δ) makes A into a Banach module over the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$ via

$$f \cdot a = S_f \circ \delta(a) \qquad \text{for} \quad f \in B(G), a \in A,$$

where $S_f : A \otimes C^*(G) \to A$ is the slice map, which we sometimes alternatively denote by $\mathrm{id} \otimes f$. Frequently we restrict the module action to the Fourier algebra $A(G)$, which is dense in $C_0(G)$.

The *Kronecker product* (see, e.g., [2, Théorème 1.5], [10, p. 118], [12, Definition A.2], [14, Definition 6.6]) of two nondegenerate homomorphisms μ and ν of $C_0(G)$ in $M(A)$ and $M(B)$, respectively, is defined by

$$\mu \times \nu := \overline{\mu \otimes \nu} \circ \alpha,$$

where $\alpha : C_0(G) \to C_b(G \times G) = M(C_0(G) \otimes C_0(G))$ is given by

$$\alpha(f)(s,t) = f(st).$$

Letting

$$u = \overline{\mu \otimes \mathrm{id}}(w_G) \qquad \text{and} \qquad v = \overline{\nu \otimes \mathrm{id}}(w_G),$$

we have

$$\overline{(\mu \times \nu) \otimes \mathrm{id}}(w_G) = u_{13}v_{23}.$$

A *covariant homomorphism* of a coaction (A, δ) is a pair $(\pi, \mu) : (A, C_0(G)) \to M(B)$ comprising nondegenerate homomorphisms $\pi : A \to M(B)$ and $\mu : C_0(G) \to M(B)$ such that

$$\overline{\pi \otimes \mathrm{id}} \circ \delta(a) = \mathrm{Ad}\, \overline{\mu \otimes \mathrm{id}}(w_G)(\pi(a) \otimes 1).$$

A *crossed product* of (A, δ) is a triple $(A \rtimes_\delta G, j_A, j_G)$ consisting of a covariant homomorphism $(j_A, j_G) : (A, C_0(G)) \to M(A \rtimes_\delta G)$ that is *universal* in the sense that for every covariant homomorphism $(\pi, \mu) : (A, C_0(G)) \to M(B)$ there is a unique nondegenerate homomorphism $\pi \times \mu : A \rtimes_\delta G \to M(B)$ making the diagram

$$\begin{array}{ccccc} A & \xrightarrow{j_A} & M(A \rtimes_\delta G) & \xleftarrow{j_G} & C_0(G) \\ & \searrow^{\pi} & \downarrow{\scriptstyle !\,\pi\times\mu} & \swarrow_{\mu} & \\ & & M(B) & & \end{array}$$

commute. It follows that $A \rtimes_\delta G = \overline{\mathrm{span}}\{j(A)j_G(C_0(G))$. The crossed product is unique up to isomorphism, and one construction is given by the *regular representation*

$$((\mathrm{id} \otimes \lambda) \circ \delta, 1 \otimes M) : (A, C_0(G)) \to M(A \otimes \mathcal{K}(L^2(G))),$$

where λ is the left regular representation of G and $M : C_0(G) \to B(L^2(G))$ is the multiplication representation.

For correspondence coactions, we follow [4], but again build in coaction-nondegeneracy:

DEFINITION 2.2. A *coaction* of G on a correspondence (A, X, B) is a nondegenerate correspondence homomorphism

$$(\delta, \sigma, \varepsilon) : (A, X, B) \to \big(M(A \otimes C^*(G)), M(X \otimes C^*(G)), M(B \otimes C^*(G))\big)$$

such that:

(i) δ and ε are coactions on A and B, respectively;

(ii) σ satisfies the coaction identity given by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\sigma\quad} & M(X \otimes C^*(G)) \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\sigma\otimes\mathrm{id}}} \\ M(X \otimes C^*(G)) & \xrightarrow[\overline{\mathrm{id}\otimes\delta_G}]{} & M(X \otimes C^*(G) \otimes C^*(G)); \end{array}$$

(iii) σ satisfies the coaction-nondegeneracy condition

$$\overline{\operatorname{span}}\{(1 \otimes C^*(G)) \cdot \sigma(X)\} = \overline{\operatorname{span}}\{\sigma(X) \cdot (1 \otimes C^*(G))\} = X \otimes C^*(G).$$

We also say that σ is $\delta - \varepsilon$ *compatible*.

REMARKS 2.3. (1) Remarks similar to those following the definition of C^*-coaction apply to correspondence coactions. For example, coaction-nondegeneracy implies that $\sigma(X) \subset M_{C^*(G)}(X \otimes C^*(G))$ and σ is nondegenerate as a correspondence homomorphism. In fact, it implies a stronger form of nondegeneracy, namely that, in addition to $\overline{\operatorname{span}}\{\sigma(X) \cdot (X \otimes C^*(G))\} = X \otimes C^*(G)$, we also have the symmetric property on the other side:

$$\overline{\operatorname{span}}\{(X \otimes C^*(G)) \cdot \sigma(X)\} = X \otimes C^*(G).$$

(2) On the other hand, nondegeneracy of σ as a correspondence homomorphism implies one half of the coaction-nondegeneracy, namely $\overline{\operatorname{span}}\{\sigma(X) \cdot (1 \otimes C^*(G))\} = X \otimes C^*(G)$, by coaction-nondegeneracy of ε.

(3) σ will be isometric since ε is injective.

Frequently we will have $A = B$ and $\delta = \varepsilon$, in which case we say that (σ, δ) is a coaction on (X, A); of course the case $X = A = B$ and $\sigma = \delta = \varepsilon$ reduces to a C^*-coaction. Being particularly nice correspondence homomorphisms, coactions on C^*-correspondences are easily shown to be Cuntz-Pimsner covariant:

LEMMA 2.4. *A coaction* (σ, δ) *of* G *on a correspondence* (X, A) *is Cuntz-Pimsner covariant as a correspondence homomorphism if and only if*

$$\delta(J_X) \subset M(A \otimes C^*(G); J_{X \otimes C^*(G)}).$$

PROOF. By definition of correspondence coaction, the correspondence homomorphism $(\sigma, \delta) : (X, A) \to (M(X \otimes C^*(G)), M(A \otimes C^*(G)))$ is nondegenerate, and the inclusion $\sigma(X) \subset M_{C^*(G)}(X \otimes C^*(G))$ trivially implies that $\sigma(X) \subset M_{A\otimes C^*(G)}(X \otimes C^*(G))$. Combining with [7, Lemma 3.2] gives the result.

However, as a consequence of the nonexactness of minimal C^*-tensor products, we will need a variation on Lemma 2.4, and we state it in abstract form, not involving coactions:

LEMMA 2.5. *Let (X, A) be a correspondence, let C be a C^*-algebra, and let $(\psi, \pi) : (X, A) \to (M_C(X \otimes C), M_C(A \otimes C))$ be a nondegenerate correspondence homomorphism. If*

$$\pi(J_X) \subset M(A \otimes C; J_X \otimes C),$$

then the composition

$$\big(\overline{k_X \otimes \mathrm{id}} \circ \psi, \overline{k_A \otimes \mathrm{id}} \circ \pi\big) : (X, A) \to M(\mathcal{O}_X \otimes C)$$

is Cuntz-Pimsner covariant.

PROOF. By checking on elementary tensors one verifies that, on the ideal $J_X \otimes C$ of $A \otimes C$, we have

$$\begin{aligned}(k_X \otimes \mathrm{id})^{(1)} \circ \varphi_{A\otimes C} &= (k_X^{(1)} \otimes \mathrm{id}) \circ (\varphi_A \otimes \mathrm{id}) \\ &= k_X^{(1)} \circ \varphi_A \otimes \mathrm{id} \\ &= k_A \otimes \mathrm{id},\end{aligned}$$

and so, by strict continuity, on $M(A \otimes C; J_X \otimes C)$ we have

$$\overline{(k_X \otimes \mathrm{id})^{(1)}} \circ \overline{\varphi_{A\otimes C}} = \overline{k_A \otimes \mathrm{id}}.$$

Thus, on J_X we have

$$\begin{aligned}\big(\overline{k_X \otimes \mathrm{id}} \circ \psi\big)^{(1)} \circ \varphi_A &= \overline{(k_X \otimes \mathrm{id})^{(1)}} \circ \psi^{(1)} \circ \varphi_A \\ &= \overline{(k_X \otimes \mathrm{id})^{(1)}} \circ \overline{\varphi_{A\otimes C}} \circ \pi,\end{aligned}$$

by [7, Lemma 3.3], since (ψ, π) is nondegenerate,

$$= \overline{k_A \otimes \mathrm{id}} \circ \pi,$$

which is Cuntz-Pimsner covariance.

Here is the connection between Lemmas 2.4 and 2.5:

LEMMA 2.6. *Let (X, A) be a correspondence, let C be a C^*-algebra, and let $(X \otimes C, A \otimes C)$ be the external-tensor-product correspondence. Then*

$$J_X \otimes C \subset J_{X \otimes C},$$

with equality if C is exact.

PROOF. We use the characterization [9, Paragraph following Definition 2.3] of J_X as the largest ideal of A that φ_A maps injectively into $\mathcal{K}(X)$, and similarly for $J_{X\otimes C}$. By [16, Corollary 3.38], for example, we have

$$\mathcal{K}(X \otimes C) = \mathcal{K}(X) \otimes C,$$

so

$$\varphi_{A\otimes C} = \varphi_A \otimes \mathrm{id}_C .$$

Since φ_A maps J_X injectively into $\mathcal{K}(X)$, $\varphi_A \otimes \mathrm{id}$ maps $J_X \otimes C$ injectively into $\mathcal{K}(X) \otimes C$. Therefore $\varphi_{A\otimes C}$ maps $J_X \otimes C$ injectively into $\mathcal{K}(X \otimes C)$, so $J_X \otimes C \subset J_{X\otimes C}$.

Now assume that C is exact, and let $x \in J_{X\otimes C}$. Since C is exact, it has the slice map property, so to show that $x \in J_X \otimes C$ it suffices to show that $(\mathrm{id} \otimes \omega)(x) \in J_X$ for all $\omega \in C^*$. To verify the first property of J_X, we have

$$\varphi_A((\mathrm{id} \otimes \omega)(x)) = (\mathrm{id} \otimes \omega) \circ (\varphi_A \otimes \mathrm{id})(x),$$

which is in $\mathcal{K}(X)$ because

$$(\varphi_A \otimes \mathrm{id})(x) = \varphi_{A\otimes C}(x) \in \mathcal{K}(X \otimes C) = \mathcal{K}(X) \otimes C.$$

For the other property of J_X, let $a \in \ker \phi_A$. Factor $\omega = c \cdot \omega'$ with $c \in C$ and $\omega' \in C^*$. Then

$$\begin{aligned}((\mathrm{id} \otimes \omega)(x))a &= (\mathrm{id} \otimes c \cdot \omega')(x(a \otimes 1)) \\ &= (\mathrm{id} \otimes \omega')(x(a \otimes c)),\end{aligned}$$

which is 0 because

$$a \otimes c \in \ker \varphi_A \otimes C = \ker \varphi_{A\otimes C}.$$

Recall from [4, Proposition 3.9] that if $(\delta, \sigma, \varepsilon)$ is a coaction of G on a correspondence (A, X, B), then the *crossed product* correspondence $(A \rtimes_\delta G, X \rtimes_\sigma G, B \rtimes_\varepsilon G)$ is defined by

$$X \rtimes_\sigma G = \overline{\mathrm{span}}\{j_X(X) \cdot j_G^B(C_0(G))\} \subset M(X \otimes \mathcal{K}(L^2(G))),$$

where

$$j_X = (\mathrm{id} \otimes \lambda) \circ \sigma$$
$$j_G^B = 1_{M(B)} \otimes M.$$

$X \rtimes_\sigma G$ is an $A \rtimes_\delta G - B \rtimes_\varepsilon G$ correspondence in a natural way when we use the regular representations

$$(j_A, j_G^A) : (A, C_0(G)) \to M(A \otimes \mathscr{K}(L^2(G)))$$
$$(j_B, j_G^B) : (B, C_0(G)) \to M(B \otimes \mathscr{K}(L^2(G))).$$

[4, Lemma 3.10] proves that there is a coaction μ of G on $\mathscr{K}(X)$ such that

- $\varphi_A : A \to M(\mathscr{K}(X))$ is $\delta - \mu$ equivariant;
- there is an isomorphism $\mathscr{K}(X \rtimes_\sigma G) \cong \mathscr{K}(X) \rtimes_\mu G$ that carries $\varphi_{A \rtimes_\delta G}$ to $\varphi_A \rtimes G$.

In fact, the an examination of the construction used in [4] reveals that the coaction on $\mathscr{K}(X)$ is none other than

$$\sigma^{(1)} : \mathscr{K}(X) \to M_{C^*(G)}(\mathscr{K}(X) \otimes C^*(G)) = M_{C^*(G)}(\mathscr{K}(X \otimes C^*(G)),$$

so that the left-module action of $A \rtimes_\delta G$ on $X \rtimes_\sigma G$ can be regarded as

$$\varphi_A \rtimes G : A \rtimes_\delta G \to M(\mathscr{K}(X) \rtimes_{\sigma^{(1)}} G).$$

REMARK 2.7. Note that

$$(j_A, j_X, j_B) : (A, X, B) \to \big(M(A \rtimes_\delta G), M(X \rtimes_\sigma G), M(B \rtimes_\varepsilon G)\big)$$

is a correspondence homomorphism. In fact, it is a bit more: since j_A and j_B are nondegenerate by the standard theory of C^*-coactions, it follows from [4, Lemma 3.10] that he correspondence homomorphism (j_A, j_X, j_B) is nondegenerate.

LEMMA 2.8. *Let (σ, δ) be a coaction of G on a correspondence (X, A). Then the canonical correspondence homomorphism $(j_X, j_A) : (X, A) \to (M(X \rtimes_\sigma G), M(A \rtimes_\delta G))$ is Cuntz-Pimsner covariant if and only if*

$$j_A(J_X) \subset M(A \rtimes_\delta G; J_{X \rtimes_\sigma G}).$$

PROOF. By Remark 2.7 and [7, Lemma 3.2], it suffices to observe that

$$j_X(X) \subset M_{A \rtimes_\delta G}(X \rtimes_\sigma G).$$

Although the following concept does not appear in [4], we will find it useful:

DEFINITION 2.9. Let $(\delta, \sigma, \varepsilon)$ be a coaction of G on a correspondence (A, X, B), let $(\pi, \psi, \rho) : (A, X, B) \to (M(D), M(Y), M(E))$ be a correspondence homomorphism, and let $\mu : C_0(G) \to M(D)$ and $\nu : C_0(G) \to M(E)$ be homomorphisms. Then $(\pi, \psi, \rho, \mu, \nu)$ is *covariant for* $(\delta, \sigma, \varepsilon)$ if

(i) (π, μ) and (ρ, ν) are covariant for (A, δ) and (B, ε), respectively;

(ii) for all $\xi \in X$ we have

$$\overline{\psi \otimes \mathrm{id}} \circ \sigma(\xi) = \overline{\mu \otimes \mathrm{id}}(w_G) \cdot (\psi(\xi) \otimes 1) \cdot \overline{\nu \otimes \mathrm{id}}(w_G)^*.$$

REMARK 2.10. Note that covariance of (π, μ) and (ρ, ν) entails that π, μ, ρ, ν are all nondegenerate.

If $A = B$, $\delta = \varepsilon$, $\pi = \rho$, and $\mu = \nu$, we say (ψ, π, μ) is covariant for (σ, δ).

3. Coactions on Cuntz-Pimsner algebras

PROPOSITION 3.1. *Let (σ, δ) be a coaction of G on a correspondence (X, A). If*

$$\delta(J_X) \subset M(A \otimes C^*(G); J_X \otimes C^*(G)),$$

then there is a unique coaction ζ of G on $\mathcal{O}_X$ making the diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{(\sigma,\delta)} & (M_{C^*(G)}(X \otimes C^*(G)), M_{C^*(G)}(A \otimes C^*(G))) \\ {\scriptstyle (k_X,k_A)}\downarrow & & \downarrow{\scriptstyle (\overline{k_X\otimes \mathrm{id}},\overline{k_A\otimes \mathrm{id}})} \\ \mathcal{O}_X & \xrightarrow[\zeta]{!} & M_{C^*(G)}(\mathcal{O}_X \otimes C^*(G)) \end{array}$$

commute.

PROOF. By definition of correspondence coaction, the correspondence homomorphism (σ, δ) is nondegenerate, and so, by Lemma 2.5, our hypothesis guarantees that the composition

$$\left(\overline{k_X \otimes \mathrm{id}} \circ \sigma, \overline{k_A \otimes \mathrm{id}} \circ \delta\right)$$

is Cuntz-Pimsner covariant. Thus there is a unique homomorphism ζ making the diagram commute, and moreover ζ is injective because δ is.

For the coaction identity, we have

$$\begin{aligned}
\overline{\zeta \otimes \mathrm{id}} \circ \zeta \circ k_X &= \overline{\zeta \otimes \mathrm{id}} \circ \zeta_X \\
&= \overline{\zeta \otimes \mathrm{id}} \circ \overline{k_X \otimes \mathrm{id}} \circ \sigma \\
&= \overline{\zeta \circ k_X \otimes \mathrm{id}} \circ \sigma \\
&= \overline{\overline{k_X \otimes \mathrm{id}} \circ \sigma \otimes \mathrm{id}} \circ \sigma \\
&= \overline{k_X \otimes \mathrm{id} \otimes \mathrm{id}} \circ \overline{\sigma \otimes \mathrm{id}} \circ \sigma \\
&= \overline{k_X \otimes \mathrm{id} \otimes \mathrm{id}} \circ \overline{\mathrm{id} \otimes \zeta_G} \circ \sigma \\
&= \overline{\mathrm{id} \otimes \zeta_G} \circ \overline{k_X \otimes \mathrm{id}} \circ \sigma \\
&= \overline{\mathrm{id} \otimes \zeta_G} \circ \zeta \circ k_X,
\end{aligned}$$

and similarly

$$\overline{\zeta \otimes \mathrm{id}} \circ \zeta \circ k_A = \overline{\mathrm{id} \otimes \zeta_G} \circ \zeta \circ k_A,$$

and it follows that

$$\overline{\zeta \otimes \mathrm{id}} \circ \zeta = \overline{\mathrm{id} \otimes \zeta_G} \circ \zeta.$$

For the coaction-nondegeneracy, routine computations show that

$$\overline{\operatorname{span}}\{\zeta_X(X)(1 \otimes C^*(G))\} = k_X(X) \otimes C^*(G),$$

and of course

$$\overline{\operatorname{span}}\{\zeta_A(A)(1 \otimes C^*(G))\} = k_A(A) \otimes C^*(G),$$

and hence the property (2.4) holds.

We now develop a few tools involving inner coactions on correspondences, for use elsewhere.

PROPOSITION 3.2. *Let X be an $A-B$ correspondence, and let $\mu : C_0(G) \to M(A)$ and $\nu : C_0(G) \to M(B)$ be nondegenerate homomorphisms, and let δ^μ and δ^ν be the associated inner coactions on A and B. Then there is a $\delta^\mu - \delta^\nu$ compatible coaction σ on X given by*

$$\sigma(\xi) = \overline{\mu \otimes \mathrm{id}}(w_G) \cdot (\xi \otimes 1) \cdot \overline{\nu \otimes \mathrm{id}}(w_G)^*.$$

PROOF. Write

$$u = \overline{\mu \otimes \mathrm{id}}(w_G) \qquad \text{and} \qquad v = \overline{\nu \otimes \mathrm{id}}(w_G).$$

Then $u \in M(A \otimes C^*(G))$, $v \in M(B \otimes C^*(G))$, and

$$X \otimes 1 \subset M(X \otimes C^*(G)),$$

so certainly σ maps into $M(X \otimes C^*(G))$.

To see that $(\delta, \sigma, \varepsilon)$ is a correspondence homomorphism, we compute, for $a \in A$ and $\xi, \eta \in X$:

$$\begin{aligned}\sigma(a \cdot \xi) &= u \cdot (a \cdot \xi \otimes 1) \cdot v^* \\ &= u \cdot \big((a \otimes 1) \cdot (\xi \otimes 1)\big) \cdot v^* \\ &= u(a \otimes 1)u^* u \cdot (\xi \otimes 1) \cdot v^* \\ &= \delta(a) \cdot \sigma(\xi),\end{aligned}$$

and

$$\begin{aligned}\langle \sigma(\xi), \sigma(\eta)\rangle &= \langle u \cdot (\xi \otimes 1) \cdot v^*, u \cdot (\eta \otimes 1) \cdot v^*\rangle \\ &= \langle (\xi \otimes 1) \cdot v^*, (\eta \otimes 1) \cdot v^*\rangle \qquad \text{(because } u \text{ is unitary)} \\ &= v\langle \xi \otimes 1, \eta \otimes 1\rangle v^* \\ &= v(\langle \xi, \eta\rangle \otimes 1)v^* \\ &= \varepsilon(\langle \xi, \eta\rangle).\end{aligned}$$

We show coaction-nondegeneracy:

$$\begin{aligned}&\overline{\text{span}}\{(1 \otimes C^*(G)) \cdot \sigma(X)\} \\ &\quad= \overline{\text{span}}\{(1 \otimes C^*(G))u \cdot (X \otimes 1) \cdot v^*\} \\ &\quad= \overline{\text{span}}\{(1 \otimes C^*(G))u \cdot (\mu(C_0(G) \cdot X \otimes 1) \cdot v^*\} \\ &\quad= \overline{\text{span}}\{(1 \otimes C^*(G))u(\mu(C_0(G)) \otimes 1) \cdot (X \otimes 1) \cdot v^*\} \\ &\quad= \overline{\text{span}}\{(1 \otimes C^*(G))(\mu(C_0(G)) \otimes 1)u \cdot (X \otimes 1) \cdot v^*\} \\ &\qquad\qquad \text{(because } u \in M(\mu(C_0(G)) \otimes C^*(G))) \\ &\quad= \overline{\text{span}}\{(\mu(C_0(G)) \otimes C^*(G))u \cdot (X \otimes 1) \cdot v^*\} \\ &\quad= \overline{\text{span}}\{(\mu(C_0(G)) \otimes C^*(G)) \cdot (X \otimes 1) \cdot v^*\} \\ &\qquad\qquad \text{(because } u \text{ is a unitary multiplier)} \\ &\quad= (X \otimes C^*(G)) \cdot v^* \\ &\quad= (X \otimes C^*(G)),\end{aligned}$$

because v is unitary, and similarly

$$\overline{\text{span}}\{\sigma(X) \cdot (1 \otimes C^*(G))\} = X \otimes C^*(G).$$

This also implies that σ is nondegenerate as a correspondence homomorphism.

For the coaction identity, we have

$$\begin{aligned}\overline{\sigma \otimes \mathrm{id}} \circ \sigma(\xi) &= u_{12} \cdot \sigma(\xi)_{13} \cdot v_{12}^* \\ &= u_{12}u_{13} \cdot (\xi \otimes 1 \otimes 1) \cdot v_{13}^* v_{12}^* \\ &= \overline{\mathrm{id} \otimes \delta_G}(u) \cdot \overline{\mathrm{id} \otimes \delta_G}(\xi \otimes 1) \cdot \overline{\mathrm{id} \otimes \delta_G}(v)^* \\ &= \overline{\mathrm{id} \otimes \delta_G} \circ \sigma(\xi),\end{aligned}$$

where the third equality expresses the fact that u and v are "corepresentations" of $C_0(G)$, and where the first equality follows from linearity, density, strict continuity, and the following computation with an elementary tensor $\eta \otimes c \in X \odot C^*(G)$:

$$\begin{aligned}\overline{\sigma \otimes \mathrm{id}}(\eta \otimes c) &= \sigma(\eta) \otimes c \\ &= u \cdot (\eta \otimes 1) \cdot v^* \otimes c \\ &= (u \otimes 1) \cdot (\eta \otimes 1 \otimes c) \cdot (v \otimes 1)^* \\ &= u_{12} \cdot (\eta \otimes c)_{13} \cdot v_{12}^*.\end{aligned}$$

DEFINITION 3.3. In the situation of Proposition 3.2, we call the coaction σ on X *inner*, and say that it is *implemented* by the pair (μ, ν).

COROLLARY 3.4. *Let (X, A) be a correspondence, let δ be a coaction of G on A, and let $\mu : C_0(G) \to \mathscr{L}(X)$ be a nondegenerate representation such that the pair (φ_A, μ) is a covariant representation of the coaction (A, δ). Define a unitary*

$$u = \overline{\mu \otimes \mathrm{id}}(w_G) \in \mathscr{L}(X \otimes C^*(G)).$$

Then there is an $\delta - \delta^1$ compatible coaction σ on X given by

$$\sigma(\xi) = u \cdot (\xi \otimes 1).$$

PROOF. Temporarily regard X as a $\mathscr{K}(X) - A$ correspondence. Letting δ^μ be the inner coaction on $\mathscr{K}(X)$ implemented by μ, by Proposition 3.2 the formula for σ defines a $\delta^\mu - \delta^1$ compatible coaction on X. Since (φ_A, μ) is covariant for (A, δ), it follows that σ is also $\delta - \delta^1$ compatible.

COROLLARY 3.5. *Let (X, A) be a correspondence, and let $\mu : C_0(G) \to \mathscr{L}(X)$ be a nondegenerate representation commuting with φ_A. Then there is a coaction ζ of G on $\mathcal{O}_X$ such that for $\xi \in X$ and $a \in A$ we have*

$$\begin{aligned}\zeta \circ k_X(\xi) &= \overline{k_X \otimes \mathrm{id}}\big(\overline{\mu \otimes 1}(w_G) \cdot (\xi \otimes 1)\big) \\ \zeta \circ k_A(a) &= k_A(a) \otimes 1.\end{aligned}$$

PROOF. Since μ commutes with φ_A, the hypotheses of Corollary 3.4 are satisfied when δ is taken to be the trivial coaction δ^1, and we let σ be the resulting $\delta^1 - \delta^1$ compatible coaction on X. Then Proposition 3.1 gives a suitable coaction ζ of G on $\mathcal{O}_X$, because the trivial coaction δ^1 maps J_X into

$$J_X \otimes 1 \subset M(A \otimes C^*(G); J_X \otimes C^*(G)).$$

4. Crossed products

LEMMA 4.1. *Let* (σ, δ) *be a coaction of* G *on a correspondence* (X, A) *such that* $\delta(J_X) \subset M(A \otimes C^*(G); J_X \otimes C^*(G))$, *and let* $(\psi, \pi, \mu) : (X, A, C_0(G)) \to M(B)$ *be a* (σ, δ)*-covariant homomorphism, with* (ψ, π) *Cuntz-Pimsner covariant. Then the pair*

$$(\psi \times \pi, \mu) : (\mathcal{O}_X, C_0(G)) \to M(B)$$

is covariant for the associated coaction ζ *of* G *on* $\mathcal{O}_X$.

PROOF. π and μ are nondegenerate, hence so is $\psi \times \pi$. Let $u = \overline{\mu \otimes \mathrm{id}}(w_G)$. We must show that for $x \in \mathcal{O}_X$ we have

$$\overline{(\psi \times \pi) \otimes \mathrm{id}} \circ \zeta(x) = \mathrm{Ad}\, u\big((\psi \times \pi)(x) \otimes 1\big),$$

and it suffices to show this on generators $k_X(\xi)$ and $k_A(a)$ for $\xi \in X$ and $a \in A$. For for the first, we have

$$\begin{aligned}
\overline{(\psi \times \pi) \otimes \mathrm{id}} \circ \zeta \circ k_X(\xi) &= \overline{(\psi \times \pi) \otimes \mathrm{id}} \circ \overline{k_X \otimes \mathrm{id}} \circ \sigma(\xi) \\
&= \overline{\psi \otimes \mathrm{id}} \circ \sigma(\xi) \\
&= u(\psi(\xi) \otimes 1)u^* \\
&= \mathrm{Ad}\, u\big((\psi \times \pi) \circ k_X(\xi) \otimes 1\big),
\end{aligned}$$

and for the second,

$$\begin{aligned}
\overline{(\psi \times \pi) \otimes \mathrm{id}} \circ \zeta \circ k_A(a) &= \overline{(\psi \times \pi) \otimes \mathrm{id}} \circ \overline{k_A \otimes \mathrm{id}} \circ \delta(a) \\
&= \overline{\pi \otimes \mathrm{id}} \circ \delta(a) \\
&= \mathrm{Ad}\, u(\pi(a) \otimes 1) \\
&= \mathrm{Ad}\, u\big((\psi \times \pi) \circ k_A(a) \otimes 1\big).
\end{aligned}$$

LEMMA 4.2. *Let* (σ, δ) *be a coaction of* G *on a correspondence* (X, A), *let* $(\psi, \pi, \mu) : (X, A, C_0(G)) \to (M_B(Y), M(B))$ *be a* (σ, δ)*-covariant correspondence homomorphism, and let* $(\rho, \tau) : (Y, B) \to (M_D(Z), M(D))$ *be a* *correspondence homomorphism with τ nondegenerate. Then the composition*

$$(\overline{\rho} \circ \psi, \overline{\tau} \circ \pi, \overline{\tau} \circ \mu) : (X, A, C_0(G)) \to (M_D(Z), M(D))$$

is covariant for (σ, δ).

PROOF. First of all, since π, μ, and τ are nondegenerate, $\overline{\tau} \circ \pi$ is also nondegenerate, and $(\overline{\tau} \circ \pi, \overline{\tau} \circ \mu)$ is covariant for (A, δ) by the standard theory of C^*-coactions.

Routine calculations show that

$$(\overline{\rho} \circ \psi, \overline{\tau} \circ \pi) : (X, A) \to (M(Z), M(D))$$

is a correspondence homomorphism. Also, since ψ and ρ map into $M_B(Y)$ and $M_D(Z)$, respectively, it is easy to see that $\overline{\rho} \circ \psi$ maps X into $M_D(Z)$.

Letting $u = \overline{\overline{\tau} \circ \mu \otimes \mathrm{id}}(w_G)$, the following calculation completes the proof: for $\xi \in X$ we have

$$\begin{aligned}\overline{(\overline{\tau} \circ \psi) \otimes \mathrm{id}} \circ \sigma(\xi) &= \overline{\tau \otimes \mathrm{id}} \circ \overline{\psi \otimes \mathrm{id}} \circ \sigma(\xi) \\ &= \overline{\tau \otimes \mathrm{id}}\big(\overline{\mu \otimes \mathrm{id}}(w_G) \cdot (\psi(\xi) \otimes 1) \cdot \overline{\mu \otimes \mathrm{id}}(w_G)^*\big) \\ &= u \cdot (\overline{\tau} \circ \psi(\xi) \otimes 1) \cdot u^*.\end{aligned}$$

COROLLARY 4.3. *Let* (σ, δ) *be a coaction of* G *on a correspondence* (X, A) *such that* $\delta(J_X) \subset M(A \otimes C^*(G); J_X \otimes C^*(G))$, *and let* $(\psi, \pi, \mu) : (X, A, C_0(G)) \to (M_B(Y), M(B))$ *be a* (σ, δ)*-covariant homomorphism, with* (ψ, π) *Cuntz-Pimsner covariant. Then the pair*

$$(\mathcal{O}_{\psi,\pi}, \overline{k_B} \circ \mu) : (\mathcal{O}_X, C_0(G)) \to M(\mathcal{O}_Y)$$

is covariant for the associated coaction ζ.

PROOF. Applying Lemma 4.2 to the Toeplitz representation $(k_Y, k_B) : (Y, B) \to \mathcal{O}_Y$, we see that

$$(\overline{k_Y} \circ \psi, \overline{k_B} \circ \pi, \overline{k_B} \circ \mu) : (X, A, C_0(G)) \to M(\mathcal{O}_Y)$$

is covariant for (σ, δ).

By [7, Theorem 3.5] the composition $(\overline{k_Y} \circ \psi, \overline{k_B} \circ \pi)$ is a Cuntz-Pimsner-covariant Toeplitz representation of (X, A) in $M(\mathcal{O}_Y)$. Then, since (ψ, π) is Cuntz-Pimsner covariant, Lemma 4.1 with $B = \mathcal{O}_Y$ tells us that

$$\big((\overline{k_Y} \circ \psi) \times (\overline{k_B} \circ \pi), \overline{k_B} \circ \mu\big) : (\mathcal{O}_X, C_0(G)) \to M(\mathcal{O}_Y)$$

is ζ-covariant. But by construction (see [7, Corollary 3.6]) we have

$$(\overline{k_Y} \circ \psi) \times (\overline{k_B} \circ \pi) = \mathcal{O}_{\psi,\pi}.$$

THEOREM 4.4. *Let (σ, δ) be a coaction of G on a correspondence (X, A) such that $\delta(J_X) \subset M(A \otimes C^*(G); J_X \otimes C^*(G))$, and let ζ be the associated coaction on $\mathcal{O}_X$, as in Proposition 3.1. If the canonical correspondence homomorphism*

$$(j_X, j_A) : (X, A) \to (M(X \rtimes_\sigma G), M(A \rtimes_\delta G))$$

is Cuntz-Pimsner covariant, then

$$\mathcal{O}_X \rtimes_\zeta G \cong \mathcal{O}_{X \rtimes_\sigma G}.$$

REMARK 4.5. We do not know whether the hypothesis of Cuntz-Pimsner covariance of (j_X, j_A) is redundant; in Corollary 4.6 below we will show that it is satisfied under certain conditions.

PROOF OF THEOREM 4.4. Our strategy is to construct a covariant homomorphism

$$(\rho, \mu) : (\mathcal{O}_X, C_0(G)) \to M(\mathcal{O}_{X \rtimes_\sigma G}),$$

and show that the integrated form $\rho \times \mu$ is an isomorphism of $\mathcal{O}_X \rtimes_\zeta G$ onto $\mathcal{O}_{X \rtimes_\sigma G}$. For the covariant homomorphism we will need a homomorphism of $\mathcal{O}_X$, and to get this we will apply functoriality: since (j_X, j_A) is Cuntz-Pimsner covariant, by [7, Corollary 3.6] there is a unique nondegenerate homomorphism

$$\mathcal{O}_{j_X, j_A} : \mathcal{O}_X \to M(\mathcal{O}_{X \rtimes_\sigma G})$$

making the diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{(j_X, j_A)} & (M_{A \rtimes_\delta G}(X \rtimes_\sigma G), M(A \rtimes_\delta G)) \\ {\scriptstyle (k_A, k_A)}\downarrow & & \downarrow{\scriptstyle (\overline{k_{X \rtimes_\sigma G}}, \overline{k_{A \rtimes_\delta G}})} \\ \mathcal{O}_X & \xrightarrow[\mathcal{O}_{j_X, j_A}]{} & M(\mathcal{O}_{X \rtimes_\sigma G}) \end{array}$$

commute.

We next show that (j_X, j_A, j_G) is covariant for (σ, δ):

$$\begin{aligned}
\overline{j_X \otimes \mathrm{id}} \circ \sigma &= \overline{\big(\overline{\mathrm{id} \otimes \lambda} \circ \sigma\big) \otimes \mathrm{id}} \circ \sigma \\
&= \overline{\mathrm{id} \otimes \lambda \otimes \mathrm{id}} \circ \overline{\sigma \otimes \mathrm{id}} \circ \sigma \\
&= \overline{\mathrm{id} \otimes \lambda \otimes \mathrm{id}} \circ \overline{\mathrm{id} \otimes \delta_G} \circ \sigma \\
&= \mathrm{Ad}\big(1 \otimes \overline{M \otimes \mathrm{id}}(w_G)\big) \circ \overline{\mathrm{id} \otimes \lambda \otimes \mathrm{id}} \circ (\sigma \otimes 1) \\
&= \mathrm{Ad}\, \overline{\overline{1 \otimes M} \otimes \mathrm{id}}(w_G) \circ \big(\overline{\mathrm{id} \otimes \lambda} \circ \sigma \otimes 1\big) \\
&= \mathrm{Ad}\, \overline{j_G \otimes \mathrm{id}}(w_G) \circ (j_X \otimes 1),
\end{aligned}$$

where the fourth equality follows by linearity, density, and strict continuity from the following computation with elementary tensors: for $\eta \in X$ and $t \in G$ we have

$$\begin{aligned}
\overline{\mathrm{id} \otimes \lambda \otimes \mathrm{id}} \circ \overline{\mathrm{id} \otimes \delta_G}(\eta \otimes t) &= \overline{\mathrm{id} \otimes \lambda \otimes \mathrm{id}}(\eta \otimes \delta_G(t)) \\
&= \eta \otimes \overline{\lambda \otimes \mathrm{id}} \circ \delta_G(t) \\
&= \eta \otimes \lambda_t \otimes t \\
&= \eta \otimes \mathrm{Ad}\, \overline{M \otimes \mathrm{id}}(w_G)(\lambda_t \otimes 1) \\
&= \mathrm{Ad}\big(1 \otimes \overline{M \otimes \mathrm{id}}(w_G)\big)(\eta \otimes \lambda_t \otimes 1)
\end{aligned}$$

where in turn the fourth equality follows from the following: for $f \in B(G)$ we have

$$\begin{aligned}
S_f\big((\lambda_t \otimes t)\overline{M \otimes \mathrm{id}}(w_G)\big) &= \lambda_t S_{f \cdot t}\big(\overline{M \otimes \mathrm{id}}(w_G)\big) \\
&= \lambda_t M_{f \cdot t} \\
&= M_f \lambda_t \\
&= S_f\big(\overline{M \otimes \mathrm{id}}(w_G)\big)\lambda_t \\
&= S_f\big(\overline{M \otimes \mathrm{id}}(w_G)(\lambda_t \otimes 1)\big),
\end{aligned}$$

so that

$$(\lambda_t \otimes t)\overline{M \otimes \mathrm{id}}(w_G) = \overline{M \otimes \mathrm{id}}(w_G)(\lambda_t \otimes 1).$$

It now follows from Corollary 4.3 that the pair

$$\big(\mathcal{O}_{j_X, j_A}, \overline{k_{A \rtimes_\delta G}} \circ j_G\big)$$

is a covariant homomorphism of the coaction $(\mathcal{O}_X, \zeta)$ in $M(\mathcal{O}_{X \rtimes_\sigma G})$, and thus we get a homomorphism

$$\Pi := \mathcal{O}_{j_X, j_A} \times \big(\overline{k_{A \rtimes_\delta G}} \circ j_G\big) : \mathcal{O}_X \rtimes_\zeta G \to M(\mathcal{O}_{X \rtimes_\sigma G}).$$

It remains to show the following:

(i) Π maps into $\mathcal{O}_{X\rtimes_\sigma G}$;

(ii) Π is surjective;

(iii) Π is injective.

For (i), for $\xi \in X$, $a \in A$, and $f \in C_0(G)$ we have

$$\begin{aligned}\mathcal{O}_{j_X,j_A} \circ k_X(\xi)\overline{k_{A\rtimes_\delta G}} \circ j_G(f) &= \overline{k_{X\rtimes_\sigma G}}(j_X(\xi))\overline{k_{A\rtimes_\delta G}}(j_G(f))\\ &= \overline{k_{X\rtimes_\sigma G}}(j_X(\xi)\cdot j_G(f))\end{aligned}$$

and

$$\begin{aligned}\mathcal{O}_{j_X,j_A} \circ k_A(a)\overline{k_{A\rtimes_\delta G}} \circ j_G(f) &= \overline{k_{A\rtimes_\delta G}}(j_A(a))\overline{k_{A\rtimes_\delta G}}(j_G(f))\\ &= \overline{k_{A\rtimes_\delta G}}\big(j_A(a)j_G(f)\big).\end{aligned}$$

For (ii), we see from the above that the image of Π contains

$$\overline{k_{X\rtimes_\sigma G}}\big(j_X(X)\cdot j_G(C_0(G))\big) \qquad \text{and} \qquad \overline{k_{A\rtimes_\delta G}}\big(j_A(A)\cdot j_G(C_0(G))\big),$$

and hence contains

$$\overline{k_{X\rtimes_\sigma G}}(X\rtimes_\sigma G) \qquad \text{and} \qquad \overline{k_{A\rtimes_\delta G}}(A\rtimes_\delta G),$$

which generate $\mathcal{O}_{X\rtimes_\sigma G}$.

For (iii) we apply [15, Proposition 3.1]: we must show that $\Pi\circ j_{\mathcal{O}_X}$ is faithful and that there is an action α of G on $\mathcal{O}_{X\rtimes_\sigma G}$ such that Π is $\hat{\zeta}-\alpha$ equivariant.

To see that $\Pi\circ j_{\mathcal{O}_X}$ is faithful, we apply the Gauge-Invariant Uniqueness Theorem: since

$$\Pi\circ j_{\mathcal{O}_X}\circ k_A = \mathcal{O}_{j_X,j_X}\circ k_A = j_A$$

is faithful, it suffices to show that for all $z\in \mathsf{T}$, $\xi\in X$, and $a\in A$ we have

$$\begin{aligned}\gamma_z\circ\Pi\circ j_{\mathcal{O}_X}\circ k_X(\xi) &= z\Pi\circ j_{\mathcal{O}_X}\circ k_X(\xi)\\ \gamma_z\circ\Pi\circ j_{\mathcal{O}_X}\circ k_A(a) &= \Pi\circ j_{\mathcal{O}_X}\circ k_A(a).\end{aligned}$$

For the first, we have

$$\begin{aligned}\gamma_z\circ\Pi\circ j_{\mathcal{O}_X}\circ k_X(\xi) &= \gamma_z\circ\mathcal{O}_{j_X,j_A}\circ k_X(\xi)\\ &= \gamma_z\circ\overline{k_{X\rtimes_\sigma G}}\circ j_X(\xi)\\ &= z\overline{k_{X\rtimes_\sigma G}}\circ j_X(\xi)\\ &= z\Pi\circ j_{\mathcal{O}_X}\circ k_X(\xi),\end{aligned}$$

where the third equality follows from

$$\gamma_z\circ k_{X\rtimes_\sigma G} = zk_{X\rtimes_\sigma G}.$$

The second is similar, this time using $\gamma_z \circ k_{A\rtimes_\delta G} = k_{A\rtimes_\delta G}$.

We now turn to the action of G. First note that there is an action β of G on $X \rtimes_\sigma G$ given by

$$\beta_t\big(j_X(\xi)\cdot j_G(f)\big) = j_X(\xi)\cdot j_G\circ \mathrm{rt}_t(f) \qquad \text{for} \quad \xi\in X,\, f\in C_0(G),$$

where rt is the action of G on $C_0(G)$ given by right translation. This in turn gives an action α of G on $\mathcal{O}_{X\rtimes_\sigma G}$ such that

$$\begin{aligned} \alpha_t\circ k_{X\rtimes_\sigma G} &= k_{X\rtimes_\sigma G}\circ\beta_t \\ \alpha_t\circ k_{A\rtimes_\delta G} &= k_{A\rtimes_\delta G}\circ\beta_t. \end{aligned}$$

Finally, we check the $\hat{\zeta}-\alpha$ covariance:

$$\begin{aligned} \alpha_t\circ\Pi\circ j_{\mathcal{O}_X} &= \alpha_t\circ\mathcal{O}_{j_X,j_A} \\ &= \alpha_t\circ\overline{k_{X\rtimes_\sigma G}}\circ j_X \\ &= \overline{k_{X\rtimes_\sigma G}}\circ\beta_t\circ j_X \\ &= \overline{k_{X\rtimes_\sigma G}}\circ j_X \\ &= \Pi\circ j_{\mathcal{O}_X} \\ &= \Pi\circ\hat{\zeta}_t\circ j_{\mathcal{O}_X}, \end{aligned}$$

and

$$\begin{aligned} \alpha_t\circ\Pi\circ j_G &= \alpha_t\circ\overline{k_{A\rtimes_\delta G}}\circ j_G \\ &= \overline{k_{A\rtimes_\delta G}}\circ\beta_t\circ j_G \\ &= \overline{k_{A\rtimes_\delta G}}\circ j_G\circ\mathrm{rt}_t \\ &= \Pi\circ j_G\circ\mathrm{rt}_t \\ &= \Pi\circ\hat{\zeta}_t\circ j_G. \end{aligned}$$

COROLLARY 4.6. *Let (σ,δ) be a coaction of G on a correspondence (X, A). If either*

(i) *G is amenable, or*

(ii) *$\varphi_A : A\to\mathcal{L}(X)$ is faithful,*

then the canonical correspondence homomorphism

$$(j_X, j_A) : (X, A)\to (M(X\rtimes_\sigma G), M(A\rtimes_\delta G))$$

is Cuntz-Pimsner covariant.

PROOF. By Lemma 2.8, it suffices to show that

$$j_A(J_X)(A\rtimes_\delta G)\subset J_{X\rtimes_\sigma G}.$$

The ideal of $A \rtimes_\delta G$ generated by $j_A(J_X)(A \rtimes_\delta G)$ is

$$I := \overline{\operatorname{span}}\{(A \rtimes_\delta G) j_A(J_X)(A \rtimes_\delta G)\},$$

so it suffices to show that $\varphi_{A \rtimes_\delta G}$ maps I injectively into $\mathcal{K}(X \rtimes_\sigma G)$. As we observed immediately before Remark 2.7, we can work with $\varphi_A \rtimes G$ and $\mathcal{K}(X) \rtimes_{\sigma^{(1)}} G$ rather than $\varphi_{A \rtimes_\delta G}$ and $\mathcal{K}(X \rtimes_\sigma G)$. To see that $\varphi_A \rtimes G$ maps I into $\mathcal{K}(X) \rtimes_{\sigma^{(1)}} G$, it suffices to observe that

$$\begin{aligned}
&(\varphi_A \rtimes G)\big(j_G^A(C_0(G)) j_A(A) j_A(J_X) j_A(A) j_G^A(C_0(G))\big) \\
&\quad = (\varphi_A \rtimes G)\big(j_G^A(C_0(G)) j_A(A J_X A) j_G^A(C_0(G))\big) \\
&\quad \subset (\varphi_A \rtimes G)\big(j_G^A(C_0(G)) j_A(J_X) j_G^A(C_0(G))\big) \\
&\quad = j_G^{\mathcal{K}(X)}(C_0(G)) \overline{j_{\mathcal{K}(X)}}(\varphi_A(J_X)) j_G^{\mathcal{K}(X)}(C_0(G)) \\
&\quad \subset j_G^{\mathcal{K}(X)}(C_0(G)) j_{\mathcal{K}(X)}(\mathcal{K}(X)) j_G^{\mathcal{K}(X)}(C_0(G)) \\
&\quad \subset \mathcal{K}(X) \rtimes_{\sigma^{(1)}} G.
\end{aligned}$$

On the other hand, to see that $\varphi_A \rtimes G$ is injective on I, we now consider each hypothesis (i) and (ii) separately. First, if φ_A is injective, then so is $\varphi_A \rtimes G$, because φ_A gives a G-equivariant isomorphism between (A, δ) and the image $(\varphi_A(A), \eta)$, where η is the corresponding coaction on $\varphi_A(A)$, and we have a commuting diagram

$$\begin{array}{ccc}
A \rtimes_\delta G & \xrightarrow{\ \cong\ } & \varphi_A(A) \rtimes_\eta G \\
& \searrow^{\pi} & \downarrow \\
& & M(\mathcal{K}(X) \rtimes_{\sigma^{(1)}} G),
\end{array}$$

where the horizontal arrow is an isomorphism and the vertical arrow is an inclusion.

Thus it remains to show that $\varphi_A \rtimes G$ is injective on I under the assumption that G is amenable. We will show that in this case J_X is a δ-invariant ideal of A in the sense that δ restricts to a coaction on J_X. It will follow that

$$I = J_X \rtimes_\delta G,$$

and since the restriction $\varphi_A| : J_X \to \mathcal{K}(X)$ is injective we will be able to conclude that

$$\varphi_A| \rtimes G : J_X \rtimes_\delta G \to \mathcal{K}(X) \rtimes_{\sigma^{(1)}} G$$

is injective as well.

To see that J_X is invariant, by [15, Proposition 2.6] it suffices to show that J_X is an $A(G)$-submodule of A. Let $f \in A(G)$ and $a \in J_X$. We must show both of the following:

(i) $\varphi_A(f \cdot a) \in \mathcal{K}(X)$;

(ii) $(f \cdot a)b = 0$ for all $b \in \ker \varphi_A$.

For (i), we have

$$\begin{aligned}
\varphi_A(f \cdot a) &= \varphi_A \circ S_f \circ \delta(a) \\
&= S_f \circ \overline{\varphi_A \otimes \mathrm{id}} \circ \delta(a) \\
&= S_f \circ \overline{\sigma^{(1)}} \circ \varphi_A(a) \\
&\subset S_f \circ \sigma^{(1)}(\mathcal{K}(X)) \\
&\subset S_f\big(M_{C^*(G)}\big(\mathcal{K}(X) \otimes C^*(G)\big)\big) \\
&\subset \mathcal{K}(X),
\end{aligned}$$

by [11, Lemma 1.5].

In preparation for (ii), we first show that $\ker \varphi_A$ is δ-invariant: if $f \in A(G)$ and $b \in \ker \varphi_A$, then

$$\varphi_A(f \cdot b) = S_f \circ \overline{\varphi_A \otimes \mathrm{id}} \circ \delta(b) = S_f \circ \overline{\sigma^{(1)}} \circ \varphi_A(b) = 0.$$

Thus δ restricts to a coaction on $\ker \varphi_A$, so

$$\overline{\operatorname{span}}\{\delta(\ker \varphi_A)(1 \otimes C^*(G))\} = \ker \varphi_A \otimes C^*(G). \tag{4.1}$$

We now verify (ii): for $f \in A(G)$, $a \in J_X$, and $b \in \ker \varphi_A$ we first factor $f = c \cdot f'$ for some $c \in C^*(G)$ and $f' \in A(G)$ (using amenability of G again), and then

$$\begin{aligned}
(f \cdot a)b &= S_f \circ \delta(a)b \\
&= S_f\big(\delta(a)(b \otimes 1)\big) \\
&= S_{c \cdot f'}\big(\delta(a)(b \otimes 1)\big) \\
&= S_{f'}\big(\delta(a)(b \otimes c)\big) \\
&\approx \sum_1^n S_{f'}\big(\delta(a)\delta(b_i)(1 \otimes c_i)\big) \\
&\qquad\qquad \text{(for some } b_i \in \ker \varphi_A \text{ and } c_i \in C, \text{ by (4.1))} \\
&\approx \sum_1^n S_{f'}\big(\delta(ab_i)(1 \otimes c_i)\big) = 0,
\end{aligned}$$

because $J_X \subset (\ker \varphi_A)^{\perp}$.

Then [14, Theorem 6.9] (see also [11, Theorem 2.9]) shows that the crossed product of a C^*-algebra A by an inner coaction of G is isomorphic to $A \otimes C_0(G)$; the following result is a version for correspondences:

PROPOSITION 4.7. *Let (A, X, B) be a correspondence, and let (A, δ) and (B, ε) be inner coactions implemented by nondegenerate homomorphisms μ and ν, respectively, and let σ be the associated coaction on X, as in Proposition 3.2. Then there is an isomorphism*

$$\Phi : X \rtimes_\sigma G \to X \otimes C_0(G)$$

given by

$$\Phi(y) = \overline{\mu \otimes \lambda}(w_G)^* \cdot y \cdot \overline{\nu \otimes \lambda}(w_G).$$

The left and right module actions are transformed by Φ as follows:

$$\Phi\big(j_A(a) j_G^A(f) \cdot y \cdot j_B(b) j_G^B(g)\big) = (a \otimes 1)(\mu \times M)(f) \cdot \Phi(y) \cdot (b \otimes 1)(\nu \times M)(g),$$

where $\mu \times M$ denotes the Kronecker product of μ and M, respectively, and similarly for $\nu \times M$.

PROOF. Note that we are identifying $C_0(G)$ with its image under the representation M on $L^2(G)$ by pointwise multiplication, i.e., $(M_f \xi)(t) = f(t)\xi(t)$ for $f \in C_0(G)$ and $\xi \in L^2(G)$. Routine calculations show

$$\begin{aligned}
\Phi \circ j_A &= \mathrm{id}_A \otimes 1 \\
\Phi \circ j_B &= \mathrm{id}_B \otimes 1 \\
\Phi \circ j_X &= \mathrm{id}_X \otimes 1 \\
\Phi \circ j_G^A &= \mu \times M \\
\Phi \circ j_G^B &= \nu \times M;
\end{aligned}$$

for the last two it helps to note that

$$\operatorname{Ad} \overline{\mathrm{id} \otimes \lambda}(w_G)^*(1 \otimes M_f) = (\mathrm{id} \times M)(f).$$

Since

$$\overline{\mu \otimes \lambda}(w_G) \in M(A \otimes \mathcal{K}(L^2(G))),$$

and similarly for $\overline{\nu \otimes \lambda}(w_G)$, clearly Φ maps $X \rtimes_\sigma G$ into $M(X \otimes L^2(G))$.

We actually have $\Phi(X \rtimes_\sigma G) = X \otimes C_0(G)$, because

$$\begin{aligned}
&\overline{\operatorname{span}}\{\Phi(j_X(X) \cdot j_G(C_0(G)))\} \\
&\quad = \overline{\operatorname{span}}\{\Phi(j_X(X \cdot B) \cdot j_G(C_0(G)))\} \\
&\quad = \overline{\operatorname{span}}\{\Phi(j_X(X) \cdot j_B(B) j_G(C_0(G)))\} \\
&\quad = \overline{\operatorname{span}}\{(X \otimes 1) \cdot \operatorname{Ad} \overline{\nu \otimes \mathrm{id}}(w_G)^*(B \rtimes_\varepsilon G)\} \\
&\quad = \overline{\operatorname{span}}\{(X \otimes 1) \cdot (B \otimes C_0(G))\} \\
&\qquad\qquad \text{(by [14, Theorem 6.9] or [11, Theorem 2.9])} \\
&\quad = X \otimes C_0(G).
\end{aligned}$$

Let (γ, α) be an action of G on a correspondence (X, A). Assume that G is amenable; in particular, there is no difference between the full and reduced crossed products $X \rtimes_\gamma G$ and $X \rtimes_{\gamma,r} G$ (and similarly for A), so we can freely apply the results of [4, Section 3.1].

As in [4, Proposition 3.5], let $\hat{\gamma}$ be the dual coaction of G on $X \rtimes_\gamma G$, determined on generators $\xi \in C_c(G, X)$ by

$$\hat{\gamma}(\xi)(t) = \xi(t) \otimes t,$$

so that $\hat{\gamma}$ is an element of $C_c(G, M^\beta(X \otimes C^*(G)))$, which in turn is embedded in $M((X \rtimes_\gamma G) \otimes C^*(G))$ via the isomorphism [4, Lemma 3.4]

$$(X \rtimes_\gamma G) \otimes C^*(G) \xrightarrow{\cong} (X \otimes C^*(G)) \rtimes_{\gamma \otimes \mathrm{id}} G$$

that extends the canonical embedding

$$C_c(G, X) \odot C^*(G) \hookrightarrow C_c(G, X \otimes C^*(G)).$$

PROPOSITION 4.8. *Let (γ, α) be an action of G on a correspondence (X, A), and assume that G is amenable. Then the dual coaction $(\hat{\gamma}, \hat{\alpha})$ on $(X \rtimes_\gamma G, A \rtimes_\alpha G)$ satisfies*

$$\hat{\alpha}(J_{X \rtimes_\gamma G}) \subset M\big((A \rtimes_\alpha G) \otimes C^*(G); J_{X \rtimes_\gamma G} \otimes C^*(G)\big). \tag{4.2}$$

PROOF. By [5, Proposition 2.7], the ideal J_X of A is α-invariant, and

$$J_{X \rtimes_\gamma G} = J_X \rtimes_\alpha G.$$

The isomorphism

$$(A \rtimes_\alpha G) \otimes C^*(G) \xrightarrow{\cong} (A \otimes C^*(G)) \rtimes_{\alpha \otimes \mathrm{id}} G,$$

of [4, Lemma A.20] clearly takes $(J_X \rtimes_\alpha G)\otimes C^*(G)$ to $(J_X\otimes C^*(G))\rtimes_{\alpha\otimes\mathrm{id}} G$.

Recall that $\hat\alpha$ takes a function $f \in C_c(G, A)$ to the function in $C_c(G, M^\beta(A\otimes C^*(G)))$ defined by

$$\hat\alpha(f)(t) = f(t)\otimes t.$$

It follows that for $g \in C_c(G, A\otimes C^*(G))$ we have

$$\begin{aligned}(\hat\alpha(f)g)(t) &= \int_G \hat\alpha(f)(s)\overline{\alpha_s\otimes \mathrm{id}}(g(s^{-1}t))\,ds\\ &= \int_G (f(s)\otimes s)\overline{\alpha_s\otimes \mathrm{id}}(g(s^{-1}t))\,ds.\end{aligned}$$

Now let $f \in C_c(G, J_X)$. For all $s \in G$, it is easy to check, by first computing with elementary tensors $a\otimes c \in A\odot C^*(G)$, that

$$(f(s)\otimes s)(A\otimes C^*(G)) \subset J_X\otimes C^*(G),$$

and it follows that

$$\hat\alpha(f)g \in C_c(G, J_X\otimes C^*(G)) \subset (J_X\otimes C^*(G))\rtimes_{\alpha\otimes\mathrm{id}} G.$$

By density, this implies that

$$\hat\alpha(J_X\rtimes_\alpha G) \subset M\big((A\otimes C^*(G))\rtimes_{\alpha\otimes\mathrm{id}} G;\ (J_X\otimes C^*(G))\rtimes_{\alpha\otimes\mathrm{id}} G\big),$$

which in turn implies (4.2).

5. Application

As an application of our techniques, we will give an alternative approach to a recent result of Hao and Ng [5, Theorem 2.10]. Given an action (γ, α) of an amenable locally compact group G on a nondegenerate correspondence (X, A), Hao and Ng construct an isomorphism

$$\mathcal{O}_{X\rtimes_\gamma G} \xrightarrow{\cong} \mathcal{O}_X \rtimes_\beta G,$$

where $X\rtimes_\gamma G$ is the crossed-product correspondence over $A\rtimes_\alpha G$ and β is the associated action of G on $\mathcal{O}_X$. In our earlier paper [7, Proposition 4.3] we suggested an alternative approach to this result, removing the amenability hypothesis on G. Namely, we construct a surjection that goes in the opposite direction:

$$\mathcal{O}_X\rtimes_\beta G \to \mathcal{O}_{X\rtimes_\gamma G}.$$

We suspect, but were unable to prove, that this is an isomorphism in general; however, at least in the amenable case, we can give a new proof of [5, Theorem 2.10] with the help of Propositions 4.8 and 3.1.

THEOREM 5.1. *Let* (γ, α) *be an action of* G *on a nondegenerate correspondence* (X, A)*, let* β *be the associated action of* G *on* $\mathcal{O}_X$*, and let*

$$\Pi := \mathcal{O}_{i_X, i_A} \times u : \mathcal{O}_X \rtimes_\beta G \to \mathcal{O}_{X \rtimes_\gamma G}$$

be the surjection from [7, Proposition 4.3]. If G *is amenable, then* Π *is an isomorphism.*

PROOF. By Propositions 4.8 and 3.1 we get a coaction ζ of G on $\mathcal{O}_{X \rtimes_\gamma G}$. Our strategy is to show that Π is $\hat{\beta} - \zeta$ equivariant and that $\mathcal{O}_{i_X, i_A}$ is injective, and then [6, Corollary 4.4] will imply that Π is injective, because by amenability of G the coaction ζ is automatically maximal.

We check the equivariance condition

$$\zeta \circ \Pi = \overline{\Pi \otimes \mathrm{id}} \circ \hat{\beta}$$

separately on generators from X, A, and G: for X we have

$$\begin{aligned}
\overline{\zeta \circ \Pi} \circ i_{\mathcal{O}_X} \circ k_X &= \overline{\zeta} \circ \overline{\Pi} \circ i_{\mathcal{O}_X} \circ k_X \\
&= \overline{\zeta} \circ \mathcal{O}_{i_X, k_A} \circ k_X \\
&= \overline{\zeta} \circ \overline{k_{X \rtimes_\gamma G}} \circ i_X \\
&= \overline{k_{X \rtimes_\gamma G} \otimes \mathrm{id}} \circ \overline{\hat{\gamma}} \circ i_X \\
&= \overline{k_{X \rtimes_\gamma G} \otimes \mathrm{id}} \circ (i_X \otimes 1) \\
&= \big(\overline{k_{X \rtimes_\gamma G}} \circ i_X\big) \otimes 1 \\
&= \mathcal{O}_{i_X, i_A} \circ k_X \otimes 1 \\
&= (\mathcal{O}_{i_X, i_A} \otimes 1) \circ k_X \\
&= (\overline{\Pi} \circ i_{\mathcal{O}_X} \otimes 1) \circ k_X \\
&= \overline{\Pi \otimes \mathrm{id}} \circ (i_{\mathcal{O}_X} \otimes 1) \circ k_X \\
&= \overline{\Pi \otimes \mathrm{id}} \circ \overline{\hat{\beta}} \circ i_{\mathcal{O}_X} \circ k_X \\
&= \overline{\overline{\Pi \otimes \mathrm{id}} \circ \hat{\beta}} \circ i_{\mathcal{O}_X} \circ k_X.
\end{aligned}$$

The verification for generators from A is parallel, using $k_A, i_A, \hat{\alpha}$ instead of $k_X, i_X, \hat{\gamma}$.

For generators from G we have

$$
\begin{aligned}
\overline{\zeta \circ \Pi} \circ i_G^{\mathcal{O}_X} &= \overline{\zeta} \circ \overline{\Pi} \circ i_G^{\mathcal{O}_X} \\
&= \overline{\zeta} \circ u \\
&= \overline{\zeta} \circ \overline{k_{A \rtimes_\alpha G}} \circ i_G^A \\
&= \overline{\zeta \circ k_{A \rtimes_\alpha G}} \circ i_G^A \\
&= \overline{\overline{k_{A \rtimes_\alpha G} \otimes \mathrm{id}} \circ \hat{\alpha}} \circ i_G^A \\
&= \overline{k_{A \rtimes_\alpha G} \otimes \mathrm{id}} \circ \overline{\hat{\alpha}} \circ i_G^A \\
&= \overline{k_{A \rtimes_\alpha G} \otimes \mathrm{id}} \circ \overline{i_G^A \otimes \mathrm{id}} \circ \delta_G \\
&= \overline{\overline{k_{A \rtimes_\alpha G}} \circ i_G^A \otimes \mathrm{id}} \circ \delta_G \\
&= \overline{u \otimes \mathrm{id}} \circ \delta_G \\
&= \overline{\overline{\Pi} \circ i_G^{\mathcal{O}_X} \otimes \mathrm{id}} \circ \delta_G \\
&= \overline{\Pi \otimes \mathrm{id}} \circ \overline{i_G^{\mathcal{O}_X} \otimes \mathrm{id}} \circ \delta_G \\
&= \overline{\Pi \otimes \mathrm{id}} \circ \overline{\hat{\beta}} \circ i_G^{\mathcal{O}_X} \\
&= \overline{\overline{\Pi \otimes \mathrm{id}} \circ \hat{\beta}} \circ i_G^{\mathcal{O}_X}.
\end{aligned}
$$

Finally, by [7, Corollary 3.6], $\mathcal{O}_{i_X, i_A}$ is injective because i_A is.

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