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CORRELATION OF PATHS BETWEEN DISTINCT VERTICES IN A RANDOMLY ORIENTED GRAPH

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Abstract

We prove that in a random tournament the events $\{s \to a\}$ (meaning that there is a directed path from s to a) and $\{t \to b\}$ are positively correlated, for distinct vertices $a, s, b, t \in K_n$. It is also proven that the correlation between the events $\{s \to a\}$ and $\{t \to b\}$ in the random graphs $G(n, p)$ and $G(n, m)$ with random orientation is positive for every fixed $p > 0$ and sufficiently large n (with $m = \lfloor p\binom{n}{2}\rfloor$). We conjecture it to be positive for all p and all n. An exact recursion for $\mathsf{P}(\{s \to a\} \cap \{t \to b\})$ in $G(n, p)$ is given.

1. Introduction

Let G be a graph on n vertices and $a, b, s, t \in V(G)$ four different vertices in the graph. Let further every edge in G be oriented either way with the same probability independently of each other. This model was first considered in [4], and a similar model was discussed in [3]. We will study the correlation between the event that there exists a directed path from s to a, $\{s \to a\}$, and the event that there exists a directed path from t to b, $\{t \to b\}$. Our main result is that these events are positively correlated for the complete graph and for two natural models of random graphs. Note however that it is easy to construct examples when the correlation will be negative, e.g. if G is the path on four vertices with edges sb, ba, at.

The events $\{s \to a\}$ and $\{s \to b\}$ can be shown to have positive correlation for any vertices in any graph G. In [1] it was proven, somewhat surprisingly, that also the events $\{s \to a\}$ and $\{b \to s\}$ have positive correlation in K_n, when $n \geq 5$, but negative correlation if G is a tree or a cycle. Further, in [2] it was shown that in the random graph models $G(n, p)$ and $G(n, m)$, see proper definition below, for a fixed probability p and large enough n the correlation between $\{s \to a\}$ and $\{b \to s\}$ is negative if p is below a critical value and positive if p is above the critical value. The critical value in $G(n, p)$ was exactly $1/2$ and in $G(n, m)$ approximately 0.799.

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The situation in this paper turns out to be different. We prove positive correlation when G is K_n and in $G(n, p)$ and $G(n, m)$ for fixed $p > 0$ and n sufficiently large. We conjecture that it is in fact non-negative for all pairs n, p.

For technical reasons we will study the complementary events $A := \{s \nrightarrow a\}$, the event that no directed path from s to a exists, and $B := \{t \nrightarrow b\}$. Note that the events A and B have the same covariance as the events $\{s \to a\}$ and $\{t \to b\}$.

The paper is organised as follows. In Section 2 we present a lower bound for $\mathsf{P}(A \cap B)$ in a random tournament and prove that A and B are positively correlated for $n \geq 4$. An intuitive explanation is that if A is true then the most likely situations are that no edges are directed from s or no edges are directed to a, thus s or a cannot be on a path from t to b. Since $\mathsf{P}(B)$ is increasing when n is decreasing this gives positive correlation. We show more precisely that the relative covariance, $(\mathsf{P}(A \cap B) - \mathsf{P}(A) \cdot \mathsf{P}(B)) / \mathsf{P}(A \cap B)$, of the two events converges to $2/3$ as $n \to \infty$.

In Section 3 we consider the random graph $G(n, p)$ on n vertices. It is a random graph model in which every edge exists with probability p independently of each other and then every existing edge is directed in either of the two directions with the same probability independently of all other edges. The two random processes can be combined in two different ways. In this paper we study the joined probability space of $G(n, p)$ and that of edge orientations, which we call $\vec{G}(n, p)$. This will be referred to as the annealed version. The other possibility, the quenched version, will be briefly discussed in Section 6. We prove that for fixed $p > 0$ and sufficiently large n the events A and B will be positively correlated in $\vec{G}(n, p)$.

In Section 4 we study the random graph model $G(n, m)$, with uniform distribution among all graphs with n vertices and m edges. Note that in this graph the edges do not exist independently of each other since the number of edges in the graph is fixed. As before every existing edge is directed in either way with equal probability independent of all other edges. We prove that for fixed $p = m / \binom{n}{2}$ the events A and B are positively correlated for sufficiently large n.

In Section 5 we give an exact recursion to compute $\mathsf{P}(A \cap B)$ in $G(n, p)$ which supports our conjecture that the correlation is positive for all values of n and p.

The problems studied here were first motivated by the, so far in vain, attempts to prove the so called bunkbed conjecture, see [5].

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2. Correlation in a random tournament

To show that the correlation between A and B is positive we need a sufficient upper bound for $\mathsf{P}(A)$ (and $\mathsf{P}(B)$) and a lower bound for $\mathsf{P}(A \cap B)$. Both an upper bound and a lower bound for $\mathsf{P}(A)$ were given in [1]:

LEMMA 2.1 (Theorem 2.1 in [1]). *For all $n \geq 2$,*

$$\left(\frac{1}{2}\right)^{n-2}\left(1-\left(\frac{1}{2}\right)^{n-1}\right) \leq \mathsf{P}(A) \leq \left(\frac{1}{2}\right)^{n-2}\left(1+3.2\cdot\left(\frac{7}{8}\right)^{n-1}\right).$$

The next lemma gives a lower bound for the probability of the event $A \cap B$.

LEMMA 2.2. *For all $n \geq 4$,*

$$\mathsf{P}(A \cap B) \geq \left(\frac{1}{2}\right)^{2n-4}\left(3-\left(\frac{1}{2}\right)^{2n-7}-\left(\frac{1}{2}\right)^{n-4}\right)$$

PROOF. Define $I_{a,b}$ to be the set of vertices in $[n] \setminus \{a, b\}$ that can reach a or b in one step, that is with a single edge directed to a or b. Similarly define $O_{s,t}$ to be the set of vertices in $[n] \setminus \{s, t\}$ that can be reached from s or t in one step. Define further I_a and I_b to be the set of vertices in $[n] \setminus \{a\}$ and $[n] \setminus \{b\}$ respectively that can reach a and b respectively in one step, and finally in the same way define O_s and O_t to be the set of vertices in $[n] \setminus \{s\}$ and $[n] \setminus \{t\}$ respectively that can be reached from s and t respectively in one step.

Each of $I_{a,b} = \emptyset$, $O_{s,t} = \emptyset$, $I_a = O_t = \emptyset$ and $I_b = O_s = \emptyset$ implies $A \cap B$. Hence we have

$$\mathsf{P}(A \cap B) \geq \mathsf{P}((I_{a,b} = \emptyset) \cup (O_{s,t} = \emptyset) \cup (I_a = O_t = \emptyset) \cup (I_b = O_s = \emptyset)).$$

By inclusion-exclusion we have

$$\begin{aligned}
&\mathsf{P}((I_{a,b} = \emptyset) \cup (O_{s,t} = \emptyset) \cup (I_a = O_t = \emptyset) \cup (I_b = O_s = \emptyset)) \\
&\quad = 2\cdot\left(\frac{1}{2}\right)^{2(n-2)} + 2\cdot\left(\frac{1}{2}\right)^{2(n-1)-1} \\
&\qquad -\left(\left(\frac{1}{2}\right)^{4(n-2)-4} + 4\cdot\left(\frac{1}{2}\right)^{3n-6}\right) + 2\cdot\left(\frac{1}{2}\right)^{4n-10} \\
&\quad = \left(\frac{1}{2}\right)^{2n-4}\left(3-\left(\frac{1}{2}\right)^{2n-7}-\left(\frac{1}{2}\right)^{n-4}\right)
\end{aligned}$$

since the events $(I_a = \emptyset)$ and $(I_b = \emptyset)$ are disjoint and so are the events $(O_s = \emptyset)$ and $(O_t = \emptyset)$. This completes the proof.

THEOREM 2.3. *The events $A = \{s \nrightarrow a\}$ and $B = \{t \nrightarrow b\}$ are positively correlated in a random tournament for $n \geq 4$.*

PROOF. From Lemmas 2.1 and 2.2 we get

$$\begin{aligned}
&\mathsf{P}(A \cap B) - \mathsf{P}(A)\mathsf{P}(B) \\
&\qquad = \mathsf{P}(A \cap B) - (\mathsf{P}(A))^2 \\
&\qquad \geq \left(\frac{1}{2}\right)^{2n-4}\left(3 - \left(\frac{1}{2}\right)^{2n-7} - \left(\frac{1}{2}\right)^{n-4} - \left(1 + 3.2 \cdot \left(\frac{7}{8}\right)^{n-1}\right)^2\right) \\
&\qquad > 0
\end{aligned}$$

when $n \geq 13$.

To complete the proof, the cases $4 \leq n \leq 12$ were checked using Lemma 2.2 and the values of $\mathsf{P}(A)$ computed by recursion in [1]. The (rounded) values used are listed below.

n	$\mathsf{P}(A)$
4	0.25
5	0.146484
6	0.076416
7	0.036942
8	0.017427
9	0.008309
10	0.004038
11	0.001988
12	0.000986

We can also give an upper bound for $\mathsf{P}(A \cap B)$ to show that $\lim_{n\to\infty} \mathsf{P}(A \cap B) \cdot 2^{2n-4} = 3$ and $\lim_{n\to\infty} \frac{\mathsf{P}(A\cap B) - \mathsf{P}(A)\cdot\mathsf{P}(B)}{\mathsf{P}(A\cap B)} = \frac{2}{3}$. These statements are special cases of Theorems 3.2 and 3.4 below.

3. Random orientations of $G(n, p)$

Let as usual $G(n, p)$ be the random graph in which every edge exists with probability p independently of the other edges. We also let every edge be directed in either way with equal probability independently of each other. We will call the corresponding random graph model $\vec{G}(n, p)$. For this section, let $x = p/2$ be the probability of one edge to exist and be directed in a certain way and let $y = 1 - x$ be the probability of an edge not to exist in a certain direction. We will adopt the usual notation $f \sim g$ to denote that the quotient of f and g goes to a constant. In [2] the following lemma was proven.

LEMMA 3.1 (Lemma 4.2 in [2]). *For any vertices s, a in $\vec{G}(n, p)$*

$$\mathsf{P}(A) \sim 2y^{n-1}.$$

Clearly, $\mathsf{P}(A) = \mathsf{P}(B)$. To find the relative covariance between A and B when n approaches infinity we need an estimate of $\mathsf{P}(A \cap B)$.

A set X of vertices in K_n is said to be an inset (outset) if all existing edges from $[n] \setminus X$ are directed to (from) X. Let I^X be the event that X is an inset. Let also

$$Z_k = \bigcup_{\substack{X: s \in X \\ a \notin X \\ |X| = k}} I^X \qquad \text{and} \qquad Z'_k = \bigcup_{\substack{X': t \in X' \\ b \notin X' \\ |X'| = k}} I^{X'}.$$

Now we have

$$\mathsf{P}(s \nrightarrow a) = \mathsf{P}\left(\bigcup_{k=1}^{n-1} Z_k\right)$$

and

$$\mathsf{P}(A \cap B) = \mathsf{P}(s \nrightarrow a, t \nrightarrow b) = \mathsf{P}\left(\bigcup_{k=1}^{n-1} Z_k \cap \bigcup_{k=1}^{n-1} Z'_k\right).$$

THEOREM 3.2. *For $p \in (0, 1]$ we have*

$$\lim_{n \to \infty} \frac{\mathsf{P}(A \cap B)}{y^{2n-4}} = 4 - p$$

REMARK 3.3. Exact computations indicate that this convergence is very slow for small p, see Figure 2 in Section 5.

PROOF OF THEOREM 3.2. First note that

$$\mathsf{P}(A \cap B) = \mathsf{P}\left(\bigcup_{k=1}^{n-1} Z_k \cap \bigcup_{k=1}^{n-1} Z'_k\right) = s_1 + s_2 + s_3 - s_4,$$

where

$$s_1 = \mathsf{P}\left(\bigcup_{k=3}^{n-3} Z_k \cap \bigcup_{k=1}^{n-1} Z'_k\right), \qquad s_2 = \mathsf{P}\left(\bigcup_{k=1}^{n-1} Z_k \cap \bigcup_{k=3}^{n-3} Z'_k\right),$$

$$s_3 = \mathsf{P}\left(\left(\bigcup_{k=1}^{2} Z_k \bigcup_{k=n-2}^{n-1} Z_k\right) \cap \left(\bigcup_{k=1}^{2} Z'_k \bigcup_{k=n-2}^{n-1} Z'_k\right)\right),$$

$$s_4 = \mathsf{P}\left(\bigcup_{k=3}^{n-3} Z_k \cap \bigcup_{k=3}^{n-3} Z'_k\right).$$

By symmetry $s_1 = s_2$ and clearly $s_4 < s_1$. We will write $\mathsf{P}_N(I^X)$ for $\mathsf{P}(I^X)$ with $|X| = N$. We show that s_1, s_2, s_4 are negligible compared to s_3, and give an estimate of s_3. Starting with s_1, first note that $\mathsf{P}_k(I^X) = y^{k(n-k)}$ and if $k < l \leq \frac{n}{2}$ we have $\mathsf{P}_l(I^Y) < \mathsf{P}_k(I^X)$. This gives us

$$
\begin{aligned}
s_1 = \mathsf{P}\Big(\bigcup_{k=3}^{n-3} Z_k \cap \bigcup_{k=1}^{n-1} Z'_k\Big) &\leq \mathsf{P}\Big(\bigcup_{k=3}^{n-3} Z_k\Big) \leq \sum_{k=3}^{n-3} \binom{n}{k-1} y^{k(n-k)} \\
&\leq 2 \cdot \sum_{k=3}^{K-1} \binom{n}{k-1} y^{k(n-k)} + \sum_{k=K}^{n-K} \binom{n}{k-1} y^{k(n-k)}
\end{aligned}
$$

Now, since p is fixed we may fix K such that $y^K < \frac{y^3}{2}$. The sum $\sum_{k=3}^{K-1} \binom{n}{k-1} \cdot y^{k(n-k)}$ is finite and it is $O(y^{3(n-3)})$ which is very small compared to y^{2n}, and hence negligible. Further we get

$$
\sum_{k=K}^{n-K} \binom{n}{k-1} y^{k(n-k)} < 2^n \cdot y^{K(n-K)} < 2^n \Big(\frac{y^3}{2}\Big)^{n-K} = O(y^{3n}).
$$

That is $s_1 \sim o(y^{2n})$ and analogously so is s_2 and s_4.

To estimate s_3, first consider $\mathsf{P}(Z_1 \cap Z'_2)$ as an example. In this case no edges will be directed from s. For the inset X' we have two subcases, either it contains s and t or t and another vertex (different from s, b). In the first case we get a total of y^{2n-3}, and for the second case we can choose X' in $n-3$ ways and no edges will be directed from X', this gives us $(n-3)y^{3n-9}(1-p)^2$. In the computations below it will always be the case that if three or more vertices are involved, then the probability will be negligible, i.e. $o(y^{2n})$.

We get four contributing cases which can be reduced to two by symmetry.

(1) $\mathsf{P}((Z_1 \cup Z_2) \cap (Z'_1 \cup Z'_2)) = y^{2n-4} + o(y^{2n})$.

(2) $\mathsf{P}((Z_{n-1} \cup Z_{n-2}) \cap (Z'_{n-1} \cup Z'_{n-2})) = y^{2n-4} + o(y^{2n})$.

(3) $\mathsf{P}(Z_1 \cap Z'_{n-1}) = y^{2n-3}$.

(4) $\mathsf{P}(Z_{n-1} \cap Z'_1) = y^{2n-3}$.

For (1) we see that if any other vertex than s and t is in the insets for s and t we will have conditions on at least $3n-9$ edges and thus a probability of size $o(y^{2n})$. All the interesting cases are when we have no restriction on the possible edge between s and t, and no edge must be directed from s, t to any other vertex. Note that our example above is a subset of this case. Case (2) is symmetric to (1).

For (3) no edge may be directed from s or to b, which imposes conditions on $2n-3$ edges. Case (4) is symmetric to (3). One can easily check that the remaining six possibilities, four cases symmetric to $Z_1 \cap Z'_{n-2}$ and two cases symmetric to $Z_2 \cap Z'_2$, all have probabilities of size $o(y^{2n})$ and can hence be ignored.

All together we end up with

$$\begin{aligned} 2y^{2n-4} + 2y^{2n-3} + o(y^{2n}) &= 2y^{2n-4}\left(1 + \left(1 - \frac{p}{2}\right)\right) + o(y^{2n}) \\ &= y^{2n-4}(4-p) + o(y^{2n}), \end{aligned}$$

as claimed.

THEOREM 3.4. *For fixed $p \in [0, 1]$*

$$\lim_{n\to\infty} \frac{\mathsf{P}(A \cap B) - \mathsf{P}(A)\mathsf{P}(B)}{\mathsf{P}(A \cap B)} = \frac{p(3-p)}{4-p}$$

PROOF. Follows from Lemma 3.1 and Theorem 3.2.

COROLLARY 3.5. *For a fixed $p \in (0, 1]$, the correlation between A and B is always positive for sufficiently large n.*

We believe that something stronger is true and we offer the following conjecture, which is supported by our calculations in Section 5.

CONJECTURE 1. *For any $n \geq 4$ and $p \in (0, 1]$, the events $\{s \to a\}$ and $\{t \to b\}$ are always positively correlated.*

4. Random orientations of $G(n, m)$

In this section we study the same problem on the random graph $G(n, m)$ where each simple graph with m edges and n vertices is equally likely. We will also here let every edge have an independent direction and call the combined probability space $\vec{G}(n, m)$. Again, let $y = 1 - \frac{p}{2}$, let further $q(l) = q(l; n, m)$ be the probability that l fixed edges in K_n does not exist in $\vec{G}(n, m)$ with given directions. In $\vec{G}(n, p)$ this corresponds to y^l. If nothing else is written the graph considered in this section is always $\vec{G}(n, m)$.

In [2] the following lemma was proven.

LEMMA 4.1 (Janson, Lemma 3.2 in [2]). *Suppose that $0 \leq m = m(n) \leq \binom{n}{2}$. Then with $p = p(n) = m(n)/\binom{n}{2}$, as $n \to \infty$,*

$$q(l; n, m) \sim y^l \exp\left(-\left(\frac{l}{n}\right)^2 \frac{p(1-p)}{(2-p)^2}\right),$$

and for any l, n, m we have $q(l; n, m) \leq q'(l; n, p)$.

This lemma together with the proof of Theorem 3.2 gives us an analogue result of Theorem 3.2 for $\vec{G}(n,m)$.

THEOREM 4.2. *In the case of* $\vec{G}(n,m)$ *for fixed* $0 < p < 1$ *we have*

$$\mathsf{P}(A \cap B) \sim 2y^{2n-4} \exp\left(-4\frac{p(1-p)}{(2-p)^2}\right) + 2y^{2n-3} \exp\left(-4\frac{p(1-p)}{(2-p)^2}\right)$$

Also we need the following lemma.

LEMMA 4.3 (Lemma 4.3 in [2]). *For fixed* $0 < p < 1$

$$\mathsf{P}(A) \sim 2y^{n-1} \exp\left(-\frac{p(1-p)}{(2-p)^2}\right).$$

We are now ready to state and prove the main theorem of this section.

THEOREM 4.4. *For fixed* $0 < p < 1$ *and sufficiently large* n*, the events* A *and* B *are positively correlated in* $\vec{G}(n,m)$ *and the relative covariance is*

$$\sim 1 - \frac{2\left(1-\frac{p}{2}\right)^2}{2-\frac{p}{2}} \cdot \exp\left(2\frac{p(1-p)}{(2-p)^2}\right).$$

PROOF. We rewrite the relative covariance as

$$\frac{\mathsf{P}(A \cap B) - \mathsf{P}(A)\mathsf{P}(B)}{\mathsf{P}(A \cap B)} = 1 - \frac{\mathsf{P}(A)\mathsf{P}(B)}{\mathsf{P}(A \cap B)}.$$

As n approaches ∞, Theorem 4.2 and Lemma 4.3 gives

$$\begin{aligned}\frac{\mathsf{P}(A)\mathsf{P}(B)}{\mathsf{P}(A \cap B)} &\sim \frac{4y^{2n-2}\exp\left(-2\frac{p(1-p)}{(2-p)^2}\right)}{2y^{2n-4}\exp\left(-4\frac{p(1-p)}{(2-p)^2}\right)\left(2-\frac{p}{2}\right)} \\ &= \frac{2\left(1-\frac{p}{2}\right)^2}{\left(2-\frac{p}{2}\right)} \exp\left(2\frac{p(1-p)}{(2-p)^2}\right)\end{aligned}$$

Let us denote this expression by f. It remains to prove that f is less than one when $0 < p < 1$. This can be proven by using the derivative of f. We have that

$$f'(p) = e^{\frac{2(1-p)p}{(2-p)^2}} \frac{p^3 + 4p^2 - 8}{(4-p)^2(2-p)}.$$

The theorem follows since the derivative is negative in this interval and $f(0) = 1$.

We conjecture the covariance to be positive at all times.

CONJECTURE 2. *The events A and B are positivelly correlated in* $\vec{G}(n, m)$ *for all* p *and all* n.

The covariance in $\vec{G}(n, p)$ is always less than the covariance in $\vec{G}(n, m)$ (see [2]). So the conjecture would also imply the correlation to be positive in $\vec{G}(n, p)$.

5. Exact recursion in $\vec{G}(n, p)$

In this section we will give an exact recursion to compute

$$\mathsf{P}_{\vec{G}(n,p)}(a \nrightarrow s, t \nrightarrow b).$$

Together with the recursion given for $f_n(p) := \mathsf{P}_{\vec{G}(n,p)}(a \nrightarrow s)$ in [2] we will be able compute the covariance for n as a rational function in p. Our computations for $n \leq 34$, using Maple, supports our Conjecture 1 that the covariance is always positive, see Figure 1.

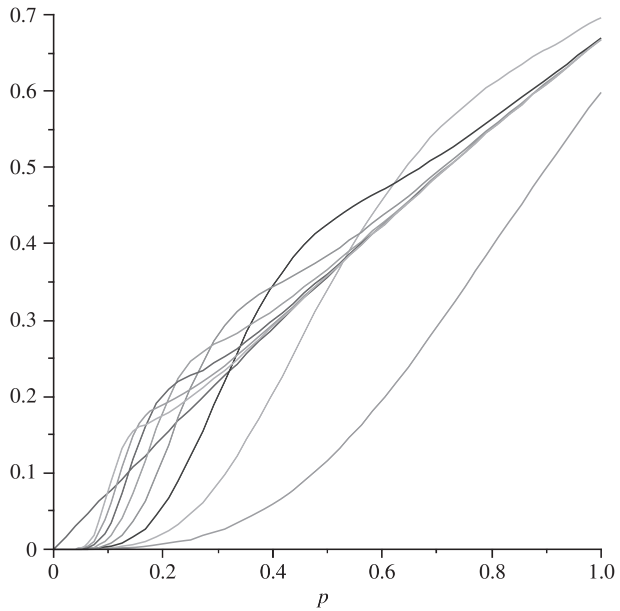

FIGURE 1. The relative covariance $\frac{\mathsf{P}(a \nrightarrow s, t \nrightarrow b) - \mathsf{P}(a \nrightarrow s)\mathsf{P}(t \nrightarrow b)}{\mathsf{P}(a \nrightarrow s, t \nrightarrow b)}$ in $\vec{G}(n, p)$ for going from right to left $n = 6, 10, 14, 18, 22, 26, 30, 34$, and the asymptote $p(3 - p)/(4 - p)$. All curves are positive for $0 < p \leq 1$.

For a vertex $v \in V(G)$, let $\overrightarrow{C}_v \subseteq V(G)$ be the (random) set of all vertices u for which there is a directed path from v to u. We say that $\overrightarrow{C}_v$ is the *out-cluster*

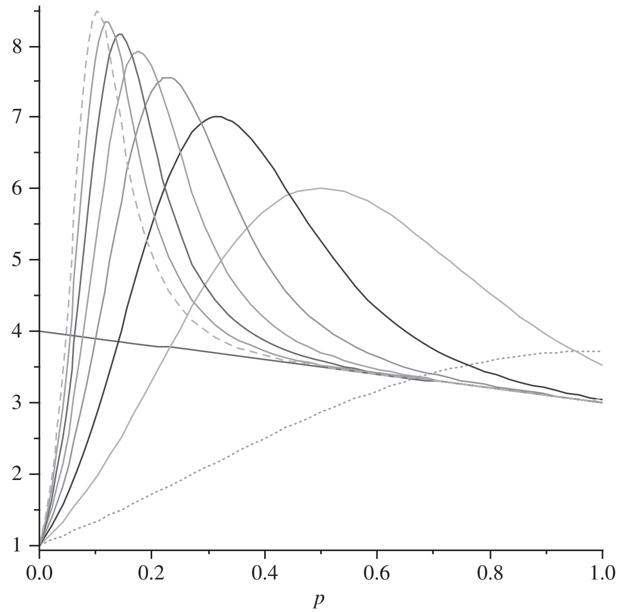

FIGURE 2. Plots of $\frac{\mathsf{P}(a\not\to s, t\not\to b)}{y^{2n-4}}$ in $\vec{G}(n,p)$ for going from right to left $n = 6$ (dotted), 10, 14, 18, 22, 26, 30, 34 (dashed), and the asymptote $4 - p$. See Theorem 3.2.

from v. Let analogously the *in-cluster*, $\overleftarrow{C}_v \subseteq V(G)$ be the (random) set of all vertices u for which there is a directed path from u to v. We will use the convention that $v \in \overleftarrow{C}_v \cap \overrightarrow{C}_v$. Let as before $y := 1 - p/2$ be the probability that an edge does not exist with a certain direction, and let $q := 1 - p$ be the probability that there is no edge at all.

For $n \geq 1$, $s \in S \subseteq [n]$ and $|S| = k$ define:

$$d_p(n,k) := \mathsf{P}_{\vec{G}(n,p)}(\overrightarrow{C}_s = S),$$

where in particular $d_p(1,1) = 1$. A recursion to compute $d_p(n,k)$ as a polynomial in p was given in [2].

LEMMA 5.1 (Lemma 5.1 in [2]). *We have the following recursions*

$$d_p(n,k) = d_p(k,k)y^{k(n-k)}, \qquad \textit{for} \quad n > k \geq 1,$$

and

$$d_p(k,k) = 1 - \sum_{i=1}^{k-1} \binom{k-1}{i-1} d_p(i,i) y^{i(k-i)}.$$

Note that, by symmetry, also $\mathsf{P}_{\vec{G}(n,p)}(\overleftarrow{C}_s = S) = d_p(n,k)$.

It turns out that the following quantity is possible to compute recursively and enables us to compute $h_n(p)$. For $n \geq 2$, $t \in T \subseteq [n]$, $a \in A \subseteq [n]$ with $|T| = \tau$, $|A| = \alpha$ and $|[n] \setminus (A \cup T)| = r$ define:

$$N_p(n, \tau, \alpha, r) := \mathsf{P}_{\vec{G}(n,p)}(\overrightarrow{C}_t = T, \overleftarrow{C}_a = A),$$

where in particular $N_p(2, 2, 2, 0) = x$ and $N_p(2, 1, 1, 0) = y$.

We will use the variable j for the size of the intersection $|A \cap T|$. If there is any intersection between A and T then $a, t \in A \cap T$, so in particular $j = \alpha + \tau - (n - r)$ can never be 1.

THEOREM 5.2. *We have the following recursions for N_p, where $\tau + \alpha > n - r \geq \tau, \alpha$ and $\tau, \alpha \geq 1$*

(i) $N_p(n, \tau, \alpha, r) = N_p(n - r, \tau, \alpha, 0) q^{r(r+\tau+\alpha-n)} y^{r(2n-2r-\tau-\alpha)}$
for $r > 0$,

(ii) $N_p(n, \tau, \alpha, r) = N_p(n, \alpha, \tau, r)$,

(iii) $$N_p(n, \tau, \alpha, 0) = \sum_{\zeta=1}^{n-\tau} \binom{n - \tau - 1}{\zeta - 1} N_p(n - \zeta, \tau, \alpha - \zeta, 0)\, d_p(\zeta, \zeta) \cdot q^{(\zeta-1)(\alpha+\tau-n)} y^{(\zeta-1)(2n-\tau-\alpha-\zeta)} (y^\tau - y^{2n-\alpha-\tau-\zeta} q^{\alpha+\tau-n}),$$
for $n > \tau, n \geq \alpha \geq 2$, $j \geq 2$,

(iv) $$N_p(n, \tau, \alpha, 0) = \sum_{\zeta=1}^{\alpha-1} \binom{\alpha - 2}{\zeta - 1} N_p(n - \zeta, \tau, \alpha - \zeta, 0)\, d_p(\zeta, \zeta) \cdot y^{(\zeta-1)(\tau+\alpha-\zeta)} y^\tau (1 - y^{\alpha-\zeta}),$$
for $\alpha \geq 2$, $j = 0$ i.e. $n = \tau + \alpha$,

(v) $$N_p(n, n, n, 0) = 1 - \sum_{j=2}^{n-1} \binom{n - 2}{j - 2} \cdot \sum_{\tau=j}^{n} \binom{n - j}{\tau - j} \sum_{\alpha=j}^{n-\tau+j} \binom{n - \tau}{\alpha - j} N_p(n, \tau, \alpha, n - \alpha - \tau + j) - \sum_{\tau=1}^{n} \binom{n - 2}{\tau - 1} \sum_{\alpha=1}^{n-\tau} \binom{n - \tau - 1}{\alpha - 1} N_p(n, \tau, \alpha, n - \alpha - \tau).$$

PROOF. For the first equation we have $r > 0$, thus $[n] \setminus (A \cup T)$ is non-empty and no vertex in that set must not have any edge directed to A or from T. Hence there must be no edge at all to $A \cap T$, which gives probability

$q^{|[n]\setminus A\cup T|\cdot|A\cap T|} = q^{r(r+\tau+\alpha-n)}$. There must not be any edge directed to $(A \setminus T)$ and there must not be any edge directed from $(T \setminus A)$. This gives the probability of $y^{|[n]\setminus A\cup T|\cdot|(A\setminus T)\cup(T\setminus A)|} = y^{r(2n-2r-\tau-\alpha)}$.

The second equation is obtained from the symmetry of reversing all directions and switching the roles of a and t.

For equation (iii) and (iv), we pick a vertex $z \in A \setminus T$, such a vertex exists by the assumption $n > \tau$ and $r = 0$. Let G be any directed graph on n vertices with $\overrightarrow{C}_t = T$ and $\overleftarrow{C}_a = A$. If we remove vertex z and all its edges from G the resulting graph will still have $\overrightarrow{C}_t = T$ since $z \notin T$, whereas $\overleftarrow{C}_s = A \setminus Z$, for some $Z \subseteq A \setminus \{a\}$ such that $Z \cap T = \emptyset$. This follows from the fact that the vertices in Z are those that have a path to a only via z and no vertex in T has a directed path leading to z by assumption. Let $\zeta = |Z|$ and sum over all possible Z. The probability is $N_p(n-\zeta, \tau, \alpha-\zeta, 0)$ that the subgraph on $[n] \setminus Z$ is as needed. The subgraph on Z must have $\overleftarrow{C}_z = Z$ which has probability $d_p(j, j)$. Let us first consider equation (iii) when $j = \tau + \alpha - n \geq 2$.

There must not be any edge between $T \cap A$ and $Z \setminus \{z\}$, since the vertices of the latter do not belong to T and have all directed paths via z. This gives a factor $q^{(\zeta-1)(\alpha+\tau-n)}$. No vertex of $Z \setminus \{z\}$ can have an edge to $A \setminus (T \cup Z)$ or from $T \setminus A$, which gives a factor $y^{(\zeta-1)(2n-\tau-\alpha-\zeta)}$. Finally, we must consider the edges of z. The main condition is that there must not be any edge from T to z. However, there must be at least one edge edge directed from z to $A \setminus Z$. This give the last factor. The case of equation (iv) when $j = 0$ is easier and obtained similarly.

Equation (v) follows from the fact that for fixed n

$$\sum_{T,A:a\in A,t\in T\subseteq[n]} \mathsf{P}_{\vec{G}(n,p)}(\overrightarrow{C}_t = T, \overleftarrow{C}_a = A) = 1.$$

Here $j = |A \cap T|$ and recall that $j = 1$ is not an option.

THEOREM 5.3. *We have the following expression for* $\mathsf{P}_{\vec{G}(n,p)}(a \not\twoheadrightarrow s, t \not\twoheadrightarrow b)$.

$$\begin{aligned}\mathsf{P}_{\vec{G}(n,p)}(a \not\twoheadrightarrow s, t \not\twoheadrightarrow b) &= \sum_{j=2}^{n-2} \binom{n-4}{j-2} \\ &\cdot \Bigg[\sum_{\tau=j}^{n-2} \binom{n-2-j}{\tau-j} \sum_{\alpha=j}^{n-\tau+j-1} \binom{n-\tau-1}{\alpha-j} N_p(n, \tau, \alpha, n-\alpha-\tau+j) \\ &+ \sum_{\tau=j+1}^{n-1} \binom{n-2-j}{\tau-j-1} \sum_{\alpha=j}^{n-\tau+j} \binom{n-\tau}{\alpha-j} N_p(n, \tau, \alpha, n-\alpha-\tau+j) \Bigg]\end{aligned}$$

$$+\sum_{\tau=1}^{n-3}\binom{n-4}{\tau-1}\sum_{\alpha=1}^{n-\tau-1}\binom{n-\tau-2}{\alpha-1}N_p(n,\tau,\alpha,n-\alpha-\tau)$$

$$+\sum_{\tau=2}^{n-2}\binom{n-4}{\tau-2}\sum_{\alpha=j}^{n-\tau}\binom{n-\tau-1}{\alpha-1}N_p(n,\tau,\alpha,n-\alpha-\tau).$$

PROOF. The equation for $\mathsf{P}_{\vec{G}(n,p)}(a \nrightarrow s, t \nrightarrow b)$ is obtained by summing over all possible pairs A, T such that $s \notin A, b \notin T$. Again $j = |A \cap T|$ and the formula is split into four cases depending on if $s \in T$ or not and if $j = 0$ or not.

Note that in $\vec{G}(n, p)$ the functions $\mathsf{P}(s \nrightarrow a)$ and $\mathsf{P}(s \nrightarrow a, t \nrightarrow b)$ are polynomials in p and hence continuous.

6. The Quenched version

For the quenched version the correlation between A and B is computed for each graph in $G(n, p)$ ($G(n, m)$) in the probability space of edge orientations and then the expected value is taken over all graphs.

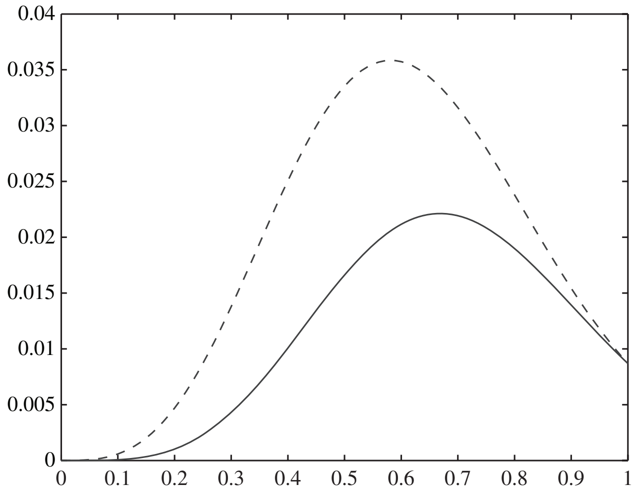

FIGURE 3. The covariance for $G(6, p)$. The dashed curve represents the annealed case and the continuous one the quenched case.

We computed the covariance between A and B for $G(n, p)$ as a function over p, in both the annealed and the quenched case for $n \leq 6$. The two cases looks quite similar, see Figure 3. Note that for $n \leq 6$ the expected covariances are positive also for small p and we conjecture it to be positive for all n. This differs from the behavior for the similar problem studied in Section 9 in [2]. It would also be interesting to find an analogue to Theorem 3.4 for the quenched version.

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