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SPECTRA OF SUB-DIRAC OPERATORS ON CERTAIN NILMANIFOLDS

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Abstract

We study sub-Dirac operators associated to left-invariant bracket-generating sub-Riemannian structures on compact quotients of nilpotent semi-direct products $G = \mathsf{R}^n \rtimes_A \mathsf{R}$. We prove that these operators admit an L^2-basis of eigenfunctions. Explicit examples of this type show that the spectrum of these operators can be non-discrete and that eigenvalues may have infinite multiplicity. In this case the sub-Dirac operator is neither Fredholm nor hypoelliptic.

1. Introduction

Spectra of sub-Laplace operators on sub-Riemannian manifolds are intensely studied. Especially interesting is the case where the distribution defining the sub-Riemannian structure is bracket generating, what we shall assume in the following. In this case the sub-Laplacian is known to be hypoelliptic [13].

Many explicit calculations of the spectrum have been done in the situation where the underlying manifold is a compact Lie group or a quotient of a Lie group by a discrete cocompact subgroup, see, for example, [3], [4], [5], [19]. In [3], [4], [19] the authors study spectral properties of sub-Laplace operators on nilpotent groups of step two and on compact quotients by discrete subgroups. They determine the heat kernels of these operators. This allows an explicit determination of the spectrum of the sub-Laplacian, which is discrete in this situation.

Less is known about differential operators on functions that contain partial derivatives of arbitrary order. However, some special cases are studied. For instance, if the geometry is locally close to sub-Riemannian Heisenberg groups, there are results of van Erp [8] including an index theorem for differential operators that have a Rockland model operator.

Here we will consider a geometric partial differential operator that acts on sections of a vector bundle rather than on functions. More exactly, we will study spectra of sub-Riemannian analogs of the classical Dirac operator. These operators are defined as follows. Let $(M, \mathscr{H}, g)$ be a sub-Riemannian manifold,

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$\dim \mathscr{H} = d$. Suppose that ∇ is a metric connection on $\mathscr{H}$. Moreover, assume that $\mathscr{H}$ is oriented and that the bundle of oriented orthonormal frames of $\mathscr{H}$ admits a reduction to $\mathrm{Spin}(d)$. Such a reduction will be called a spin structure of $\mathscr{H}$. Then we can associate a spinor bundle S with this spin structure. Moreover, using the connection ∇ we can define a sub-Riemannian Dirac operator, which acts on sections in S.

In the definition of the sub-Dirac operator the following difficulty occurs: In contrast to the Riemannian case where we have the Levi-Civita connection as a preferred connection, in general, there is no connection canonically associated with a sub-Riemannian structure. Only in special geometric situations a canonical connection exists. Hence the definition of the sub-Dirac operator depends on the choice of the connection on $\mathscr{H}$.

In general, i.e., for arbitrary metric connections in $\mathscr{H}$, the sub-Dirac is not symmetric. We will characterize the symmetry of this operator by a simple condition on the connection.

Several variants of sub-Dirac operators can be found in the literature. On sub-Riemannian manifolds of contact type Petit defined an operator of this kind using the generalized Tanaka-Webster connection and Spin^c-structures, see [18]. This operator is called Kohn-Dirac operator.

On manifolds with foliations Dirac operators are studied by Brüning, Kamber, Prokhorenkov and Richardson, see e.g. [6], [20]. If $\mathscr{F}$ is a Riemannian foliation and if $\mathscr{V}$ denotes the tangent distribution to $\mathscr{F}$, then the transversal Dirac operator defined in [6], Section 2.4, coincides with our sub-Dirac operator on the spinor bundle associated with $\mathscr{H} := \mathscr{V}^\perp$. However, while the transversal Dirac operator in [6] is considered as an operator on so-called basic sections only, we want to let the sub-Riemannian Dirac operator act on arbitrary sections of the spinor bundle S.

Finally, we want to mention the paper [2] by Ammann and Bär although it does not consider sub-Riemannian structures explicitly. However, in a certain sense, one can consider the limit case of collapsing circle bundles as a sub-Riemannian structure.

Studying the sub-Riemannian Dirac operator the following natural questions arise: Is this operator hypoelliptic? Which structure does its spectrum have? How does the spectrum depend on the sub-Riemannian geometry of the manifold and on the spin structure of $\mathscr{H}$? How do the sub-Dirac operator and its spectrum depend on the chosen connection?

Here we will answer some of these questions for sub-Dirac operators on certain nilmanifolds. More precisely, we study manifolds of the form $M = \Gamma \setminus G$ where $G = \mathsf{R}^n \rtimes_A \mathsf{R}$ is a semi-direct product defined by a one-parameter subgroup $A(t)$ of unipotent matrices in $\mathrm{GL}(n, \mathsf{R})$ and Γ is the subgroup $\mathsf{Z}^n \rtimes_A \mathsf{Z}$. These manifolds M can be interpreted as a suspension of the diffeomorphism of the torus $\mathsf{R}^n/\mathsf{Z}^n$ induced by $A(1)$. This is also the starting point of [14] where the spectrum of the Laplacian on left-invariant differential forms on M is considered. Our sub-Dirac operators will be associated with sub-Riemannian structures $(\dot{\mathscr{H}}, \dot{g})$ on $\Gamma \backslash G$ coming from a left-invariant and bracket-generating distribution $(\mathscr{H}, g)$ on G. We choose a metric connection in H such that D is symmetric.

Our approach is to give an explicit decomposition of the regular representation of G. Roughly speaking, it turns out that the sub-Dirac operator is unitarily equivalent to an orthogonal sum of elliptic operators on the real line, each having a discrete spectrum. This shows that D on $\Gamma \setminus G$ has pure point spectrum.

We apply our results to compute the spectrum of D explicitly for two classes of two-step nilmanifolds of the above form. First we consider three-dimensional Heisenberg manifolds. Secondly, we study a class of five-dimensional two-step nilpotent nilmanifolds with a three-dimensional distribution. Finally, we discuss a three-step nilpotent example of dimension four with a two-dimensional distribution. In this case the spectrum can be expressed in terms of the spectra of the family of operators $P_c = \partial_t^2 + (t^2 + c)^2 \pm 2t$, $c \in \mathsf{R}$. In all three examples, the multiplicities of the eigenvalues of D can be read off from the coadjoint orbit picture.

The examples will show that

- the spectrum of the sub-Dirac operator on a compact 2-step nilmanifold is not necessarily a discrete subset of R; its eigenvalues may have infinite multiplicity, contrary to the results for the spectrum of the sub-Laplacian on compact 2-step nilmanifolds;
- the kernel of the sub-Dirac operator on such manifolds can be infinite-dimensional;
- in general, sub-Dirac operators are not Rockland operators in the following sense.

Let be given a left-invariant sub-Riemannian structure on a Lie group G. The sub-Dirac operator acting on $C^\infty(G, \Delta)$ corresponds to an element $D_{\mathrm{sub}} \in \mathscr{U}(\mathfrak{g}) \otimes \mathrm{End}(\Delta)$. As a natural generalization of the Rockland condition for operators acting on functions introduced in [21], one might take that $(d\rho \otimes \mathrm{id})(D_{\mathrm{sub}})$ is injective for every irreducible representation ρ of G. However, our explicit examples show that this condition does not hold true in general. For example, it is not satisfied for the sub-Dirac operator associated to any left-invariant sub-Riemannian structure $(\mathscr{H}, g, P_{\mathrm{Spin},\varepsilon}, \nabla, \mu)$ on the three-dimensional Heisenberg group, see Section 4.2.

The results on the spectrum mentioned above imply that there exists a uniform discrete subgroup Γ and a left-invariant sub-Riemannian structure

$(\mathscr{H}, g, P_{\mathrm{Spin}}, \nabla, \mu)$ on a simply connected two-step nilpotent Lie group G with a distribution of codimension two in the tangent bundle of G such that the associated sub-Dirac operator D on $\Gamma \setminus G$ is symmetric but neither hypoelliptic nor Fredholm (see Section 4.5).

It follows that in this situation $D_{\mathrm{sub}}^2 = \Delta_{\mathrm{sub}} + P$, P a differential operator of first order, is not hypoelliptic. In particular, we observe that hypoellipticity is not preserved by adding operators of lower order.

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2. Sub-Riemannian Dirac operators

2.1. Definition of sub-Dirac operators

Let M be a smooth manifold and let $\mathscr{H} \subset TM$ be a smooth distribution, where $\dim \mathscr{H}_x = d$ for all $x \in M$. Let $\Gamma(\mathscr{H})$ denote the space of smooth sections of $\mathscr{H}$. We assume that $\mathscr{H}$ is bracket-generating. That means, that for each $x \in M$ there is a $J \in \mathsf{N}$ such that the sequence

$$\Gamma_0 := \Gamma(\mathscr{H}), \qquad \Gamma_{j+1} := \Gamma_j + [\Gamma_0, \Gamma_j]$$

satisfies $\{X(x) \mid X \in \Gamma_J\} = T_x M$. If g is a Riemannian metric on $\mathscr{H}$, then the pair $(\mathscr{H}, g)$ is called a sub-Riemannian structure on M and $(M, \mathscr{H}, g)$ is called a sub-Riemannian manifold.

A sub-Riemannian manifold is said to be regular if for each $j = 1, \dots, J$ the dimension of $\{X(x) \mid X \in \Gamma_j\}$ does not depend on the point $x \in M$. Examples are contact structures and left-invariant sub-Riemannian structures on Lie groups. In case of a regular sub-Riemannian structure one has the notion of an intrinsic volume form, i.e., of a nowhere vanishing section of $\Lambda^{\dim M}(TM)$ naturally defined by the sub-Riemannian structure so that one can define an intrinsic Laplacian, see [1].

Let $\nabla : \Gamma(\mathscr{H}) \otimes \Gamma(\mathscr{H}) \to \Gamma(\mathscr{H})$ be a metric connection on $\mathscr{H}$. Note that here we consider only derivations by vector fields in $\mathscr{H}$. Suppose that $\mathscr{H}$ is oriented and that it admits a spin structure, i.e., that there is a $\mathrm{Spin}(d)$-reduction $P_{\mathrm{Spin}}(\mathscr{H})$ of the principal $\mathrm{SO}(d)$-bundle $P_{\mathrm{SO}}(\mathscr{H})$ of oriented orthonormal frames of $(\mathscr{H}, g)$. We consider the complex representation of $\mathrm{Spin}(d)$ which is obtained by restriction of (one of) the complex irreducible representation(s) of the Clifford algebra $\mathscr{C}l(d) := \mathscr{C}l(\mathsf{R}^d)$. We will call it spinor representation and denote it by Δ_d. The associated bundle $P_{\mathrm{Spin}}(\mathscr{H}) \times_{\mathrm{Spin}(d)} \Delta_d$ is called spinor bundle S of $(\mathscr{H}, g)$. The space of smooth sections in S is denoted by $\Gamma(S)$. The connection ∇ defines a connection $\nabla^S : \Gamma(\mathscr{H}) \times \Gamma(S) \to \Gamma(S)$ in the following way. Let $s_1, \ldots, s_d$ be a local orthonormal frame of $\mathscr{H}$ and consider the local connection forms $\omega_{ij} = g(\nabla s_i, s_j)$. Then we define

$$\nabla^S_X \varphi := X(\varphi) + \frac{1}{2} \sum_{i<j} \omega_{ji}(X) s_i \cdot s_j \cdot \varphi,$$

where '$\cdot$' denotes the Clifford multiplication.

Now we can define a sub-Riemannian Dirac operator, or sub-Dirac operator for short, by

$$D = \sum_i s_i \cdot \nabla^S_{s_i} : \Gamma(S) \longrightarrow \Gamma(S), \tag{1}$$

where again $s_1, \ldots, s_d$ is a local orthonormal frame of $\mathscr{H}$. Note, that the definition of D depends on the choice of the connection ∇ on $\mathscr{H}$ and that, in general, this choice is far from being canonical in contrast to the Riemannian case, where we have the Levi-Civita connection as a preferred connection.

A large class of metric connections in $\mathscr{H}$ can be obtained in the following way. Suppose we are given a further distribution $\mathscr{V} \subset TM$ such that $TM = \mathscr{H} \oplus \mathscr{V}$. Then this decomposition of TM gives us a projection $\mathrm{pr} : TM \to \mathscr{H}$ and we can define a connection ∇ by the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad + g(\mathrm{pr}[X,Y], Z) - g(\mathrm{pr}[X,Z], Y) - g(\mathrm{pr}[Y,Z], X), \end{aligned} \tag{2}$$

where $X, Y, Z \in \Gamma(\mathscr{H})$. In this case ∇ is uniquely determined by the vanishing of $\nabla_X Y - \nabla_Y X - \mathrm{pr}[X,Y]$. The latter condition is equivalent to saying that the torsion $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ is vertical for all $X, Y \in \Gamma(\mathscr{H})$.

2.2. Symmetry of the sub-Dirac operator

Let $(M, \mathscr{H}, g)$ be an oriented sub-Riemannian manifold and ω an arbitrary volume form on M. Define the divergence of a vector field X on M by $\mathscr{L}_X \omega = (\operatorname{div} X) \cdot \omega$. Let ∇ be a metric connection on $\mathscr{H}$. Suppose that $\mathscr{H}$ admits a spin structure and define $D : \Gamma(S) \to \Gamma(S)$ as above. Let $\langle \cdot, \cdot \rangle$ be a hermitian inner product on Δ_d for which the Clifford multiplication is antisymmetric. This inner product is unique up to scale. It induces a hermitian inner product on S, which together with ω gives an L^2-inner product $(\cdot, \cdot)$ on the space $\Gamma_0(S)$ of sections in S with compact support.

It is easy to find examples of three-dimensional Heisenberg manifolds with two-dimensional distribution $\mathscr{H}$ and metric connection for which D is not symmetric, see Section 4.2.

The following lemma states that the sub-Dirac operator is symmetric if and only if the divergence defined by the sub-Riemannian structure coincides with the divergence given by the connection, compare also [9] for the Riemannian case.

LEMMA 2.1. *Under the above conditions, D is symmetric if and only if*

$$\operatorname{div} X = \sum_{i=1}^{d} g(\nabla_{s_i} X, s_i) \tag{3}$$

holds for all $X \in \Gamma(\mathscr{H})$ and for one (and therefore every) local orthonormal basis $s_1, \dots, s_d$ of $\mathscr{H}$.

If, in addition, $\mathscr{V}$ is a complement of $\mathscr{H}$ in TM and ∇ is defined as in (2)*, then* (3) *is equivalent to the following condition. For one* (*and therefore for all*) *sets $\{\xi_1, \dots, \xi_l\}$, $l = \dim M - k$, of local sections of $\mathscr{V}$ that satisfy $\omega(s_1, \dots, s_d, \xi_1, \dots, \xi_l) = 1$ the equation*

$$\eta_1([X, \xi_1]) + \cdots + \eta_l([X, \xi_l]) = 0$$

holds for all $X \in \Gamma(\mathscr{H})$, where $\eta_1, \dots, \eta_l \in \Gamma(T^*M)$ are defined to be zero on $\mathscr{H}$ and dual to $\xi_1, \dots, \xi_l$.

In particular, if $\operatorname{codim} \mathscr{H} = 1$*, then D is symmetric if and only if $[\Gamma(\mathscr{H}), \xi_1] \subset \Gamma(\mathscr{H})$.*

PROOF. Consider sections $\varphi, \psi \in \Gamma_0(S)$ and define $f : \mathscr{H} \to \mathsf{C}$ by

$$f(w) := \langle \varphi, w \cdot \psi \rangle. \tag{4}$$

Moreover, define $u \in \Gamma_0(\mathscr{H} \otimes \mathsf{C})$ by

$$g^{\mathsf{C}}(u, w) = f(w) \tag{5}$$

for all $w \in \Gamma(\mathscr{H})$, where g^{C} denotes the the complex bilinear extension of g. Choose a local orthonormal frame $s_1, \dots, s_d$ of $\mathscr{H}$. Then

$$\langle D\varphi, \psi \rangle - \langle \varphi, D\psi \rangle = \sum_{i=1}^{d} \big(f(\nabla_{s_i} s_i) - s_i(f(s_i)) \big) = \sum_{i=1}^{d} g^{\mathsf{C}}(\nabla_{s_i} u, s_i),$$

thus

$$\begin{aligned}(D\varphi, \psi) - (\varphi, D\psi) &= \int_M \Big(\sum_{i=1}^{d} g^{\mathsf{C}}(\nabla_{s_i} u, s_i) \Big) \omega \\ &= \int_M \Big(\sum_{i=1}^{d} g^{\mathsf{C}}(\nabla_{s_i} u, s_i) - \operatorname{div}(u) \Big) \omega.\end{aligned}$$

In particular, (3) is sufficient for the symmetry of D. On the other hand, any section $u_1 \in \Gamma_0(\mathscr{H})$ is the real part of a section $u \in \Gamma_0(\mathscr{H} \otimes \mathbb{C})$ that satisfies (4) and (5) for some $\varphi, \psi \in \Gamma_0(S)$. Indeed, choose ψ such that $\langle \psi(x), \psi(x) \rangle = 1$ for all $x \in \operatorname{supp} u_1$ and put $\varphi := u_1 \cdot \psi$. Define u by (4) and (5). Then

$$g(u_1, w) = \mathrm{Re}\langle u_1 \cdot \psi, w \cdot \psi \rangle = \mathrm{Re}\langle \varphi, w \cdot \psi \rangle = \mathrm{Re}\, f(w)$$

for all $w \in \Gamma(\mathscr{H})$, hence $u_1 = \mathrm{Re}\, u$. Consequently, the symmetry of D implies

$$\int_M \left(\sum_{i=1}^{d} g(\nabla_{s_i} u, s_i) - \mathrm{div}(u) \right) \omega = 0$$

for all $u \in \Gamma_0(\mathscr{H})$. Since the integrand is $C_0^\infty(M)$-linear in u, Equation (3) follows.

The second part of the lemma now follows from

$$\begin{aligned}
\mathrm{div}(u) &= (\mathscr{L}_u \omega)(s_1, \ldots, s_d, \xi_1, \ldots, \xi_l) \\
&= -\sum_{i=1}^{d} \omega(s_1, \ldots, [u, s_i], \ldots, s_d, \xi_1, \ldots, \xi_l) \\
&\quad - \sum_{j=1}^{l} \omega(s_1, \ldots, s_d, \xi_1, \ldots, [u, \xi_j], \ldots, \xi_l) \\
&= -\sum_{i=1}^{d} g(\mathrm{pr}[u, s_i], s_i) - \sum_{j=1}^{l} \eta_j([u, \xi_j]) \\
&= \sum_{i=1}^{d} g(\nabla_{s_i} u, s_i) - \sum_{j=1}^{l} \eta_j([u, \xi_j]),
\end{aligned}$$

where the last equality is a consequence of Equation (2).

2.3. Sub-Dirac operators on Lie groups and compact quotients

Let G be a simply connected Lie group and $\Gamma \subset G$ a uniform discrete subgroup. Let ω denote the volume form of G defining the Haar measure. Let $\mathscr{H} \subset TG$ be a left-invariant oriented distribution and g a left-invariant Riemannian metric on $\mathscr{H}$. Obviously, there exist left-invariant positively oriented orthonormal vector fields $s_1, \ldots, s_d$ which span $\mathscr{H}$. In particular, the frame bundle $P_{\mathrm{SO}}(\mathscr{H})$ is trivial and the unique spin structure of $\mathscr{H}$ equals $P_{\mathrm{Spin}}(\mathscr{H}) = G \times \mathrm{Spin}(d)$. Let ∇ be a left-invariant connection on $\mathscr{H}$.

The triple $(\mathscr{H}, g, \nabla)$ induces a sub-Riemannian structure on $\Gamma \setminus G$, which we will denote by $(\dot{\mathscr{H}}, \dot{g}, \dot{\nabla})$. The frame bundle $P_{\mathrm{SO}}(\dot{\mathscr{H}})$ can be identified with

$$P_{\mathrm{SO}}(\dot{\mathscr{H}}) = G \times_\Gamma \mathrm{SO}(d),$$

where Γ acts by left multiplication on G and trivially on $\mathrm{SO}(d)$. There is a one-to-one correspondence between homomorphisms $\varepsilon : \Gamma \to \mathbb{Z}_2 = \{0, 1\}$ and spin structures of $\dot{\mathscr{H}}$ given by

$$\varepsilon \longmapsto P_{\mathrm{Spin},\varepsilon}(\dot{\mathscr{H}}) = G \times_\Gamma \mathrm{Spin}(d),$$

where $\gamma \in \Gamma$ acts by multiplication by $e^{i\pi\varepsilon(\gamma)}$ on $\mathrm{Spin}(d)$. Spinor fields are sections of the associated spinor bundle $P_{\mathrm{Spin},\varepsilon}(\dot{\mathscr{H}}) \times_{\mathrm{Spin}(d)} \Delta_d \cong G \times_\Gamma \Delta_d$ or, equivalently, maps $\psi : G \to \Delta_d$ that satisfy $\psi(\gamma g) = e^{i\pi\varepsilon(\gamma)}\psi(g)$ for all $\gamma \in \Gamma$, $g \in G$.

In the left-invariant situation, the symmetry of the sub-Dirac operator D associated to $(\mathscr{H}, g, P_{\mathrm{Spin}(\mathscr{H}),\varepsilon}, \nabla)$ or $(\dot{\mathscr{H}}, \dot{g}, P_{\mathrm{Spin},\varepsilon}(\dot{\mathscr{H}}), \dot{\nabla})$ can be characterized as follows.

Lemma 2.2. *The sub-Dirac operator is symmetric with respect to the L^2-norm if and only if*

$$0 = \sum_{i=1}^{d} g(\nabla_{s_i} s_j, s_i) \tag{6}$$

holds for all j.

Proof. If Equation (3) holds true, then (6) follows by choosing $X = s_j$ because the divergence of any left-invariant vector field is zero as G is unimodular.

Conversely, (6) implies (3) for all left-invariant $X \in \Gamma(\mathscr{H})$. Furthermore, it holds

$$\begin{aligned}\mathrm{div}(fX) = X(f) + f\,\mathrm{div}(X) &= \sum_i \big(s_i(f) g(X, s_i) + f g(\nabla_{s_i} X, s_i)\big) \\ &= \sum_i g(\nabla_{s_i}(fX), s_i).\end{aligned}$$

This yields the claim.

Condition (6) is not satisfied in general, see Subsection 4.2.

3. Sub-Riemannian structures on $\Gamma \setminus (\mathsf{R}^n \rtimes_A \mathsf{R})$

3.1. The standard model

Let $A(t) = \exp(tB)$ be a one-parameter subgroup of $\mathrm{GL}(n, \mathsf{R})$. We consider the simply-connected solvable Lie group $\mathsf{R}^n \rtimes_A \mathsf{R}$ with group law

$$(x, s)(y, t) = (x + A(s)y, s + t).$$

In particular, $(0, t)(x, 0)(0, t)^{-1} = (A(t)x, 0)$. In addition, we assume $A(1) \in \mathrm{SL}(n, \mathsf{Z})$ so that the set $\mathsf{Z}^n \times \mathsf{Z}$ becomes a uniform discrete subgroup.

A Lie group G is called exponential if the exponential map gives a diffeomorphism of its Lie algebra $\mathfrak{g}$ onto G. Then G is simply connected and solvable. A simply connected solvable Lie group is exponential if and only if no operator $\mathrm{ad}(X)$ of the adjoint representation of $\mathfrak{g}$ admits a non-zero purely imaginary eigenvalues, compare Theorem 1 of [16]. In particular every simply connected nilpotent Lie group belongs to this class.

The pair $(\mathsf{R}^n \rtimes_A \mathsf{R}, \mathsf{Z}^n \rtimes_A \mathsf{Z})$ serves as a standard model in the following sense.

LEMMA 3.1. *Let G be an exponential Lie group admitting a connected abelian normal subgroup N of codimension one. Let Γ be a uniform discrete subgroup of G such that $\Gamma \cap N$ is uniform in N. Then there exists a one-parameter subgroup A of $\mathrm{GL}(n, \mathsf{R})$, $n = \dim N$, with $A(1) \in \mathrm{SL}(n, \mathsf{Z})$, and an isomorphism Φ of $\mathsf{R}^n \rtimes_A \mathsf{R}$ onto G mapping $\mathsf{Z}^n \rtimes_A \mathsf{Z}$ onto Γ.*

PROOF. We fix generators $v_1, \ldots, v_n$ of the lattice $\Gamma \cap N$ of the vector group N and consider the linear isomorphism M of R^n onto N given by $M(e_j) = v_j$. On the other hand, the assumption on Γ implies that ΓN is closed in G and that $\Gamma N/N$ is a discrete subgroup of G/N. Hence there exists $b \in \mathfrak{g}$ with $\exp(b) \in \Gamma$ and such that $\exp(b)N$ is a generator of $\Gamma N/N$.

Put $A(t)x = M^{-1}(\exp(tb)M(x)\exp(tb)^{-1})$. Now it follows that $\Phi(x, t) = M(x)\exp(tb)$ is an isomorphism of $\mathsf{R}^n \rtimes_A \mathsf{R}$ onto G with $\Phi(\mathsf{Z}^n \times \mathsf{Z}) = \Gamma$. In particular, $\mathsf{Z}^n \rtimes_A \mathsf{Z}$ is a subgroup of $\mathsf{R}^n \rtimes_A \mathsf{R}$. This means that Z^n is $A(l)$-invariant for all $l \in \mathsf{Z}$ so that $A(l) \in \mathrm{SL}(n, \mathsf{Z})$.

The condition $A(1) \in \mathrm{SL}(n, \mathsf{Z})$ implies $B \in \mathfrak{sl}(n, \mathsf{R})$ and $A(t) \in \mathrm{SL}(n, \mathsf{R})$ for all $t \in \mathsf{R}$. This reflects the fact that locally compact groups admitting a uniform discrete subgroup are unimodular, compare Theorem 7.1.7 of [23].

The Lie algebra of $G := \mathsf{R}^n \rtimes_A \mathsf{R}$ is isomorphic to $\mathfrak{g} = \mathsf{R}^n \rtimes_B \mathsf{R}$, and $B = \mathrm{ad}(b)|\mathfrak{n}$, where $b = (0, 1)$ and $\mathfrak{n} = \mathsf{R}^n \times \{0\}$. Note that G is exponential if and only if B has no non-zero purely imaginary eigenvalues.

It is evident that

$$\pi(x,t) = \begin{pmatrix} & & & x_1 \\ & A(t) & & \vdots \\ & & & x_n \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

defines a representation, which is faithful provided that G is exponential and not abelian.

EXAMPLE 3.2. Fix $r \in \mathsf{Z}_+$ and set $B = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$ so that $A(t) = \exp(tB) = I + tB$. Since $A(l) \in \mathrm{SL}(2, \mathsf{Z})$ for $l \in \mathsf{Z}$, $\Gamma = \mathsf{Z}^2 \rtimes_A \mathsf{Z}$ is a subgroup of $G = \mathsf{R}^2 \rtimes_A \mathsf{R}$. On the other hand,

$$\pi(x_1, x_2, t) = \begin{pmatrix} 1 & rt & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

gives an isomorphism from G onto the three-dimensional Heisenberg group $H(1)$ in its standard realisation as a group of matrices mapping Γ onto

$$\Gamma_r = \left\{ \begin{pmatrix} 1 & rl & k_1 \\ 0 & 1 & k_2 \\ 0 & 0 & 1 \end{pmatrix} : l, k_1, k_2 \in \mathsf{Z} \right\}.$$

In particular, the above construction yields all uniform discrete subgroups of $H(1)$ and hence all three-dimensional Heisenberg manifolds, compare Section 2 of [11].

Heisenberg manifolds and certain generalisations of them will be discussed in Section 4.2 and 4.3 in greater detail.

Define $\Gamma := \mathsf{Z}^n \rtimes_A \mathsf{Z}$. The spin structures of distributions of $T(\Gamma \setminus G)$ induced by a left-invariant distribution of $T(G)$ are determined as follows.

LEMMA 3.3. *A map $\varepsilon : \Gamma \to \mathsf{Z}_2$ is a homomorphism if and only if $\varepsilon(k, l) = \varepsilon'(k) + \dot{\varepsilon}(l)$ for some homomorphism $\dot{\varepsilon} : \mathsf{Z} \to \mathsf{Z}_2$ and a homomorphism $\varepsilon' : \mathsf{Z}^n \to \mathsf{Z}_2$ satisfying*

$$\sum_\mu \varepsilon'(e_\mu)(A(1) - I)_{\mu\nu} \in 2\mathsf{Z} \tag{7}$$

for all ν. Here we identify Z_2 with $\{0, 1\} \subset \mathsf{R}$.

PROOF. Any homomorphism $\varepsilon : \Gamma \to \mathsf{Z}_2$ defines homomorphisms $\dot{\varepsilon} : \mathsf{Z} \to \mathsf{Z}_2$, $\dot{\varepsilon}(l) = \varepsilon(0, l)$, and $\varepsilon' : \mathsf{Z}^n \to \mathsf{Z}_2$, $\varepsilon'(k) = \varepsilon(k, 0)$, where ε' satisfies

$$\varepsilon'(A(l)k) = \varepsilon(A(l)k, 0) = \varepsilon((0, l)(k, 0)(0, l)^{-1}) = \varepsilon(k, 0) = \varepsilon'(k)$$

for all $l \in \mathsf{Z}$ and $k \in \mathsf{Z}^n$. The latter condition reduces to $\varepsilon'(A(1)k) = \varepsilon'(k)$ for all k, and hence to $\varepsilon'(e_\nu) = \varepsilon'(A(1)e_\nu) = \sum_\mu A_{\mu\nu}(1)\varepsilon'(e_\mu)$ in Z_2 for all ν, which is equivalent to (7). It is easy to check that the converse holds also true.

Since Γ is uniform and discrete, there exists a unique normalised right G-invariant Radon measure μ on $\Gamma \setminus G$ which can be obtained as follows: Let $I_n = [0, 1]^n$ denote the unit cube of R^n. Then

$$\int_{\Gamma\setminus G} \varphi \, d\mu = \int_0^1 \left(\int_{I_n} \varphi(x, s)\, dx \right) ds \tag{8}$$

for all $\varphi \in C(\Gamma \setminus G)$. This can be proved using that the Haar measure of G equals the Lebesgue measure of R^{n+1} and that $F = [0, 1)^{n+1}$ is a fundamental set for Γ on G.

3.2. Decomposition of the right-regular representation

Let $A(t)$ be a one-parameter subgroup of $\mathrm{GL}(n, \mathsf{R})$ such that $G = \mathsf{R}^n \rtimes_A \mathsf{R}$ is exponential and $\Gamma = \mathsf{Z}^n \rtimes_A \mathsf{Z}$ is a subgroup of G. Let $\varepsilon : \Gamma \to \mathsf{Z}_2$ be a group homomorphism. Our aim is to decompose the right regular representation of G on $L^2(G, \varepsilon)$.

Let $C(G, \varepsilon)$ denote the space of all continuous C-valued functions φ on G satisfying $\varphi(gy) = e^{i\pi\varepsilon(g)}\varphi(y)$ for all $g \in \Gamma$ and $y \in G$, and $L^2(G, \varepsilon)$ the completion of $C(G, \varepsilon)$ with respect to the Hilbert space norm

$$|\varphi|^2_{L_2} = \int_{\Gamma\setminus G} |\varphi|^2 \, d\mu. \tag{9}$$

Now right translation $(\rho(x)\varphi)(y) = \varphi(yx)$ gives rise to a unitary representation of G on $L^2(G, \varepsilon)$. This is precisely the definition of the induced representation $\rho = \mathrm{ind}_\Gamma^G\, e^{i\pi\varepsilon}$ of the unitary character $e^{i\pi\varepsilon}$ of Γ. By Theorem 7.2.5 of [23], ρ can be written as a countable orthogonal sum of irreducible subrepresentations with finite multiplicity. We will give such a decomposition explicitly, generalizing the results of [2] for the three-dimensional Heisenberg group, which motivated this article.

To this end, we consider partial Fourier transformation with respect to the first n variables: If $\varphi \in C(G, \varepsilon)$, then

$$\varphi((k, l)(x, t)) = \varphi(k + A(l)x, l + t) = e^{i\pi\varepsilon(k,l)}\varphi(x, t)$$

for all $(k, l) \in \Gamma$. In particular, $\varphi(2k + x, t) = e^{i\pi\varepsilon(2k,0)}\varphi(x, t) = \varphi(x, t)$ which shows that $x \mapsto \varphi(x, t)$ is $2\mathsf{Z}^n$-invariant. For such functions it is natural to consider

$$\widehat{\varphi}(\xi, t) = \int_{I_n} \varphi(2x, t)e^{-2\pi i\langle\xi,x\rangle}\, dx$$

for $\xi \in \mathsf{Z}^n$. Clearly φ is uniquely determined by its Fourier coefficient functions.

It follows from (8) that the restriction of $\varphi \in L^2(G, \varepsilon)$ to $2I_n \times [0, 1]$ is L^2- and hence L^1-integrable with respect to the Lebesgue measure. In particular, the integral in the definition of $(T_\xi \cdot \varphi)(t) := \widehat{\varphi}(\xi, t)$ makes sense for almost all t.

PROPOSITION 3.4. *For $\varphi \in L^2(G, \varepsilon)$ there holds the Plancherel formula*

$$|\varphi|_{L^2}^2 = \sum_{\xi \in \mathsf{Z}^n} \int_0^1 |\widehat{\varphi}(\xi, t)|^2 \, dt.$$

PROOF. By the Plancherel theorem for L^2-functions on the torus, we obtain from (8) and (9) that

$$|\varphi|_{L^2}^2 = \int_0^1 \left(\int_{I_n} |\varphi(x, t)|^2 \, dx \right) dt = \int_0^1 \sum_{\xi \in \mathsf{Z}^n} |\widehat{\varphi}(\xi, t)|^2 \, dt,$$

where summation and integration can be interchanged.

Taking into account

$$\int_{I_n} g(Mx) \, dx = \int_{I_n} g(x) \, dx$$

for integrable Z^n-invariant functions g on R^n and $M \in \mathrm{SL}(n, \mathsf{Z})$, one can prove that the ε-equivariance of φ entails the following conditions on its Fourier transform.

LEMMA 3.5. *For $\varphi \in L^2(G, \varepsilon)$ and $(k, l) \in \Gamma$ it holds*

$$e^{i\pi\varepsilon(k,l)} \widehat{\varphi}(\xi, t) = e^{\pi i \langle A(-l)^\top \xi, k \rangle} \widehat{\varphi}(A(-l)^\top \xi, l + t),$$

where $A(l)^\top$ denotes the transpose of the operator $A(l)$ with respect to the standard inner product on R^n.

The one-parameter subgroup A represents the restriction of the adjoint representation of the subgroup $\mathsf{R} \cong \{0\} \times \mathsf{R}$ of G to the ideal $\mathsf{R}^n \times \{0\}$ of $\mathfrak{g}$. Identifying the linear dual of this Lie algebra with R^n by means of the standard inner product, we see that $t \mapsto A(t)^\top$ is the coadjoint representation. The preceding lemma reveals the importance of this group action in the present context. Any $\xi \in \mathsf{R}^n$ has a Z-orbit $\theta = \{A(l)^\top \xi : l \in \mathsf{Z}\}$ and an R-orbit $\omega = \{A(t)^\top \xi : t \in \mathsf{R}\}$.

Let $\varphi \in L^2(G,\varepsilon)$ and $\xi \in \mathsf{Z}^n$. The equality $\widehat{\varphi}(A(l)^\top\xi, t) = e^{i\pi\varepsilon(0,-l)}\widehat{\varphi}(\xi, l+t)$ shows that $\widehat{\varphi}(\xi,\cdot)$ determines $\widehat{\varphi}(\eta,\cdot)$ for all $\eta \in \theta$. In particular, $\operatorname{supp}\widehat{\varphi} := \{\xi \in \mathsf{Z}^n : \widehat{\varphi}(\xi,\cdot) \neq 0\}$ is a Z-invariant subset.

LEMMA 3.6. *The set*

$$\Sigma_{\varepsilon'} := \left\{\xi \in \mathsf{Z}^n : \xi_\nu \in 2\mathsf{Z} + \varepsilon'(e_\nu) \textit{ for all } 1 \le \nu \le n\right\}$$

is Z*-invariant and contains* $\operatorname{supp}\widehat{\varphi}$ *for every* $\varphi \in L^2(G,\varepsilon)$.

PROOF. Let $\xi \in \Sigma_{\varepsilon'}$ and $l \in \mathsf{Z}$. Since $\varepsilon'(A(l)^\top e_\nu) = \varepsilon'(e_\nu)$ and $\langle \xi, k\rangle \in 2\mathsf{Z}+\varepsilon'(k)$ for all $k \in \mathsf{Z}^n$, it follows $\langle A(l)^\top\xi, e_\nu\rangle = \langle \xi, A(l)e_\nu\rangle \in 2\mathsf{Z}+\varepsilon'(e_\nu)$ and $A(l)^\top\xi \in \Sigma_{\varepsilon'}$. This proves $\Sigma_{\varepsilon'}$ to be Z-invariant. Let $\varphi \in L^2(G,\varepsilon)$ and $\xi \notin \Sigma_{\varepsilon'}$. Then there is $1 \le \nu \le n$ such that $\xi_\nu \notin 2\mathsf{Z} + \varepsilon'(e_\nu)$ and hence $e^{\pi i\varepsilon'(e_\nu)} \neq e^{\pi i\langle \xi, e_\nu\rangle}$. On the other hand, by Lemma 3.5 we have $e^{i\pi\varepsilon(k,0)}\widehat{\varphi}(\xi,t) = e^{\pi i\langle \xi,k\rangle}\widehat{\varphi}(\xi,t)$ for all k. This implies $\widehat{\varphi}(\xi,\cdot) = 0$.

Note that $\xi \in \Sigma_{\varepsilon'}$ implies $e^{i\pi\varepsilon(k,0)} = e^{i\pi\langle\xi,k\rangle}$ for all $k \in \mathsf{Z}^n$.

Let $\mathsf{Z}\setminus\Sigma_{\varepsilon'}$ denote the set of all Z-orbits in $\Sigma_{\varepsilon'}$. For $\theta \in \mathsf{Z}\setminus\Sigma_{\varepsilon'}$, it follows that

$$U_\theta := \bigcap_{\xi\notin\theta} \ker T_\xi = \left\{\varphi \in L^2(G,\varepsilon) : \operatorname{supp}\widehat{\varphi} \subset \theta\right\} \neq 0$$

is a closed subspace of $L^2(G,\varepsilon)$. It will be shown below that U_θ is $\rho(G)$-invariant.

PROPOSITION 3.7. *These subspaces form an orthogonal decomposition*

$$L^2(G,\varepsilon) = \overline{\bigoplus_{\theta\in\mathsf{Z}\setminus\Sigma_{\varepsilon'}}} U_\theta.$$

PROOF. Let $\theta_1, \theta_2 \in \mathsf{Z}\setminus\Sigma_{\varepsilon'}$ be distinct orbits. By Proposition 3.4 and the polarisation identity, we obtain

$$\langle\varphi_1,\varphi_2\rangle_{L^2} = \sum_{\xi\in\mathsf{Z}^n}\int_0^1 \widehat{\varphi_1}(\xi,t)\overline{\widehat{\varphi_2}(\xi,t)}\,dt = 0$$

for $\varphi_1 \in U_{\theta_1}$ and $\varphi_2 \in U_{\theta_2}$. Hence U_{θ_1} and U_{θ_2} are orthogonal. It remains to be shown that the direct sum of the U_θ is dense. Since $C^\infty(G,\varepsilon)$ is dense in $L^2(G,\varepsilon)$, it suffices to prove that every smooth ε-equivariant function can be approximated by a finite sum of functions in the U_θ. By the decay of the Fourier transform of $\varphi \in C^\infty(G,\varepsilon)$, it follows that

$$\varphi_\theta(x,t) = \sum_{\xi\in\theta}\widehat{\varphi}(\xi,t)e^{\pi i\langle\xi,x\rangle}$$

is a smooth function in U_θ and that $\varphi = \sum_{\theta \in \mathsf{Z} \setminus \Sigma_{\varepsilon'}} \varphi_\theta$ converges uniformly on $\mathsf{R}^n \times [0, 1]$. In particular, $\left|\varphi - \sum_{\theta \in J} \varphi_\theta\right|_{L^2} \to 0$ for $J \subset \mathsf{Z} \setminus \Sigma_{\varepsilon'}$ finite and increasing. (This also proves that $U_\theta \cap C^\infty(G, \varepsilon)$ is dense in U_θ.)

The right regular representation $\rho(x, s)\varphi(y, t) = \varphi(y + A(t)x, t + s)$ on $L^2(G, \varepsilon)$ is compatible with partial Fourier transform in the sense that

$$(\rho(x, s)\varphi)\widehat{\ }(\xi, t) = e^{\pi i \langle A(t)^\top \xi, x\rangle} \widehat{\varphi}(\xi, t + s). \tag{10}$$

In particular, $\widehat{\varphi}(\xi, \cdot) = 0$ implies $(\rho(x, s)\varphi)\widehat{\ }(\xi, \cdot) = 0$ for all $(x, s) \in G$ which proves U_θ to be $\rho(G)$-invariant. We define $\rho_\theta = \rho|_{U_\theta}$.

Let $L^2(\mathsf{R}, \dot{\varepsilon})$ denote the L^2-completion of the vector space $C(\mathsf{R}, \dot{\varepsilon})$ of all continuous functions satisfying $f(t + k) = e^{\pi i \dot{\varepsilon}(k)} f(t)$ for all $k \in \mathsf{Z}$. Here $\dot{\varepsilon} : \mathsf{Z} \to \mathsf{Z}_2 \cong \{0, 1\}$ is defined as in Lemma 3.3. Put $(T_\xi \cdot \varphi)(t) = \widehat{\varphi}(\xi, t)$. From Lemma 3.5 we get

$$\begin{aligned}(T_\xi \cdot \varphi)(l + t) &= \widehat{\varphi}(\xi, l + t) = \widehat{\varphi}(A(-l)^\top \xi, l + t) \\ &= e^{i\pi \dot{\varepsilon}(l)} \widehat{\varphi}(\xi, t) = e^{i\pi \dot{\varepsilon}(l)} (T_\xi \cdot \varphi)(t)\end{aligned}$$

for all $\varphi \in L^2(G, \varepsilon)$. Furthermore, Proposition 3.4 implies $|T_\xi \cdot \varphi|_{L^2} \le |\varphi|_{L_2}$ for all φ. This shows that $T_\xi : L^2(G, \dot{\varepsilon}) \to L^2(\mathsf{R}, \dot{\varepsilon})$ is a continuous linear operator.

Let $\theta \subset \Sigma_{\varepsilon'}$ be a Z-orbit. We claim that the restriction of T_ξ gives a unitary isomorphism of U_θ onto $L^2(\mathsf{R}, \dot{\varepsilon})$. To prove this we must distinguish two cases.

Let ω be the unique R-orbit containing θ. The stabilizer $\dot{H}_\omega := \{t \in \mathsf{R} : A(t)^\top \xi = \xi\}$ does not depend on the choice of the point $\xi \in \omega$. Since $t \mapsto A(t)^\top$ is the coadjoint representation of an exponential Lie group, we know that the closed subgroup $\dot{H}_\omega$ is connected, see p. 49 of [16]. Thus there are only two possibilities, either $\dot{H}_\omega = \mathsf{R}$ or $\dot{H}_\omega = \{0\}$.

First we consider the case $\dot{H}_\omega = \mathsf{R}$. This implies that $\omega = \theta = \{\xi\}$ is a fixed point. We obtain

$$|T_\xi \cdot \varphi|_{L^2}^2 = \int_0^1 |\widehat{\varphi}(\xi, t)|^2 \, dt = |\varphi|_{L^2}^2$$

by Proposition 3.4. If $\psi \in C(\mathsf{R}, \dot{\varepsilon})$, then $\varphi(x, t) = \psi(t) e^{\pi i \langle \xi, x\rangle}$ is in $C(G, \varepsilon)$ by Lemma 3.6, and $T_\xi \cdot \varphi = \psi$. Thus T_ξ is a unitary isomorphism of U_θ onto $L^2(\mathsf{R}, \dot{\varepsilon})$. From Equation (10) it follows that the representation $\rho_\xi(x, s) = T_\xi \rho_\theta(x, s) T_\xi^*$ on $L^2(\mathsf{R}, \dot{\varepsilon})$ is given by

$$\rho_\xi(x, s)\psi(t) = e^{\pi i \langle \xi, x\rangle} \psi(t + s).$$

Define $\varepsilon^\sharp(t) = e^{\pi i\dot\varepsilon(1)t}$ for $t \in \mathsf{R}$. Then $\varepsilon^\sharp$ is a unitary character of R extending $l \mapsto e^{\pi i\dot\varepsilon(l)}$. Any such extension has the form $t \mapsto \varepsilon^\sharp(t)e^{2\pi imt}$ for some $m \in \mathsf{Z}$. By means of the unitary isomorphism $(U \cdot \psi)(t) = \varepsilon^\sharp(-t)\psi(t)$ of $L^2(\mathsf{R}, \dot\varepsilon)$ onto $L^2(\mathsf{Z} \setminus \mathsf{R})$, we see that ρ_ξ is unitarily equivalent to

$$\tilde\rho_\xi(x, s)\psi(t) = e^{\pi i\langle\xi,x\rangle}\varepsilon^\sharp(s)\psi(t + s) \tag{11}$$

on $L^2(\mathsf{Z} \setminus \mathsf{R})$. For $m \in \mathsf{Z}$ we consider the unitary character of G given by

$$\chi_{\varepsilon^\sharp,\omega,m}(x, s) = e^{\pi i\langle\xi,x\rangle}e^{2\pi ims}\varepsilon^\sharp(s).$$

Finally, using the Fourier transformation and the Plancherel theorem for L^2-functions on the torus, we conclude that ρ_θ is unitarily equivalent to an orthogonal sum

$$\rho_\theta \cong \bigoplus_{m\in\mathsf{Z}} \chi_{\varepsilon^\sharp,\omega,m}$$

of 1-dimensional subrepresentations. Note that up to isomorphism this decomposition does not depend on the choice of the extension $\varepsilon^\sharp$.

Now we consider the second case where $\dot H_\omega = \{0\}$. Then ω is not relatively compact and the Z-orbit θ is an infinite set.

LEMMA 3.8. *For every $\eta \in \omega$ there exists a unitary isomorphism T_η of U_θ onto $L^2(\mathsf{R})$ which intertwines ρ_θ and*

$$\rho_\eta(x, s)\psi(t) = e^{\pi i\langle A(t)^\top\eta,x\rangle}\psi(t + s). \tag{12}$$

PROOF. Let $\xi \in \theta$ and $r \in \mathsf{R}$ such that $\eta = A(r)^\top\xi$. We claim that $(T_\eta\varphi)(t) = \widehat\varphi(\xi, t + r)$ is a unitary isomorphism satisfying our needs: First of all, Proposition 3.4 implies

$$\begin{aligned}|T_\eta\varphi|^2_{L^2} &= \int_{-\infty}^{+\infty} |\widehat\varphi(\xi, t + r)|^2\,dt = \int_{-\infty}^{+\infty} |\widehat\varphi(\xi, t)|^2\,dt\\ &= \sum_{l\in\mathsf{Z}}\int_0^1 |\widehat\varphi(\xi, l + t)|^2\,dt = \sum_{l\in\mathsf{Z}}\int_0^1 |\widehat\varphi(A(l)^\top\xi, t)|^2\,dt = |\varphi|^2_{L^2}\end{aligned}$$

which shows that $T_\eta\varphi \in L^2(\mathsf{R})$ is well-defined and that T_η is isometric. If $\psi \in C_0(\mathsf{R})$, then the sum

$$\varphi(x, t) = \sum_{l\in\mathsf{Z}} e^{-i\pi\dot\varepsilon(l)}\psi(l + t - r)e^{\pi i\langle A(l)^\top\xi,x\rangle}$$

is locally finite in t and $\varphi \in C(G, \varepsilon)$ by Lemma 3.6. Using $A(l)^\top \xi \neq \xi$ for $l \neq 0$, we conclude $T_\eta \varphi = \psi$. This proves T_η to be surjective. Finally, we observe that (12) is a consequence of (10).

Let $X^\infty_{\varepsilon'}$ denote the set of all R-orbits which intersect the subset $\Sigma_{\varepsilon'}$ of Z^n and which are not relatively compact. Let $X^0_{\varepsilon'}$ be the set of all orbits of the form $\omega = \{\, \xi \,\}$ for a fixed point $\xi \in \Sigma_{\varepsilon'}$. If $\omega \in X^\infty_{\varepsilon'}$, $\eta \in \omega$ is arbitrary, and θ_1, θ_2 are Z-orbits contained in $\omega \cap \Sigma_{\varepsilon'}$, then $\rho_{\theta_1} \cong \rho_\eta \cong \rho_{\theta_2}$ by Lemma 3.8. This implies

$$\bigoplus_{\theta \in \mathsf{Z} \backslash (\omega \cap \Sigma_{\varepsilon'})} \rho_\theta \cong m_{\varepsilon,\omega} \rho_\omega$$

where $m_{\varepsilon,\omega}$ is the number of Z-orbits contained in $\omega \cap \Sigma_{\varepsilon'}$, which is apriori known to be finite, and ρ_ω is the common unitary equivalence class of the representations ρ_θ for $\theta \in \mathsf{Z} \setminus (\omega \cap \Sigma_{\varepsilon'})$.

Summing up the preceding conclusions, we obtain

THEOREM 3.9. *Let $A(t)$ be a one-parameter group of* $\mathrm{GL}(n, \mathsf{R})$ *with $A(1) \in$* $\mathrm{SL}(n, \mathsf{Z})$ *and such that $G = \mathsf{R}^n \rtimes_A \mathsf{R}$ is exponential. Let $\varepsilon : \Gamma \to \mathsf{Z}_2$ be a homomorphism. Then the right regular representation ρ of G in $L^2(G, \epsilon)$ decomposes as follows:*

$$\rho \cong \bigoplus_{\theta \in \mathsf{Z} \backslash \Sigma_{\varepsilon'}} \rho_\theta \cong \Bigg(\bigoplus_{\omega \in X^0_{\varepsilon'}} \bigoplus_{m \in \mathsf{Z}} \chi_{\dot\varepsilon,\omega,m} \Bigg) \oplus \Bigg(\bigoplus_{\omega \in X^\infty_{\varepsilon'}} m_{\varepsilon,\omega} \rho_\omega \Bigg)$$

where the multiplicities $m_{\epsilon,\omega} = \#\mathsf{Z} \setminus (\omega \cap \Sigma_{\epsilon'})$ are finite, the

$$\chi_{\dot\varepsilon,\omega,m}(x, s) = e^{\pi i \langle \xi, x \rangle} e^{\pi i (2m + \dot\varepsilon(1)) s}$$

are characters of G, and the ρ_ω are irreducible on $L^2(\mathsf{R})$. For every $\eta \in \omega$,

$$\rho_\eta(x, s)\psi(t) = e^{\pi i \langle A(t)^\top \eta, x \rangle} \psi(t + s)$$

is a representative for the unitary equivalence class of ρ_ω. Moreover, the representations $\{\chi_{\dot\varepsilon,\omega,m} : \omega \in X^0_{\varepsilon'}, m \in \mathsf{Z}\} \cup \{\rho_\omega : \omega \in X^\infty_{\varepsilon'}\}$ are mutually inequivalent.

PROOF. It remains to verify the last assertion and the irreducibility of ρ_ω. Clearly characters are unitarily equivalent if and only if they are equal, and not unitarily equivalent to a representation on $L^2(\mathsf{R})$. Let $C^*(N)$ be the enveloping C^*-algebra of the group algebra $L^1(N)$ of $N = \mathsf{R}^n \times \{0\}$. Recall that $C^*(N)$ is isomorphic to $C_\infty(\hat N)$ via Fourier transformation. The above formula for ρ_η shows that the C^*-kernel of the integrated form of $\rho_\omega|N$ consists of all $g \in C^*(N)$ whose Fourier transform vanishes on ω. Since the R-orbits are locally closed, it follows that ρ_{ω_1} and ρ_{ω_2} are inequivalent whenever $\omega_1 \neq \omega_2$. If U is a closed $\rho_\eta(G)$-invariant subspace of $L^2(\mathsf{R})$, then U is invariant under translations and multiplication by bounded continuous functions. Thus it follows $U = \{0\}$ or $U = L^2(\mathsf{R})$.

LEMMA 3.10. *Suppose that $G = \mathsf{R}^n \rtimes_A \mathsf{R}$ is a nilpotent Lie group. Let $\omega \in X^\infty_{\varepsilon'}$ and $\theta \in \mathsf{Z}\backslash(\omega\cap\Sigma_{\varepsilon'})$. For $\eta \in \omega$ let T_η denote the unitary isomorphism of U_θ onto $L^2(\mathsf{R})$ defined in the proof of Lemma 3.8. Then for every Schwartz function $\psi \in \mathscr{S}(\mathsf{R})$ there exists a (unique) $\varphi \in U_\theta \cap C^\infty(G,\varepsilon)$ such that $T_\eta\varphi = \psi$.*

PROOF. Given $\psi \in \mathscr{S}(\mathsf{R})$ we consider

$$\varphi(x,t) = \sum_{l\in\mathsf{Z}} e^{-i\pi\dot{\varepsilon}(l)}\psi(l+t-r)e^{\pi i\langle A(l)^\top\xi,x\rangle}$$

whose formal derivatives are

$$(\partial_t^\alpha\partial_x^\beta\varphi)(x,t) = \sum_{l\in\mathsf{Z}} e^{-i\pi\dot{\varepsilon}(l)}(\pi i)^{|\beta|}(A(l)^\top\xi)^\beta(\partial_t^\alpha\psi)(l+t-r)e^{\pi i\langle A(l)^\top\xi,x\rangle}.$$

Since B is nilpotent, the expression $A(l)^\top\xi = \exp(lB)^\top\xi$ is polynomial in l. Hence for each multi-index β there exist constants $N \in \mathsf{N}$ and $C_0 > 0$ such that

$$|(A(l)^\top\xi)^\beta| \le C_0(1+l^2)^N.$$

for all $l \in \mathsf{Z}$. On the other hand, since $\psi \in \mathscr{S}(\mathsf{R})$, for each α there are $C_1, C_2 > 0$ such that

$$|(\partial_t^\alpha\psi)(l+t-r)| \le C_1|1+(l+t-r)^2|^{-(N+1)} \le C_2|1+l^2|^{-(N+1)}$$

for all l, and for t ranging over a compact subset K of R. This implies that the above series converge absolutely and uniformly on $\mathsf{R}^n \times K$ so that $\varphi \in C^\infty(G,\varepsilon)$ is well-defined. Clearly $\varphi \in U_\theta$ and $T_\eta\varphi = \psi$.

3.3. *Operators with discrete spectrum*

Let $(H, \langle\cdot,\cdot\rangle)$ be a real vector space with inner product and $(\Delta, \langle\cdot,\cdot\rangle)$ a complex vector space with a hermitian inner product. Suppose that Δ carries a $\mathscr{C}l(H)$-module structure such that $\langle x\cdot v, w\rangle = -\langle v, x\cdot w\rangle$ for all $x \in H \subset \mathscr{C}l(H)$ and $v, w \in \Delta$. Let $s_1, \dots, s_d$ be an orthonormal basis of H and $a \in H$ a non-zero multiple of s_d. Furthermore, let $\Omega : \mathsf{R} \to \mathrm{span}\{s_1,\dots,s_{d-1}\}$ be a non-constant polynomial function. We consider the operator

$$P = a\partial_t + i\Omega(t)$$

on the domain $\mathscr{S}(\mathsf{R}, \Delta)$. Here $a, \Omega(t) \in \mathscr{C}l(H)$ are understood as operators acting by pointwise multiplication. Clearly P is symmetric with respect to the L^2-inner product and densely defined in the Hilbert space $L^2(\mathsf{R}, \Delta)$. Thus P is closable. The closure $\bar{P}$ of P is again a symmetric operator. Let P^* denote the adjoint of P. On its domain

$$\operatorname{dom}(P^*) = \big\{\psi \in L^2(\mathsf{R}, \Delta) : \\ \varphi \mapsto \langle P\varphi, \psi\rangle_{L^2} \text{ is continuous w.r.t. to the } L^2\text{-norm}\big\}$$

we consider the norm $|\cdot|_P$ given by $|\psi|_P^2 = |\psi|_{L^2}^2 + |P^*\psi|_{L^2}^2$. Our aim is to prove the following result.

Proposition 3.11. *The closure $\bar{P}$ of P is self-adjoint and has discrete spectrum.*

Proof. We can assume $|a| = 1$ what will simplify the estimates below.

To prove the first assertion, we imitate the proof of the essential selfadjointness of the Dirac operator, compare Theorem 5.7 of [15] and Proposition 1.3.5 of [10]. As a basic fact we know that it suffices to verify $\ker(P^* \pm iI) = \{0\}$. Moreover, since $\bar{P}$ is symmetric, it is enough to show that $\ker(P^* \pm iI) \subset \operatorname{dom}(\bar{P})$. To begin with, we note that, if $f \in \mathscr{S}(\mathsf{R})$ and $\psi \in \operatorname{dom}(P^*)$, then $f\psi \in \operatorname{dom}(P^*)$ and

$$P^*(f\psi) = f(P^*\psi) + (\partial_t f)a \cdot \psi.$$

Let $\psi \in \ker(P^* \pm iI)$. If $\hat{P}$ denotes the extension of P to tempered distributions, then we get, as P is symmetric, $(\hat{P} \pm iI)\psi = P^*\psi \pm i\psi = 0$. Since the principal symbol $p(\xi) = \xi\, a$ of $\hat{P} \pm iI$ is invertible for $\xi \neq 0$, the regularity theorem for elliptic differential operators implies that ψ is a smooth function. Choose $h \in C_0^\infty(\mathsf{R})$ satisfying $0 \le h \le 1$ and $h(0) = 1$, and put $h_k(t) = h(t/k)$ for $k \ge 1$. By definition $h_k \to 1$ and $h_k\psi \in C_0^\infty(\mathsf{R}, \Delta) \subset \operatorname{dom}(P)$. Since

$$\big|(\partial_t h_k)a \cdot \psi\big|_{L^2} \le |\partial_t h_k|_\infty |a| |\psi|_{L^2} \le \tfrac{1}{k} |\partial_t h|_\infty |\psi|_{L^2},$$

it follows that

$$\begin{aligned} |\psi - h_k\psi|_P^2 &= |\psi - h_k\psi|_{L^2}^2 + |P^*\psi - P^*(h_k\psi)|_{L^2}^2 \\ &\le |\psi - h_k\psi|_{L^2}^2 + \big(|P^*\psi - h_k(P^*\psi)|_{L^2} + |(\partial_t h_k)a \cdot \psi|_{L^2}\big)^2 \end{aligned}$$

converges to 0 for $k \to \infty$ by dominated convergence. Hence $\psi \in \operatorname{dom}(\bar{P})$. This establishes the essential selfadjointness of P.

To prove that the spectrum of $\bar{P}$ is discrete, we need the following two lemmata.

LEMMA 3.12. *Let $(\Delta, \langle\cdot,\cdot\rangle)$ be a $\mathscr{C}l(H)$-module as above and $\Omega : \mathsf{R} \to H$ a continuous function satisfying $|\Omega(t)| \to \infty$ for $|t| \to \infty$. Then*

$$X := \{\varphi \in L^2(\mathsf{R}, \Delta) : \partial_t\varphi \in L^2(\mathsf{R}, \Delta) \text{ and } \Omega \cdot \varphi \in L^2(\mathsf{R}, \Delta)\}$$

becomes a Hilbert space when endowed with the norm $\|\varphi\|^2 = |\varphi|^2_{L^2} + |\partial_t\varphi|^2_{L^2} + |\Omega \cdot \varphi|^2_{L^2}$, and the inclusion $X \to L^2(\mathsf{R}, \Delta)$ is a compact operator.

PROOF. The first assertion is obvious. Let (φ_m) be a bounded sequence in X. We prove that (φ_m) has a subsequence which is Cauchy in $L^2(\mathsf{R}, \Delta)$. For every $n \in \mathsf{N} \setminus \{0\}$ there exists $r_n > 0$ such that $|\Omega(t)| \geq n$ for $|t| \geq r_n$. We can assume $r_n < r_{n+1}$ and $r_n \to \infty$ for $n \to \infty$. Using $|\Omega(t) \cdot \varphi(t)| = |\Omega(t)||\varphi(t)| \geq n|\varphi(t)|$, we obtain

$$\int_{|t|\geq r_n} |\varphi(t)|^2\, dt \leq \frac{1}{n^2} \int_{|t|\geq r_n} |\Omega(t) \cdot \varphi(t)|^2\, dt \leq \frac{1}{n^2} |\Omega \cdot \varphi|^2_{L^2}$$

for $\varphi \in X$. For the moment, we fix the parameter n. Let $\chi \in C_0^\infty(\mathsf{R})$ be such that $0 \leq \chi \leq 1$ and $\chi(t) = 1$ for $|t| \leq r_n$. By Rellich's theorem, applied to the bounded sequence $\chi\varphi_m$ in $H^1(\mathsf{R}, \Delta)$, we conclude that there exists a subsequence $(\varphi_{m_{n,k}})$ such that

$$\int_{-r_n}^{r_n} |\varphi_{m_{n,k}}(t) - \varphi_{m_{n,l}}(t)|^2\, dt \to 0$$

for $k, l \to \infty$. Proceeding by induction, we establish $\{m_{n+1,k} : k \in \mathsf{N}\} \subset \{m_{n,k} : k \in \mathsf{N}\}$ for all n. We define $m_k = m_{k,k}$. Now it is easy to see that (φ_{m_k}) is Cauchy w.r.t. the L^2-norm.

LEMMA 3.13. *The domain of $\bar{P}$ is contained in X and the inclusion* $\mathrm{dom}(\bar{P}) \to X$ *is continuous.*

PROOF. Since $\mathrm{dom}(P) = \mathscr{S}(\mathsf{R}, \Delta)$ is contained in X and dense in $\mathrm{dom}(\bar{P})$ w.r.t. the norm $|-|_P$, it suffices to prove that there exists $K > 0$ such that $\|\varphi\| \leq K|\varphi|_P$ for all $\varphi \in \mathscr{S}(\mathsf{R}, \Delta)$. Using $a \cdot (\partial_t\varphi) = \partial_t(a \cdot \varphi)$ and $\Omega a = -a\Omega$ in $\mathscr{C}l(H)$, we compute

$$\langle a \cdot (\partial_t\varphi), i\Omega \cdot \varphi\rangle_{L^2} = -\langle a \cdot \varphi, i\partial_t(\Omega \cdot \varphi)\rangle_{L^2}$$

and

$$\langle i\Omega \cdot \varphi, a \cdot (\partial_t\varphi)\rangle_{L^2} = \langle a \cdot \varphi, i\Omega \cdot (\partial_t\varphi)\rangle_{L^2}.$$

As $\partial_t(\Omega \cdot \varphi) = (\partial_t\Omega) \cdot \varphi + \Omega \cdot (\partial_t\varphi)$, it follows

$$|P\varphi|^2_{L^2} = |a \cdot (\partial_t\varphi) + i\Omega \cdot \varphi|^2_{L^2} = |\partial_t\varphi|^2_{L^2} - \langle a \cdot \varphi, i(\partial_t\Omega) \cdot \varphi\rangle_{L^2} + |\Omega \cdot \varphi|^2_{L^2}$$

for all Schwartz functions. Since Ω is a polynomial function, there exists $r > 0$ such that $|(\partial_t\Omega)(t)| \leq |\Omega(t)|$ for all $|t| \geq r$. Fix $C > 1$ such that $|(\partial_t\Omega)(t)| \leq C$ for $|t| \leq r$. From

$$\begin{aligned}
|\langle a\cdot\varphi, i(\partial_t\Omega)\cdot\varphi\rangle_{L^2}| &\leq \int_{-r}^{r} |(\partial_t\Omega)(t)||\varphi(t)|^2\,dt + \int_{|t|\geq r} |(\partial_t\Omega)(t)||\varphi(t)|^2\,dt \\
&\leq C\int_{-r}^{r} |\varphi(t)|^2\,dt + \int_{|t|\geq r} |\varphi(t)||\Omega(t)\cdot\varphi(t)|\,dt \\
&\leq C|\varphi|_{L^2}^2 + |\varphi|_{L^2}|\Omega\cdot\varphi|_{L^2}
\end{aligned}$$

it then follows

$$\begin{aligned}
|\varphi|_P^2 = |\varphi|_{L^2}^2 + |P\varphi|_{L^2}^2 &\geq \frac{1}{2C}\big(C|\varphi|_{L^2}^2 + \tfrac{1}{4}|\varphi|_{L^2}^2 + |P\varphi|_{L^2}^2\big) \\
&\geq \frac{1}{2C}\big(\tfrac{1}{4}|\varphi|_{L^2}^2 + |\partial_t\varphi|_{L^2}^2 + |\Omega\cdot\varphi|_{L^2}^2 - |\varphi|_{L^2}|\Omega\cdot\varphi|_{L^2}\big) \\
&\geq \frac{1}{2C}|\partial_t\varphi|_{L^2}^2
\end{aligned}$$

for $\varphi \in \mathscr{S}(\mathsf{R}, \Delta)$. Moreover, $i\Omega\cdot\varphi = P\varphi - a\cdot(\partial_t\varphi)$ gives

$$|\Omega\cdot\varphi|_{L^2} \leq |P\varphi|_{L^2} + |\partial_t\varphi|_{L^2} \leq \big(1+\sqrt{2C}\,\big)|\varphi|_P.$$

Altogether, we obtain

$$\|\varphi\|^2 = |\varphi|_{L^2}^2 + |\partial_t\varphi|_{L^2}^2 + |\Omega\cdot\varphi|_{L^2}^2 \leq \big(1 + 2C + \big(1+\sqrt{2C}\,\big)^2\big)|\varphi|_P^2$$

proving the lemma.

Now we can prove the second assertion of Proposition 3.11. As $\sigma(\bar{P}) \subset \mathsf{R}$, we know that $\bar{P} - iI$ is bijective. Put $R := (\bar{P} - iI)^{-1}$. Since $\bar{P} - iI$ is continuous w.r.t. the complete norm $|\cdot|_P$ on $\operatorname{dom}(\bar{P})$ and the L^2-norm, the open-mapping theorem implies that $R : L^2(\mathsf{R}, \Delta) \to \operatorname{dom}(\bar{P})$ is continuous. Moreover, since the inclusion $\operatorname{dom}(\bar{P}) \to X \to L^2(\mathsf{R}, \Delta)$ is compact by Lemma 3.12 and 3.13, it follows that R is a compact normal operator on $L^2(\mathsf{R}, \Delta)$ with $\ker R = \{0\}$. By the spectral theorem there exists an orthonormal basis $\{\varphi_n : n \in \mathsf{N}\}$ of $L^2(\mathsf{R}, \Delta)$ with $R\varphi_n = \mu_n\varphi_n$ for suitable $0 \neq \mu_n \in \mathsf{C}$. This implies $\bar{P}\varphi_n = \big(\lambda + \frac{1}{\mu_n}\big)\varphi_n$ for all n. If $\{\mu_n : n \in \mathsf{N}\}$ happens to be an infinite set, then $\mu_n \to 0$ and hence $\big|\lambda + \frac{1}{\mu_n}\big| \to \infty$ for $n \to \infty$. This proves the spectrum of $\bar{P}$ to be discrete.

Now we resume the assumptions of Section 3.1: Let $A(t) = \exp(tB)$ be a one-parameter group of $\mathrm{GL}(n, \mathsf{R})$ with $A(1) \in \mathrm{SL}(n, \mathsf{Z})$. Suppose that B does not possess any purely imaginary eigenvalues. Let $\mathfrak{g}$ denote the Lie algebra of G, and $\mathfrak{n}$ the Lie algebra of $N = \mathsf{R}^n \times \{0\}$. As before we identify $\mathfrak{g}$ with the tangent space at the identity element $e = (0, 0)$ of G and denote by b the $(n+1)$th basis vector of $\mathfrak{g} \cong \mathsf{R}^n \rtimes_B \mathsf{R}$.

Let $\mathscr{H}$ be a left-invariant oriented distribution on G such that $b \in \mathscr{H}_e$. Suppose that $\mathscr{H}$ carries a left-invariant Riemannian metric g such that b is orthogonal to $\mathscr{H}_e \cap \mathfrak{n}$ w.r.t. the inner product $\langle\cdot,\cdot\rangle := g_e$ on $\mathscr{H}_e$. We assume that $\mathscr{H}$ is bracket-generating. With $C^0(\mathscr{H}_e) = \mathscr{H}_e$ and $C^k(\mathscr{H}_e) = [\mathscr{H}_e, C^{k-1}(\mathscr{H}_e)]$ for $k \geq 1$, this means $\mathscr{H} = \sum_{k=0}^n C^k(\mathscr{H}_e)$. In particular, it follows

$$[\mathfrak{g}, \mathfrak{g}] = \sum_{k=1}^{n} C^k(\mathscr{H}_e). \tag{13}$$

The latter condition is crucial for the proof of Theorem 3.14 but we do not claim that (13) has significance for left-invariant distributions $\mathscr{H}$ on Lie groups G which do not have the form $G = \mathsf{R}^n \rtimes_A \mathsf{R}$.

Let ∇ be a left-invariant metric connection on $\mathscr{H}$ satisfying condition (3) of Lemma 2.1, which guarantees the symmetry of the sub-Dirac operator. Here the divergence in (3) is defined w.r.t. the left-invariant volume form corresponding to the Haar measure of G. If $\dot{\mathscr{H}}$ is the distribution on $\Gamma \setminus G$ defined by $\mathscr{H}$, then the Riemannian metric on $\dot{\mathscr{H}}$ induced by g is denoted by $\dot{g}$, and the connection on $\dot{\mathscr{H}}$ by $\dot{\nabla}$. Let $\varepsilon : \Gamma \to \mathsf{Z}_2$ be a homomorphism defining a spin structure $P_{\mathrm{Spin},\varepsilon}(\dot{\mathscr{H}}) \cong G \times_\Gamma \mathrm{Spin}(d)$ of $(\dot{\mathscr{H}}, \dot{g})$, where $d = \dim \mathscr{H}$.

We fix a positively oriented orthonormal basis $s_1, \dots, s_d$ of $\mathscr{H}_e$ with $s_1, \dots, s_{d-1} \in \mathscr{H}_e \cap \mathfrak{n}$ and such that s_d is a positive multiple of b. Denoting the corresponding left-invariant vector fields again by $s_1, \dots, s_d$, the sub-Dirac operator, as acting on smooth sections of the spinor bundle $S(\dot{\mathscr{H}}, \varepsilon)$, is given by $D = \sum_i s_i \cdot \dot{\nabla}^S_{s_i}$.

Let Δ be the representation space of the complex spinor representation of $\mathrm{Spin}(\mathscr{H}_e)$. Identifying $\Gamma(S(\dot{\mathscr{H}}, \varepsilon))$ with $C^\infty(G, \varepsilon) \otimes \Delta$, we see that D is given by

$$D = \sum_i d\rho(s_i) \otimes s_i + \frac{1}{4} \sum_{i,j,k} \Gamma_{ij}^k I \otimes s_i s_j s_k =: P + W \tag{14}$$

where the s_i's in the second factor of the tensor products are understood as operators on Δ. Furthermore, the constants $\Gamma_{ij}^k = g(\nabla_{s_i} s_j, s_k)$ are the Christoffel symbols of ∇ w.r.t. the orthonormal frame $s_1, \dots, s_d$ of $\mathscr{H}$, and $d\rho$ is the derivative of the right regular representation ρ of G on $L^2(G, \varepsilon)$. By (3) the second sum in (14) reduces to a sum over all pairwise distinct indices i, j, k.

THEOREM 3.14. *If, in addition to the above assumptions,* $G = \mathsf{R}^n \rtimes_A \mathsf{R}$ *is nilpotent, then the closure of the operator* D *on* $\Gamma \setminus G$ *has a pure point spectrum.*

PROOF. By Proposition 3.7 we know that $L^2(G, \varepsilon) \otimes \Delta$ is a direct sum of the orthogonal subspaces $\{U_\theta \otimes \Delta : \theta \in \mathsf{Z} \setminus \Sigma_{\varepsilon'}\}$ which are invariant under the action of $\rho(G) \otimes \mathscr{C}l(\mathscr{H}_e)$. Let $U_\theta^\infty := U_\theta \cap C^\infty(G, \varepsilon)$. Then $U_\theta^\infty \otimes \Delta$ is D-invariant. To prove that $\bar{D}$ has a pure point spectrum, it suffices to prove that the closure of $D_\theta := D|U_\theta^\infty \otimes \Delta$ has a pure point spectrum for all θ. If $\theta = \{\xi\}$ is a fixed point, then, according to Equation (11), the subspace U_θ is an orthogonal sum of one-dimensional $\rho(G)$-invariant subspaces of U_θ^∞. Thus $U_\theta \otimes \Delta$ is an orthogonal sum of two-dimensional D_θ-invariant subspaces of $\operatorname{dom}(D_\theta)$ which shows that D_θ has a pure point spectrum. Thus we are left with the case where θ is an infinite set and U_θ is isomorphic to $L^2(\mathsf{R})$. Fix $\xi \in \theta$. Lemma 3.8 implies that there exists a unitary isomorphism $T_\xi : U_\theta \to L^2(\mathsf{R})$ such that $\rho_\xi := T_\xi \rho_\theta T_\xi^*$ is given by

$$\rho_\xi(x, s)\psi(t) = e^{\pi i \langle A(t)^\top \xi, x\rangle}\psi(t+s). \tag{15}$$

By Lemma 3.10 we may define $D_\xi = (T_\xi \otimes I)D_\theta(T_\xi^* \otimes I) \,|\, \mathscr{S}(\mathsf{R}, \Delta)$. Since $d\rho_\xi(s_j) = \pi i \langle A(t)^\top \xi, s_j\rangle$ for $1 \le j \le d-1$ and $d\rho_\xi(s_d) = |b|\partial_t$, it follows that the operator $P_\xi := \sum_{j=1}^d d\rho_\xi(s_j) \otimes s_j$ on $\mathscr{S}(\mathsf{R}, \Delta)$ has the form

$$P_\xi = a\partial_t + i\Omega_\xi(t) \tag{16}$$

with $a = |b|^{-1}s_d$ and $\Omega_\xi(t) = \pi \sum_{j=1}^{d-1} \langle A(t)^\top \xi, s_j\rangle\, s_j \in \mathscr{H}_e \subset \mathscr{C}l(\mathscr{H}_e)$. Note that $\langle A(t)^\top \xi, s_j\rangle = \langle \xi, \exp(tB)s_j\rangle$ is a polynomial in t because B is nilpotent. Since

$$[\mathfrak{g}, \mathfrak{g}] = \sum_{k=1}^n B^k(\mathscr{H}_e \cap \mathfrak{n}) = \sum_{k=1}^n \sum_{j=1}^{d-1} \mathsf{R} \cdot (B^k s_j)$$

by (13) and $\langle \xi, [\mathfrak{g}, \mathfrak{g}]\rangle \neq 0$ for non-fixed points, it follows that at least one of the components of Ω_ξ is not constant. Thus Proposition 3.11 implies that $\bar{P}_\xi$ has discrete spectrum. In other words, the essential spectrum of $\bar{P}_\xi$ is empty.

The operator $W_\xi := \frac{1}{4}\sum_{i,j,k} \Gamma_{ij}^k\, s_i s_j s_k$ is bounded on $L^2(\mathsf{R}, \Delta)$. In particular, W_ξ is relatively $\bar{P}_\xi$-compact in the sense that $\operatorname{dom}(\bar{P}_\xi) \subset \operatorname{dom}(W_\xi)$ and $W_\xi(\bar{P}_\xi - iI)^{-1}$ is compact. By the Kato-Rellich theorem we know that $\bar{D}_\xi = \bar{P}_\xi + W_\xi$ is selfadjoint. Moreover, Weyl's theorem which asserts the stability of the essential spectrum under relatively compact perturbations, and for which we refer to Theorem 14.6 of [12], implies that the essential spectrum of $\bar{D}_\xi = \bar{P}_\xi + W_\xi$ is empty. This shows that $\bar{D}_\xi$ and hence $\bar{D}_\theta$ have discrete spectrum. The proof of the theorem is complete.

REMARK 3.15. In general, the eigenvalues of the sub-Dirac operator D do not have finite multiplicity and the spectrum of D is not a discrete subset of R. A relevant example is given in Section 4.3.

3.4. Two- and three-dimensional distributions

In this subsection we will compute the spectra of the operators D_θ arising in the proof of Theorem 3.14 provided that $G = \mathsf{R}^n \rtimes_A \mathsf{R}$ is 2-step nilpotent and $\dim \mathscr{H} = 2$ or 3. The explicit formulas that will be given below in the non-fixed point case are a consequence of the following result.

PROPOSITION 3.16. *Let $\alpha, \beta \in \mathsf{R}$ and $\omega(t) = a\omega_1 t + \omega_0$ where $a > 0$, $\omega_0, \omega_1 \in \mathsf{C}$ and $|\omega_1| = 1$. Then the spectrum of the operator*

$$D = \alpha I + \beta \begin{pmatrix} i\partial_t & \bar{\omega} \\ \omega & -i\partial_t \end{pmatrix} \tag{17}$$

on $\mathscr{S}(\mathsf{R}, \mathsf{C}^2)$ is discrete. More precisely, $\sigma(D) = \{\lambda_0\} \cup \{\lambda_k^\pm : k \in \mathsf{N} \setminus \{0\}\}$, where

$$\lambda_0 := \alpha + \beta \operatorname{Im}(\omega_0\bar{\omega}_1) \qquad \text{and} \qquad \lambda_k^\pm := \alpha \pm \beta\big(2ak + \operatorname{Im}(\omega_0\bar{\omega}_1)^2\big)^{1/2}.$$

If λ_0 and the $\lambda_k^\pm$ are pairwise distinct, then all eigenvalues are simple.

PROOF. We can assume $\alpha = 0$ and $\beta = 1$. Instead of D, we consider the operator $S := Q^* D Q$, where Q is the unitary matrix

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i\bar{\omega}_1 \\ -i\omega_1 & 1 \end{pmatrix},$$

which diagonalizes D^2 and does not depend on t. Obviously the spectra of D and S coincide. We have

$$\begin{aligned} S &= \begin{pmatrix} \operatorname{Im}(\omega_1\bar{\omega}) & \bar{\omega}_1(\partial_t + \operatorname{Re}(\omega_1\bar{\omega})) \\ \omega_1(-\partial_t + \operatorname{Re}(\omega_1\bar{\omega})) & -\operatorname{Im}(\omega_1\bar{\omega}) \end{pmatrix} \\ &= \begin{pmatrix} -\operatorname{Im}(\omega_0\bar{\omega}_1) & \bar{\omega}_1(\partial_t + a\,t + \operatorname{Re}(\omega_0\bar{\omega}_1)) \\ \omega_1(-\partial_t + a\,t + \operatorname{Re}(\omega_0\bar{\omega}_1)) & \operatorname{Im}(\omega_0\bar{\omega}_1) \end{pmatrix}. \end{aligned}$$

To detect S-invariant subspaces, we start with the orthonormal basis $\{h_k : k \in \mathsf{N}\}$ of $L^2(\mathsf{R})$ given by the (normalized) Hermite functions

$$h_k(x) = (2^k \pi^{1/2} k!)^{-1/2} H_k(x) e^{-x^2/2}.$$

Here $H_k(x) = (-1)^k e^{x^2}\partial_x^k[e^{-x^2}]$ is the kth Hermite polynomial. Put $b = 2a\,\mathrm{Re}(\omega_0\bar{\omega}_1)$. Using the unitary isomorphism

$$(U \cdot w)(t) := a^{1/4} w\left(a^{1/2}\left(t + \frac{b}{2a^2}\right)\right)$$

of $L^2(\mathsf{R})$, we then define $u_k = U \cdot h_k$. Recall that the creation operator $\Lambda_+ = -\partial_x + x$ and the annihilation operator $\Lambda_- = \partial_x + x$ satisfy $\Lambda_+(h_k) = \sqrt{2(k+1)}\,h_{k+1}$ and $\Lambda_-(h_k) = \sqrt{2k}h_{k-1}$. Since

$$U\Lambda_\pm U^* = a^{-1/2}\left(\mp\partial_t + a\,t + \frac{b}{2a}\right),$$

we thus obtain

$$\begin{aligned}
\left(-\partial_t + at + \frac{b}{2a}\right)u_k &= \sqrt{2a(k+1)}u_{k+1} &&\text{for}\quad k \geq 0,\\
\left(\partial_t + at + \frac{b}{2a}\right)u_k &= \sqrt{2ak}u_{k-1} &&\text{for}\quad k \geq 1,\\
\left(\partial_t + at + \frac{b}{2a}\right)u_0 &= 0.
\end{aligned}$$

This shows

$$S \cdot \begin{pmatrix} 0 \\ u_0 \end{pmatrix} = \begin{pmatrix} \bar{\omega}_1(\partial_t + at + \mathrm{Re}(\omega_0\bar{\omega}_1))u_0 \\ \mathrm{Im}(\omega_0\bar{\omega}_1)u_0 \end{pmatrix} = \mathrm{Im}(\omega_0\bar{\omega}_1)\begin{pmatrix} 0 \\ u_0 \end{pmatrix},$$

$$S \cdot \begin{pmatrix} u_{k-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -\mathrm{Im}(\omega_0\bar{\omega}_1)u_{k-1} \\ \omega_1(-\partial_t + a\,t + \mathrm{Re}(\omega_0\bar{\omega}_1))u_{k-1} \end{pmatrix} = \begin{pmatrix} -\mathrm{Im}(\omega_0\bar{\omega}_1)u_{k-1} \\ \sqrt{2ak}\,\omega_1 u_k \end{pmatrix},$$

$$S \cdot \begin{pmatrix} 0 \\ u_k \end{pmatrix} = \begin{pmatrix} \bar{\omega}_1(\partial_t + at + \mathrm{Re}(\omega_0\bar{\omega}_1))u_k \\ \mathrm{Im}(\omega_0\bar{\omega}_1)u_k \end{pmatrix} = \begin{pmatrix} \sqrt{2ak}\,\bar{\omega}_1 u_{k-1} \\ \mathrm{Im}(\omega_0\bar{\omega}_1)u_k \end{pmatrix}.$$

In particular, the subspaces

$$V_0 := \mathsf{C}\begin{pmatrix} 0 \\ u_0 \end{pmatrix}$$

and

$$V_k := \mathrm{span}\left\{\varphi_k := \begin{pmatrix} u_{k-1} \\ 0 \end{pmatrix}, \psi_k := \begin{pmatrix} 0 \\ u_k \end{pmatrix}\right\}, \qquad k \geq 1,$$

are S-invariant, and the restriction of S to V_k, $k \geq 1$, is given by the matrix

$$\begin{pmatrix} -\operatorname{Im}(\omega_0\bar{\omega}_1) & \sqrt{2ak}\,\bar{\omega}_1 \\ \sqrt{2ak}\,\omega_1 & \operatorname{Im}(\omega_0\bar{\omega}_1) \end{pmatrix}$$

with respect to the basis φ_k, ψ_k. Since $L^2(\mathsf{R}, \mathsf{C}^2)$ is the direct sum of the V_k, $k \geq 0$, the assertion follows.

Assume that $G = \mathsf{R}^n \rtimes_A \mathsf{R}$ is 2-step nilpotent. Let $(\mathscr{H}, g, \nabla)$ be as in the preceding subsection with $2 \leq \dim \mathscr{H} \leq 3$. First we will determine the spectrum of $D_\theta := D|U_\theta^\infty \otimes \Delta$ when θ does not consist of a single point.

Suppose that $\dim \mathscr{H} = 2$. Let $s_1 \in \mathscr{H}_e \cap \mathfrak{n}$ and $s_2 \in \mathscr{H}_e$ be a positive multiple of b such that s_1, s_2 is a positively oriented orthonormal basis of $\mathscr{H}_e$. In particular, $s_2 = |b|^{-1}b$. By (3) we have $\Gamma_{11}^1 + \Gamma_{21}^2 = 0$ and $\Gamma_{12}^1 + \Gamma_{22}^2 = 0$ which implies that all Christoffel symbols of ∇ vanish. Up to isomorphism, there exists only one simple $\mathscr{C}l(\mathscr{H}_e)$-module. Let $\Delta = \mathsf{C}^2$ be the one such that s_1 and s_2, represented as operators on Δ, are given by

$$s_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad s_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Let $\theta \in \mathsf{Z} \setminus \Sigma_{\varepsilon'}$ and $\xi \in \theta$ be a non-fixed point. If $T_\xi : U_\theta \to L^2(\mathsf{R})$ is a unitary isomorphim as in the proof of Theorem 3.14 and $D_\xi = (T_\xi \otimes I) D_\theta (T_\xi^* \otimes I)|\mathscr{S}(\mathsf{R}, \Delta)$, then we know by (16) that D_ξ has the form

$$D_\xi = |b|^{-1} \begin{pmatrix} i\partial_t & \bar{\omega}_\xi \\ \omega_\xi & -i\partial_t \end{pmatrix}$$

with $\omega_\xi(t) = \pi i |b| \langle A(t)^\top \xi, s_1 \rangle$. Since $B^2 = 0$, we have $A(t) = I + tB$ so that

$$\omega_\xi(t) = \pi i |b| \big(\langle B^\top \xi, s_1 \rangle t + \langle \xi, s_1 \rangle \big)$$

is a non-constant affine-linear function. Thus Proposition 3.16 implies that $\bar{D}_\xi$ has discrete spectrum. Moreover, the eigenvalues of D_ξ can be computed as follows: Put $\alpha = 0$, $\beta = |b|^{-1}$, $a = \pi|b||\langle B^\top \xi, s_1 \rangle|$, $\omega_1 = i \operatorname{sgn}\langle B^\top \xi, s_1 \rangle$ and $\omega_0 = \pi i \langle \xi, s_1 \rangle$. Note that $\operatorname{Im}(\omega_0 \bar{\omega}_1) = 0$. Hence it follows that

$$\lambda_0(\xi) = 0 \qquad \text{and} \qquad \lambda_k^\pm(\xi) = \pm\big(2\pi|b|^{-1}|\langle B^\top \xi, s_1 \rangle| k\big)^{1/2} \tag{18}$$

with $k \in \mathsf{N} \setminus \{0\}$ are the eigenvalues of D_ξ. This completes the case $\dim \mathscr{H} = 2$.

Suppose that $\dim \mathscr{H} = 3$. Choose $s_1, s_2 \in \mathscr{H}_e \cap \mathfrak{n}$ and $s_3 = |b|^{-1}b$ such that s_1, s_2, s_3 becomes a positively oriented orthonormal basis of $\mathscr{H}_e$. Up to isomorphism, there exist two simple $\mathscr{C}l(\mathscr{H}_e)$-modules. Let $\Delta = \mathsf{C}^2$ be the one given by the representation

$$s_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad s_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Note that $s_1 s_2 = s_3$ and $s_i s_j + s_j s_i = -2\delta_{ij}$ for all $1 \leq i, j \leq 3$, as operators on Δ. Using this and that ∇ is metric, we conclude that the second sum in (14) simplifies to $W = -\frac{1}{2}(\Gamma^3_{12} + \Gamma^1_{23} + \Gamma^2_{31}) I \otimes I$.

Let $\theta \in \mathsf{Z} \setminus \Sigma_{\varepsilon'}$ and $\xi \in \theta$ be a non-fixed point. If T_ξ is a unitary isomorphism of U_θ onto $L^2(\mathsf{R})$ such that $\rho_\xi = T_\xi \rho_\theta T_\xi^*$ is given by Equation (10), then the restriction D_ξ of $(T_\xi \otimes I) D_\theta (T_\xi^* \otimes I)$, when realized in $L^2(\mathsf{R}, \Delta)$, to Schwartz functions has the form

$$D_\xi = -\frac{1}{2}(\Gamma^3_{12} + \Gamma^1_{23} + \Gamma^2_{31}) I + |b|^{-1} \begin{pmatrix} i\partial_t & \bar{\omega}_\xi \\ \omega_\xi & -i\partial_t \end{pmatrix}$$

where ω_ξ, as G is 2-step nilpotent, is a non-constant affine linear function given by

$$\begin{aligned} \omega_\xi(t) &= \pi i |b| \big(i \langle A(t)^\top \xi, s_1 \rangle + \langle A(t)^\top \xi, s_2 \rangle \big) \\ &= -\pi |b| \big(\big(\langle B^\top \xi, s_1 \rangle - i \langle B^\top \xi, s_2 \rangle \big) t + \langle \xi, s_1 \rangle - i \langle \xi, s_2, \rangle \big). \end{aligned}$$

Hence Proposition 3.16 implies that the eigenvalues of D_ξ are

$$\text{(19)} \quad \lambda_0(\xi) = -\frac{1}{2}(\Gamma^3_{12} + \Gamma^1_{23} + \Gamma^2_{31}) - \pi \frac{\langle B^\top \xi, s_1 \rangle \langle \xi, s_2 \rangle - \langle \xi, s_1 \rangle \langle B^\top \xi, s_2 \rangle}{\big(\langle B^\top \xi, s_1 \rangle^2 + \langle B^\top \xi, s_2 \rangle^2 \big)^{1/2}}$$

and

$$\begin{aligned} \text{(20)} \quad & \lambda_k^\pm(\xi) \\ &= -\frac{1}{2}(\Gamma^3_{12} + \Gamma^1_{23} + \Gamma^2_{31}) \pm \Bigg(2\pi k |b|^{-1} \big(\langle B^\top \xi, s_1 \rangle^2 + \langle B^\top \xi, s_2 \rangle^2 \big)^{1/2} \\ &\qquad + \pi^2 \frac{\big(\langle B^\top \xi, s_1 \rangle \langle \xi, s_2 \rangle - \langle \xi, s_1 \rangle \langle B^\top \xi, s_2 \rangle \big)^2}{\langle B^\top \xi, s_1 \rangle^2 + \langle B^\top \xi, s_2 \rangle^2} \Bigg)^{1/2}. \end{aligned}$$

In (18)–(20) we rediscover the fact that the eigenvalues $\lambda_k(\xi)$ do not depend on the choice of the point ξ on the orbit. More precisely, since $B^2 = 0$ and $A(t)B = B$, it follows, in accordance with Lemma 3.8, that $\lambda_k^\pm(A(t)^\top \xi) = \lambda_k^\pm(\xi)$ for all $t \in \mathsf{R}$ and $\xi \in \mathsf{R}^n \setminus [\mathfrak{g}, \mathfrak{g}]^\perp$. This completes the case $\dim \mathscr{H} = 3$.

Finally we compute the spectrum of D_θ when $\theta = \{\xi\}$ is a fixed point. For this purpose, we can drop the assumption that G is nilpotent.

By (11) we know that U_θ is an orthogonal sum of 1-dimensional subspaces $\{U_{\theta,k} : k \in \mathsf{Z}\}$ of $U_\theta \cap C^\infty(G,\varepsilon)$ on which $(x,s) \in G$ acts by multiplication with

$$\chi_{\varepsilon^\sharp,\theta,k}(x,s) = e^{\pi i \langle \xi, x\rangle} e^{\pi i (2k+\dot\varepsilon(1))s}.$$

Suppose that $\dim \mathscr{H} = 2$. Let s_1, s_2 and Δ be as above. In this case the sub-Dirac operator reads $D = d\rho(s_1) \otimes s_1 + d\rho(s_2) \otimes s_2$. Since $d\chi_{\varepsilon^\sharp,\theta,k}(s_1) = \pi i \langle \xi, s_1\rangle$ and $d\chi_{\varepsilon^\sharp,\theta,k}(s_2) = \pi i |b|^{-1}(2k + \dot\varepsilon(1))$, it follows that $D_{\theta,k} := D_\theta | U_{\theta,k} \otimes \mathsf{C}^2$ is unitarily equivalent to

$$D_{\xi,k} := \alpha I + \beta \begin{pmatrix} 2k + \dot\varepsilon(1) & \bar\omega_\xi \\ \omega_\xi & -2k - \dot\varepsilon(1) \end{pmatrix} \tag{21}$$

on C^2, where $\alpha = 0$, $\beta = -\pi|b|^{-1}$ and $\omega_\xi = -i|b|\langle \xi, s_1\rangle$ are constants. Obviously, $D_{\xi,k}$ admits the eigenvalues

$$\mu_k^\pm(\xi) = \pm\pi \left(|b|^{-2}(2k + \dot\varepsilon(1))^2 + \langle \xi, s_1\rangle^2\right)^{1/2}, \qquad k \in \mathsf{Z}. \tag{22}$$

Suppose that $\dim \mathscr{H} = 3$. Let s_1, s_2, s_3 and Δ be as above. In this case $D = P + W$ where $W = \alpha\, I \otimes I$ and $\alpha = -\frac{1}{2}(\Gamma^3_{12} + \Gamma^1_{23} + \Gamma^2_{31})$. Hence $D_{\theta,k}$ is unitarily equivalent to $D_{\xi,k}$ as in (21) with $\beta = -\pi|b|^{-1}$ and $\omega_\xi = |b|(\langle \xi, s_1\rangle - i\langle \xi, s_2\rangle)$. Thus $D_{\xi,k}$ has the eigenvalues

$$\begin{aligned} \mu_k^\pm(\xi) = {} & -\frac{1}{2}(\Gamma^3_{12} + \Gamma^1_{23} + \Gamma^2_{31}) \\ & \pm \pi\left(|b|^{-2}(2k + \dot\varepsilon(1))^2 + \langle \xi, s_1\rangle^2 + \langle \xi, s_2\rangle^2\right)^{1/2}. \end{aligned} \tag{23}$$

This shows that D_θ has discrete spectrum in the fixed point case.

4. Examples of spectra of sub-Dirac operators

4.1. A preliminary remark

To compute the spectrum of the sub-Dirac operator D, it remains, by the results in the preceding section for the spectra of the D_θ, to determine a set of representatives for the set of all Z-orbits contained in $\Sigma_{\varepsilon'}$. More precisely, in view of Theorem 3.9, we carry out the following steps:

(1) Describe all homomorphisms $\varepsilon : \Gamma \to \mathsf{Z}_2$.
(2) Find a set of representatives $\mathscr{R}_{\varepsilon'}$ for all R-orbits ω intersecting $\Sigma_{\varepsilon'}$.
(3) Compute the number of Z-orbits contained in $\omega \cap \Sigma_{\varepsilon'}$.
(4) Determine the spectrum of D_ξ for some $\xi \in \omega$.

This requires a detailed knowledge of the orbit picture of the coadjoint representation. In the following examples, the eigenvalues of the sub-Dirac operator including their multiplicities will be determined completely.

4.2. Three-dimensional Heisenberg manifolds

As we will see next, the results of this section comprise Theorem 3.1 of [2] concerning the spectrum of the Dirac operator on three-dimensional Heisenberg manifolds.

Let $G = \mathsf{R}^2 \rtimes_A \mathsf{R}$ be the Heisenberg group as discussed in Example 3.2 and $\Gamma = \mathsf{Z}^2 \rtimes_A \mathsf{Z}$. Then $\mathfrak{g} = \operatorname{span}\{e_1, e_2, b\}$ with $[b, e_2] = re_1$. For positive real numbers d and T we consider the orientation and the Riemannian metric g on $\mathscr{H} := TG$ such that $s_1 = \frac{1}{T}e_1$, $s_2 = -de_2$ and $s_3 = \frac{d}{r}b$ becomes a positively oriented orthonormal frame. The constants are chosen in accordance with [2], where the collapse of Heisenberg manifolds $M(r, d, T)$ for $T \to 0$ is studied. Let ∇ be the Levi-Civita connection of g. In particular, ∇ satisfies (3) and $-\Gamma^3_{12} = \Gamma^1_{23} = \Gamma^2_{31} = \frac{d^2T}{2}$. Let $\varepsilon' : \mathsf{Z}^2 \to \mathsf{Z}_2$ be a homomorphism. We abbreviate $\varepsilon'(e_\mu)$ to ε_μ. Then (7) is satisfied if and only if $\varepsilon_1 r$ is even. It is easy to see that the disjoint union $\mathscr{R}_{\varepsilon'}$ of

$$\mathscr{R}^{(1)}_{\varepsilon'} = \{\xi \in \Sigma_{\varepsilon'} : \xi_1 = 0\} \quad \text{and} \quad \mathscr{R}^{(2)}_{\varepsilon'} = \{\xi \in \Sigma_{\varepsilon'} : \xi_1 \neq 0 \text{ and } \xi_2 = \varepsilon_2\}$$

is a set of representatives for the set of all R-orbits intersecting $\Sigma_{\varepsilon'}$. The set $\mathscr{R}^{(1)}_{\varepsilon'}$ consists of all fixed points in $\Sigma_{\varepsilon'}$. If ω is the R-orbit represented by $(\xi_1, \varepsilon_2) \in \mathscr{R}^{(2)}_{\varepsilon'}$, then $\omega \cap \Sigma_{\varepsilon'}$ contains $|\xi_1 r|/2$ distinct Z-orbits.

We compute $|b| = \frac{r^2}{d^2}$, $\langle \xi, s_1\rangle = \frac{1}{T}\xi_1$, $\langle \xi, s_2\rangle = -d\xi_2$, $\langle B^\top \xi, s_1\rangle = 0$ and $\langle B^\top \xi, s_2\rangle = -dr\xi_1$. Inserting this into (23) gives

$$\mu_k^\pm(0, \xi_2) = -\frac{d^2T}{4} \pm \pi\left((2k + \dot{\varepsilon}(1))^2\frac{d^2}{r^2} + d^2\xi_2^2\right)^{1/2}. \tag{24}$$

Similarly, for $\xi \in \Sigma_{\varepsilon'}$ with $\xi_1 \neq 0$, Equations (19) and (20) yield

$$\lambda_0(\xi_1, \varepsilon_2) = -\frac{d^2T}{4} - \frac{\pi}{T}|\xi_1| \tag{25}$$

and

$$\lambda_k^\pm(\xi_1, \varepsilon_2) = -\frac{d^2T}{4} \pm \left(2\pi d^2 k|\xi_1| + \frac{\pi^2}{T^2}\xi_1^2\right)^{1/2}. \tag{26}$$

We will use the following notation for the description of the spectrum of the Dirac operator D on $\Gamma \setminus G$. We define the spectral multiplicity function

$$m(D) : \mathsf{R} \to \mathsf{N} \cup \{|\mathsf{N}|\}, \qquad m(D)(\lambda) = \dim\ker(D - \lambda I).$$

Moreover, $\delta = \delta(\lambda)$ denotes the function that takes the value 1 in λ and that is zero on $\mathsf{R} \setminus \{\lambda\}$.

Suppose we are given a spin structure corresponding to a homomorphism $\varepsilon : \mathsf{Z}^2 \rtimes_A \mathsf{Z} \to \mathsf{Z}_2$ with $\varepsilon_1 = 0$. Summation of (24), (25) and (26) over $\xi_\nu \in 2\mathsf{Z} + \varepsilon_\nu$ gives

$$m(D) = m_1^+(D) + m_1^-(D) + m_2^+(D) + m_2^0(D) + m_2^-(D),$$

where

$$m_1^\pm(D) = \sum_{k\in\mathsf{Z}}\sum_{l\in\mathsf{Z}} \delta\Big(-\frac{d^2T}{4} \pm \frac{\pi d}{r}\big((2k+\acute{\varepsilon}(1))^2 + r^2(2l+\varepsilon_2)^2\big)^{1/2}\Big),$$

$$m_2^0(D) = \sum_{l=1}^{\infty} 2rl\delta\Big(-\frac{d^2T}{4} - \frac{2\pi l}{T}\Big),$$

$$m_2^\pm(D) = \sum_{l=1}^{\infty} 2rl \sum_{k=1}^{\infty} \delta\Big(-\frac{d^2T}{4} \pm \Big(4\pi d^2 kl + \frac{4\pi^2 l^2}{T^2}\Big)^{1/2}\Big).$$

Now assume $\varepsilon_1 = 1$, which is only possible if r is even. Then $\mathscr{R}^{(1)}_{\varepsilon'} = \emptyset$ and we obtain

$$m(D) = m_2^+(D) + m_2^0(D) + m_2^-(D),$$

where now

$$m_2^0(D) = \sum_{l=0}^{\infty} (2l+1)r\delta\Big(-\frac{d^2T}{4} - \frac{\pi(2l+1)}{T}\Big),$$

$$m_2^\pm(D) = \sum_{l=0}^{\infty} (2l+1)r \sum_{k=1}^{\infty} \delta\Big(-\frac{d^2T}{4} \pm \Big(2\pi d^2 k(2l+1) + \frac{\pi^2(2l+1)^2}{T^2}\Big)^{1/2}\Big).$$

Now let us turn to the sub-Riemannian case and suppose $\mathscr{H} = \mathrm{span}\{s_2, s_3\}$, where again s_2 and s_3 are orthonormal. Let ∇ be defined by (2) for a leftinvariant complement $\mathscr{V} := \mathsf{R}\cdot u$, $u \in \mathfrak{g}$, of $\mathscr{H}$. Since we wish to get a symmetric sub-Dirac operator, the only possible choice is $\mathscr{V} := \mathsf{R}\cdot s_1$. Indeed, otherwise $[\Gamma(\mathscr{H}), u] \not\subset \Gamma(\mathscr{H})$, thus D is not symmetric by Lemma 2.1. We proceed as above, now using Equations (22) and (18).

For a spin structure that corresponds to a homomorphism $\varepsilon : \mathsf{Z}^2 \rtimes_A \mathsf{Z} \to \mathsf{Z}_2$ with $\varepsilon_1 = 0$ we obtain

$$m(D) = |\mathsf{N}|\cdot\delta(0) + m_1^+(D) + m_1^-(D) + m_2^+(D) + m_2^-(D),$$

where

$$m_1^{\pm}(D) = \sum_{k\in\mathsf{Z}} \sum_{l\in\mathsf{Z}} \delta\Big(\pm\frac{\pi d}{r}\big((2k+\dot{\varepsilon}(1))^2 + r^2(2l+\varepsilon_2)^2\big)^{1/2}\Big),$$

$$m_2^{\pm}(D) = \sum_{l=1}^{\infty} 2rl \sum_{k=1}^{\infty} \delta\big(\pm(4\pi d^2 kl)^{1/2}\big).$$

If $\varepsilon_1 = 1$, then

$$m(D) = |\mathsf{N}| \cdot \delta(0) + \sum_{l=0}^{\infty}(2l+1)r \sum_{k=1}^{\infty}\Big(\delta\big((2\pi d^2 k(2l+1))^{1/2}\big) + \delta\big(-(2\pi d^2 k(2l+1))^{1/2}\big)\Big).$$

4.3. A five-dimensional two-step nilpotent example

We start by considering 2-step nilpotent Lie groups which are isomorphic to a standard model $G = \mathsf{R}^{2p} \rtimes_A \mathsf{R}$ as described in Lemma 4.1 and therefore generalise the example from the preceding subsection. We will describe the orbits of R and Z acting on R^{2p} by $A^\top$. Then we will specialise to $\dim G = 5$ for the computation of the spectrum of the sub-Dirac operator on $\Gamma \setminus G$.

Lemma 4.1. *Let G be a simply connected Lie group satisfying $[G, G] = Z(G)$ and admitting a connected abelian normal subgroup N of codimension 1. Let Γ be a uniform discrete subgroup of G such that $\Gamma \cap N$ is uniform in N. Then there exist $p \geq 1$, a one-parameter subgroup of* $\mathrm{GL}(2p, \mathsf{R})$ *of the form*

$$A(t) = \begin{pmatrix} I & tR \\ 0 & I \end{pmatrix}$$

with $R = \mathrm{diag}(r_1, \ldots, r_p)$ and positive integers r_ν such that $r_{\nu+1}|r_\nu$ for $\nu = 1, \ldots, p-1$, and an isomorphism Φ of G onto $\mathsf{R}^{2p} \rtimes_A \mathsf{R}$ such that $\Phi(\Gamma) = \mathsf{Z}^{2p} \times_A \mathsf{Z}$.

Proof. Put $p = \dim Z(G) = \frac{1}{2}\dim N$. Since $\Gamma \cap Z(G)$ is uniform in $Z(G)$, we find generators $v_1, \ldots, v_{2p}$ of $\Gamma \cap N$ such that $\Gamma \cap Z(G) = \mathsf{Z}v_1 + \cdots + \mathsf{Z}v_p$. As in the proof of Lemma 3.1, we consider the linear isomorphism $M : \mathsf{R}^{2p} \to N$ given by $M(e_\nu) = v_\nu$, choose $b \in \mathfrak{g}$ such that $\exp(b) \in \Gamma$ and $\exp(b)N$ generates $\Gamma N/N$, and define $A_0(t) \in \mathrm{GL}(2p, \mathsf{R})$ such that $\Phi_0(x, t) = M(x)\exp(tb)$ becomes an isomorphism of G onto $\mathsf{R}^{2p} \rtimes_{A_0} \mathsf{R}$. Since $\Phi_0(Z(G)) = \mathsf{R}^p \times \{0\} \times \{0\}$, we have

$$A_0(t) = \begin{pmatrix} I & tR_0 \\ 0 & I \end{pmatrix}$$

with $R_0 \in \mathrm{GL}(p,\mathsf{Z})$. Recall that R_0 can be brought into Smith normal form, i.e., there exist $Q_1, Q_2 \in \mathrm{GL}(p,\mathsf{Z})$ such that $R := Q_1 R_0 Q_2^{-1} = \mathrm{diag}(r_1,\dots,r_p)$ is diagonal with positive integers r_ν such that $r_{\nu+1}|r_\nu$. Clearly $\Psi(x',x'',t) := (Q_1x', Q_2x'', t)$ gives an isomorphism of $\mathsf{R}^{2p} \rtimes_{A_0} \mathsf{R}$ onto $\mathsf{R}^{2p} \rtimes_A \mathsf{R}$ with $A(t)(x',x'') = (x' + tRx'', x'')$. Finally, it follows that the assertion of the lemma holds for $\Phi := \Psi\Phi_0$.

Let $G = \mathsf{R}^{2p} \rtimes_A \mathsf{R}$ be as in Lemma 4.1 with uniform discrete subgroup $\Gamma = \mathsf{Z}^{2p} \rtimes_A \mathsf{Z}$. In particular, $A(t)e_\nu = e_\nu$ and $A(t)e_{p+\nu} = e_{p+\nu} + r_\nu t\, e_\nu$ for all $1 \le \nu \le p$.

Let $\dot\varepsilon : \mathsf{Z} \to \mathsf{Z}_2$ and $\varepsilon' : \mathsf{Z}^{2p} \to \mathsf{Z}_2$ be homomorphisms. As before, we put $\varepsilon_\nu = \varepsilon'(e_\nu)$ for $\nu = 1,\dots,2p$. By Lemma 3.3 it follows that $\varepsilon(k,l) := \varepsilon'(k) + \dot\varepsilon(l)$ is a homomorphism of Γ if and only if $r_\nu\varepsilon_\nu \in 2\mathsf{Z}$ for $1 \le \nu \le p$. Note that the latter condition implies $\varepsilon_\nu = 0$ whenever r_ν is odd.

Next we will describe the coadjoint orbits. First of all,

$$\langle A(t)^\top \xi, e_\nu\rangle = \xi_\nu \qquad \text{and} \qquad \langle A(t)^\top \xi, e_{p+\nu}\rangle = \xi_{p+\nu} + r_\nu \xi_\nu t \tag{27}$$

for $1 \le \nu \le p$. To formulate the subsequent result, a little more notation is needed. If $\xi \in \mathsf{Z}^{2p}$, then $\bar\xi \in \mathsf{Z}^p$ denotes the projection of ξ onto the first p variables. For $\eta \in \mathsf{Z}^p$, the subset $\{\xi \in \mathsf{Z}^{2p} : \bar\xi = \eta\}$ is Z-invariant. In particular, $\{\xi : \bar\xi = 0\}$ is the set of all points remaining fixed under the coadjoint action. Put $J_\eta = \{\nu : \eta_\nu \ne 0\}$. For $\eta \ne 0$, let $d_\eta > 0$ be the greatest common divisor of the integers $|r_1\eta_1|,\dots,|r_p\eta_p|$. We choose $j_\eta = \min J_\eta$ and set $q_\eta = |r_{j_\eta}\eta_{j_\eta}|/d_\eta$.

Let $\bar\Sigma_{\varepsilon'}$ be the image of $\Sigma_{\varepsilon'}$ under projection. If $\eta \in \bar\Sigma_{\varepsilon'} \setminus \{0\}$, then d_η is even because η_ν is even whenever r_ν is odd. Furthermore, we define $\mathscr{R}_{\varepsilon',0} = \{\xi \in \Sigma_{\varepsilon'} : \bar\xi = 0\}$ and $\mathscr{R}_{\varepsilon',\eta} = \{\xi \in \Sigma_{\varepsilon'} : \bar\xi = \eta \text{ and } 0 \le \xi_{p+j_\eta} \le 2q_\eta - 1\}$ for $\eta \in \bar\Sigma_{\varepsilon'}$ non-zero. Note that $\mathscr{R}_{\varepsilon',0}$ is empty if $\varepsilon_\nu = 1$ for some $1 \le \nu \le p$.

LEMMA 4.2. *In this situation, the following holds true:*

(i) *The disjoint union $\mathscr{R}_{\varepsilon'} := \bigcup_{\eta\in\bar\Sigma_{\varepsilon'}} \mathscr{R}_{\varepsilon',\eta}$ is a set of representatives for the set of all R-orbits intersecting $\Sigma_{\varepsilon'}$.*

(ii) *Let ω be an R-orbit which intersects $\Sigma_{\varepsilon'}$. Then $\eta := \bar\xi$ does not depend on the choice of $\xi \in \omega \cap \Sigma_{\varepsilon'}$. If ω is not a fixed point, then $\omega \cap \Sigma_{\varepsilon'}$ consists of $d_\eta/2$ distinct Z-orbits.*

PROOF. Let $\xi \in \Sigma_{\varepsilon'}$ such that $\bar\xi \ne 0$. By (27) we know that $A(t)^\top\xi \in \Sigma_{\varepsilon'}$ if and only if $r_\nu\xi_\nu t \in 2\mathsf{Z}$ for all $\nu \in J_{\bar\xi}$. This proves

$$\{t \in \mathsf{R} : A(t)^\top \xi \in \Sigma_{\varepsilon'}\} = \bigcap_{\nu\in J_{\bar\xi}} \frac{2}{|r_\nu\xi_\nu|}\mathsf{Z} = \frac{2}{d_{\bar\xi}}\mathsf{Z} \tag{28}$$

To prove (i), let ω be an R-orbit and $\xi \in \omega \cap \Sigma_{\varepsilon'}$. Clearly $\eta := \bar{\xi}$ does not depend on the choice of ξ. We can assume $\bar{\xi} \neq 0$. Define d_η and $j = j_\eta$ as above. Since $\langle A(t)^\top \xi, e_{p+j} \rangle = e_{p+j} + r_j \xi_j\, t$, it follows from (28) that there exists $t \in \frac{2}{d_\eta}\mathsf{Z}$ such that $A(t)^\top \xi \in \Sigma_{\varepsilon'}$ and $0 \leq \langle A(t)^\top \xi, e_{p+j} \rangle \leq 2q_\eta - 1$. This proves $A(t)^\top \xi \in \omega \cap \mathscr{R}_{\varepsilon',\eta}$ because $\overline{A(t)^\top \xi} = \bar{\xi}$. We claim that $\omega \cap \mathscr{R}_{\varepsilon',\eta}$ consists of a single point: If $\xi, \xi^* \in \omega \cap \mathscr{R}_{\varepsilon',\eta}$, then, again by (28), there exists a $t \in \frac{2}{d_\eta}\mathsf{Z}$ such that $\xi^* = A(t)^\top \xi$. In particular $\xi^*_{p+j} = \xi_{p+j} + r_j \xi_j t$. Since $0 \leq \xi_{p+j}, \xi^*_{p+j} \leq 2q_\eta - 1$, it follows $t = 0$ and hence $\xi^* = \xi$. This proves $\mathscr{R}_{\varepsilon'}$ to be a set of representatives.

Let $\xi \in \omega \cap \Sigma_{\varepsilon'}$ be an arbitary non-fixed point. Then $f : \mathsf{R} \to \omega$, $f(t) = A(t)^\top \xi$, is bijective and R-equivariant. By (28) it holds $f^{-1}(\omega \cap \Sigma_{\varepsilon'}) = \frac{2}{d_\eta}\mathsf{Z}$. Since $d_\eta/2$ is an integer, it follows

$$\#\,\mathsf{Z} \setminus \omega \cap \Sigma_{\varepsilon'} = \#\,\mathsf{Z} \setminus \frac{2}{d_\eta}\mathsf{Z} = \frac{d_\eta}{2}.$$

More precisely, the points $\left\{ A\left(\frac{2k}{d_\eta}\right)^\top \xi : 0 \leq k < \frac{d_\eta}{2} \right\}$ are representatives for the set of all Z-orbits in $\omega \cap \Sigma_{\varepsilon'}$.

We point out that the choice of the set $\mathscr{R}_{\varepsilon'}$ is in no way canonical. For example, any choice of indices $j_\eta \in J_\eta$ leads to a set of representatives.

Now let us restrict ourselves to $p = 2$. Then canonical basis $e_1, \ldots, e_4, b$ of the Lie algebra $\mathfrak{g} \cong \mathsf{R}^4 \rtimes_B \mathsf{R}$ of G satisfies the relations $[b, e_3] = r_1 e_1$ and $[b, e_4] = r_2 e_2$.

Put $s_1 = e_3$, $s_2 = e_4$ and $s_3 = b$. As before, the corresponding left-invariant vector fields are denoted by the same symbol. The left-invariant distribution $\mathscr{H} := \operatorname{span}\{s_1, s_2, s_3\}$ is given the orientation and Riemannian metric g such that s_1, s_2, s_3 becomes a positively oriented, orthonormal frame. In particular, $|b| = 1$. Note that $\mathscr{H}$ is bracket-generating.

Remark 4.3. In general, when $\mathscr{H}$ is a left-invariant 3-dimensional distribution on a Lie group G, the affine space of all left-invariant metric connections in $\mathscr{H}$ satisfying (3) has dimension 6. However, in the present example, the left-invariant connections which are defined by a left-invariant projection pr onto $\mathscr{H}$ and the Koszul formula (2) and which satisfy (3) form a 3-dimensional space.

Let ∇ be a left-invariant metric connection in $\mathscr{H}$ satisfying (3). For example, we could take the connection given by projection onto $\mathscr{H}$ along $\mathscr{V} := \operatorname{span}\{e_1, e_2\}$, which, according to (2), satisfies $\Gamma^k_{ij} = 0$ for all i, j, k because $[\mathfrak{g}, \mathfrak{g}] \subset \mathscr{V}$. Let $\varepsilon : \Gamma \to \mathsf{Z}_2$ be a homomorphism giving a spin structure of $\mathscr{H}$. By Lemma 2.1 the sub-Dirac operator D defined by $(\mathscr{H}, g, \nabla, \varepsilon)$ is symmetric. We compute its spectrum. To this end, we note that the coadjoint representation is given by

$$B^\top \xi = \begin{pmatrix} 0 \\ 0 \\ r_1\xi_1 \\ r_2\xi_2 \end{pmatrix} \qquad \text{and} \qquad A(t)^\top \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 + r_1\xi_1 t \\ \xi_4 + r_2\xi_2 t \end{pmatrix}.$$

In particular, we get $\langle \xi, s_\nu\rangle = \langle \xi, e_{2+\nu}\rangle = \xi_{2+\nu}$ and $\langle B^\top\xi, s_\nu\rangle = r_\nu\xi_\nu$ for $\nu = 1, 2$. Put $\alpha = -\frac{1}{2}(\Gamma^3_{12}+\Gamma^1_{23}+\Gamma^2_{31})$. By (23), (19) and (20), the eigenvalues of D_ξ are of the form

$$\mu_k^\pm(\xi) = \alpha \pm \pi\big((2k+\dot\varepsilon(1))^2 + \xi_3^2 + \xi_4^2\big)^{1/2}$$

for fixed points, and

$$\lambda_0(\xi) = \alpha - \pi\frac{r_1\xi_1\xi_4 - r_2\xi_2\xi_3}{(r_1^2\xi_1^2 + r_2^2\xi_2^2)^{1/2}}$$

or

$$\lambda_k^\pm(\xi) = \alpha \pm \left(2\pi k(r_1^2\xi_1^2 + r_2^2\xi_2^2)^{1/2} + \pi^2\frac{(r_1\xi_1\xi_4 - r_2\xi_2\xi_3)^2}{r_1^2\xi_1^2 + r_2^2\xi_2^2}\right)^{1/2}$$

else. We want to decompose the set $\mathcal{R}_{\varepsilon'}$ of representatives into a disjoint union of sets that we can describe explicitly. To this end, consider $\eta = \bar\xi \in \bar\Sigma_{\varepsilon'}$ and assume $\eta \neq 0$. If $\eta_1 \neq 0$ and $\eta_2 = 0$, then $j_\eta = 1$, $d_\eta = |r_1\eta_1|$ and $q_\eta = 1$. Similarly, if $\eta_1 = 0$ and $\eta_2 \neq 0$, then $j_\eta = 2$, $d_\eta = |r_2\eta_2|$ and $q_\eta = 1$. For $\eta_1\eta_2 \neq 0$ we get $j_\eta = 1$, and obtain $d_\eta = \gcd(|r_1\eta_1|, |r_2\eta_2|)$ and $q_\eta = |r_1\eta_1|/d_\eta$. This leads to a decomposition of $\mathcal{R}_{\varepsilon'}$ into the following subsets:

$$\begin{aligned}
\mathcal{R}^{(1)}_{\varepsilon'} &= \mathcal{R}_{\varepsilon',0},\\
\mathcal{R}^{(2)}_{\varepsilon'} &= \big\{\xi \in \Sigma_{\varepsilon'} : \xi_1 \neq 0, \xi_2 = 0, \xi_3 = \varepsilon_3\big\},\\
\mathcal{R}^{(3)}_{\varepsilon'} &= \big\{\xi \in \Sigma_{\varepsilon'} : \xi_1 = 0, \xi_2 \neq 0, \xi_4 = \varepsilon_4\big\},\\
\mathcal{R}^{(4)}_{\varepsilon'} &= \big\{\xi \in \Sigma_{\varepsilon'} : \xi_1 \neq 0, \xi_2 \neq 0, 0 \le \xi_3 \le 2q_{(\xi_1,\xi_2)} - 1\big\}.
\end{aligned}$$

We have $\mathcal{R}^{(1)}_{\varepsilon'} = \emptyset$ if $\varepsilon_1 = 1$ or $\varepsilon_2 = 1$, $\mathcal{R}^{(2)}_{\varepsilon'} = \emptyset$ if $\varepsilon_2 = 1$, and $\mathcal{R}^{(3)}_{\varepsilon'} = \emptyset$ if $\varepsilon_1 = 1$. Recall that for $\nu = 1, 2$ the case $\varepsilon_\nu = 1$ can occur only if r_ν is even.

The spectrum of D depends on the spin structure given by ε. It holds $m(D) = \sum_{i=1}^4 m_i$ where $m_i = \sum_{\xi\in\mathcal{R}^{(i)}_{\varepsilon'}} m(D_\xi)$. Note that $m_i = 0$ if $\mathcal{R}^{(i)}_{\varepsilon'} = \emptyset$.

Otherwise, m_i is given as follows, where the sums are meant to be taken over $\xi_\nu \in 2\mathsf{Z} + \varepsilon_\nu$ and $\xi_5 \in 2\mathsf{Z} + \dot{\varepsilon}(1)$.

$$m_1 = \sum_{\xi_3,\xi_4,\xi_5} \delta(\alpha + \pi(\xi_3^2 + \xi_4^2 + \xi_5^2)^{1/2}) + \delta(\alpha - \pi(\xi_3^2 + \xi_4^2 + \xi_5^2)^{1/2})$$

$$m_2 = \sum_{\xi_1 \neq 0} \frac{|r_1\xi_1|}{2} \sum_{\xi_4} \Big(\delta(\alpha - \pi \operatorname{sgn}(r_1\xi_1)\xi_4) + \sum_{k=1}^{\infty} \big(\delta(\alpha + (2\pi k|r_1\xi_1| + \pi^2\xi_4^2)^{1/2}) + \delta(\alpha - (2\pi k|r_1\xi_1| + \pi^2\xi_4^2)^{1/2})\big)\Big)$$

$$m_3 = \sum_{\xi_2 \neq 0} \frac{|r_2\xi_2|}{2} \sum_{\xi_3} \Big(\delta(\alpha + \pi \operatorname{sgn}(r_2\xi_2)\xi_3) + \sum_{k=1}^{\infty} \big(\delta(\alpha + (2\pi k|r_2\xi_2| + \pi^2\xi_3^2)^{1/2}) + \delta(\alpha - (2\pi k|r_2\xi_2| + \pi^2\xi_3^2)^{1/2})\big)\Big)$$

$$m_4 = \sum_{\xi_1 \neq 0} \sum_{\xi_2 \neq 0} \frac{\gcd(|r_1\xi_1|, |r_2\xi_2|)}{2} \cdot \sum_{0 \leq \xi_3 \leq 2q_{(\xi_1,\xi_2)}-1} \sum_{\xi_4} \Big(\delta(\lambda_0(\xi)) + \sum_{k=1}^{\infty} \big(\delta(\lambda_k^+(\xi)) + \delta(\lambda_k^-(\xi))\big)\Big)$$

In particular, if $\varepsilon_\nu = 0$ for $\nu = 1$ or 2, then the numbers $\{\alpha + (2k + \varepsilon_{2+\nu})\pi : k \in \mathsf{Z}\}$ are eigenvalues of D and each of them has infinite multiplicity.

In this example, the spectrum of D is a non-discrete subset of R, no matter which homomorphism $\varepsilon : \Gamma \to \mathsf{Z}_2$ defining the underlying spin structure is chosen. Indeed, $\alpha^* := \alpha + \pi \operatorname{sgn}(r_2)\varepsilon_3$ is an accumulation point of $\sigma(D)$. To see this, we consider the sequence $\xi_n \in \mathscr{R}_{\varepsilon'}^{(4)}$ given by $\xi_{n1} = 2 + \varepsilon_1$, $\xi_{n2} = 2n + \varepsilon_2$, $\xi_{n3} = \varepsilon_3$ and $\xi_{n4} = \operatorname{sgn}(r_1r_2)(2 + \varepsilon_4)$. Then $\lambda_0(\xi_n) \neq \alpha^*$ and $\lambda_0(\xi_n) \to \alpha^*$ for $n \to +\infty$.

4.4. A three-step nilpotent example

Let $r_1, r_2 \in \mathsf{Z} \setminus \{0\}$ be such that r_1r_2 is even. Define a Lie algebra structure on $\mathfrak{g} := \operatorname{span}\{e_1, e_2, e_3, b\}$ such that $\mathfrak{n} := \operatorname{span}\{e_1, e_2, e_3\}$ is an abelian ideal and $[b, X] = B(X)$ for $X \in \mathfrak{n}$, where $B : \mathfrak{n} \to \mathfrak{n}$ is given by

$$B = \begin{pmatrix} 0 & r_1 & 0 \\ 0 & 0 & r_2 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis e_1, e_2, e_3 of $\mathfrak{n}$. Let G be the simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then $G = \mathsf{R}^3 \rtimes_A \mathsf{R}$ with

$$A(t) = \exp tB = \begin{pmatrix} 1 & tr_1 & t^2 r_1 r_2/2 \\ 0 & 1 & tr_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $A(1)$ is in $SL(2, \mathsf{Z})$, the subset $\Gamma := \mathsf{Z}^3 \rtimes_A \mathsf{Z}$ a uniform discrete subgroup of G. Let $(\mathscr{H}, g)$ be the oriented sub-Riemannian structure having $s_1 := e_3$, $s_2 := b$ as a positively oriented orthonormal frame. Then $\mathscr{H}$ is bracket generating.

The spin structures of $\dot{\mathscr{H}}$ correspond to homomorphisms $\varepsilon : \Gamma \to \mathsf{Z}_2$. As above we write $\varepsilon(k, l) = \varepsilon'(k) \cdot \dot{\varepsilon}(l)$, where $\dot{\varepsilon} : \mathsf{Z} \to \mathsf{Z}_2$ is an arbitrary homomorphism and $\varepsilon' : \mathsf{Z}^3 \to \mathsf{Z}_2$ is a homomorphism satisfying (7), which, in this example, means that $r_1\varepsilon_1$ and $r_1r_2\varepsilon_1/2 + r_2\varepsilon_2$ are both even. More precisely, this shows: If r_1 and r_2 are both even, then ε_1 and ε_2 are arbitrary. If r_1 is odd and r_2 is even, then $\varepsilon_1 = 0$ and ε_2 is arbitrary. Now suppose that r_2 is odd. If, in addition, r_1 is odd, then $\varepsilon_1 = \varepsilon_2 = 0$. If r_1 is even but not divisble by 4, then either $\varepsilon_1 = \varepsilon_2 = 0$ or $\varepsilon_1 = \varepsilon_2 = 1$. Finally, if r_1 is divisible by 4, then $\varepsilon_2 = 0$.

Clearly $\mathscr{V} := \operatorname{span}\{e_1, e_2\}$ is a complement of $\mathscr{H}$ in the tangent bundle TG. Using the projection onto $\mathscr{H}$ along $\mathscr{V}$, we define a left-invariant connection ∇ in $\mathscr{H}$ by the Koszul formula (3). Since $\mathrm{pr}[s_1, s_2] = 0$, all Christoffel symbols Γ_{ij}^k vanish. In particular, the sub-Dirac operator is symmetric and equals

$$D = s_1 \cdot \partial_{s_1} + s_2 \cdot \partial_{s_2},$$

where we use the simple $\mathscr{C}l(\mathscr{H}_e)$-module structure on C^2 defined by

$$s_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad s_2 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

On the other hand, we have

$$A^\top(t)\xi = \begin{pmatrix} \xi_1 \\ \xi_2 + tr_1\xi_1 \\ \xi_3 + tr_2\xi_2 + t^2 r_1 r_2 \xi_1/2 \end{pmatrix}. \tag{29}$$

In particular, the sets

$$\begin{aligned} R^{(1)} &:= \{\xi \in \mathsf{R}^3 \mid \xi_1 = \xi_2 = 0\}, \\ R^{(2)} &:= \{\xi \in \mathsf{R}^3 \mid \xi_1 = 0,\ \xi_2 \neq 0\}, \\ R^{(3)} &:= \{\xi \in \mathsf{R}^3 \mid \xi_1 \neq 0\} \end{aligned}$$

are invariant under $A^\top(t)$ for all $t \in \mathsf{R}$.

Let us first consider D_θ for the orbit $\theta = \{\xi\}$ of an element $\xi \in R^{(1)}$. Then, according to (22), the spectrum of D_θ consists of the eigenvalues

$$\mu_k^\pm(\xi) = \pm\pi\big((2k+\dot{\varepsilon}(1))^2 + \xi_3^2\big)^{1/2}, \qquad k \in \mathsf{Z}.$$

Now consider $\xi \in R^{(2)}$. Then D_ξ has the form

$$\begin{pmatrix} i\partial_t & \bar{\omega} \\ \omega & -i\partial_t \end{pmatrix} \tag{30}$$

with $\omega(t) = a\omega_1 t + \omega_0$, where

$$a = \pi|r_2\xi_2|, \qquad \omega_1 = \operatorname{sgn}(r_2\xi_2)\cdot i, \qquad \omega_0 = \pi i\xi_3.$$

According to (18) the spectrum of D_ξ consists of the eigenvalues $\lambda_0 = 0$ and

$$\lambda_k^\pm = \pm(2\pi|r_2\xi_2|k)^{1/2}, \qquad k \in \mathsf{N}\setminus\{0\}.$$

Finally, take $\xi \in R^{(3)}$. Then D_ξ is of the form (30) where $\omega(t) = i\pi(\xi_1 r_1 r_2 t^2/2 + \xi_2 r_2 t + \xi_3)$. Hence

$$D_\xi^2 = \begin{pmatrix} -\partial_t^2 - \omega(t)^2 & -i\omega'(t) \\ -i\omega'(t) & -\partial_t^2 - \omega(t)^2 \end{pmatrix}.$$

Obviously, D_ξ^2 is time-independent diagonalisable. More exactly, D_ξ^2 is conjugate to

$$\begin{pmatrix} -\partial_t^2 - \omega(t)^2 - i\omega'(t) & 0 \\ 0 & -\partial_t^2 - \omega(t)^2 + i\omega'(t) \end{pmatrix}.$$

The operators $-\partial_t^2 - \omega(t)^2 \mp i\omega'(t)$ are of the form

$$P_{a,b,c}^\pm := \partial_t^2 + (at^2+bt+c)^2 \pm (2at+b)$$

for

$$a = \pi\xi_1 r_1 r_2 \neq 0, \qquad b = \pi\xi_2 r_2, \qquad c = \pi\xi_3.$$

We consider the bijection

$$L^2(\mathsf{R}) \longrightarrow L^2(\mathsf{R}), \qquad \varphi \longmapsto \tilde{\varphi},\ \ \tilde{\varphi}(t) = \frac{1}{x^2}\varphi(xt+y),$$

where $x = a^{1/3}$, $y = ba^{-2/3}/2$.

We define $P_c^\pm := P_{1,0,c}^\pm$.

CLAIM. The equation $P^{\pm}_{a,b,c}\tilde{\varphi} = \tilde{\lambda}\tilde{\varphi}$ is equivalent to $P^{\pm}_{c_1}\varphi = \lambda\varphi$, where

$$c_1 = -b^2 a^{-4/3}/2 + ca^{-1/3}, \qquad \tilde{\lambda} = a^{2/3}\lambda.$$

Indeed, assume that $P^{\pm}_{c_1}\varphi = \lambda\varphi$. Then $\varphi''(t) = \big((t^2+c_1)^2 \pm 2t - \lambda\big)\varphi(t)$ holds. Hence

$$\begin{aligned}(P^{\pm}_{a,b,c}\tilde{\varphi})(t) &= -(\partial_t^2\tilde{\varphi})(t) + \big((at^2+bt+c)^2 \pm (2at+b)\big)\tilde{\varphi}(t)\\ &= \big(-x^2\big(((xt+y)^2+c_1)^2 \pm 2(xt+y) - \lambda\big)\\ &\qquad + (at^2+bt+c)^2 \pm (2at+b)\big)\tilde{\varphi}(t)\\ &= x^2\lambda\tilde{\varphi}(t) = a^{2/3}\lambda\tilde{\varphi}.\end{aligned}$$

The converse can be proven similarly using $\varphi(t) = x^2\tilde{\varphi}(t/x - y/x)$.

It is well known that the Schrödinger operator $P^{\pm}_c$ having a polynomial potential of degree 4 has the following properties [7], [22]. The spectrum of $P^{\pm}_c$ is discrete. All eigenvalues are real and simple. They can be arranged into an increasing sequence $\lambda_0 < \lambda_1 < \cdots \to \infty$ and satisfy

$$\lambda_k \sim \left(\frac{\sqrt{\pi}\,\Gamma(7/4)\cdot k}{\Gamma(5/4)}\right)^{4/3}.$$

Obviously, P^{+}_c and P^{-}_c have the same eigenvalues. We will denote these eigenvalues by $\lambda_k(c)$, $k \in \mathsf{N}$.

Since $\dim \mathscr{H}$ is even the spectrum of D_ξ is symmetric. We conclude that $\mathrm{spec}(D_\xi)$ consists of the eigenvalues

$$\pm\big(a^{2/3}\lambda_k(-4b^2a^{-4/3} + ca^{-1/3})\big)^{1/2}, \qquad k \in \mathsf{N},$$

where $a = \pi\xi_1 r_1 r_2/2$, $b = \pi\xi_2 r_2$, $c = \pi\xi_3$.

Next we determine a set of representatives of the R-orbits in R^3 that intersect $\Sigma_{\varepsilon'}$ and the number of Z-orbits that are contained in them. Obviously,

$$\mathscr{R}^{(1)} := R^{(1)} \cap \Sigma_{\varepsilon'}$$

is the set of fixed points in $\Sigma_{\varepsilon'}$ and

$$\mathscr{R}^{(2)} := \big\{\xi \in \Sigma_{\varepsilon'} \mid \xi_1 = 0, \xi_2 \neq 0, \xi_3 = \varepsilon_3\big\}$$

is a set of representatives of the R-orbits in $R^{(2)}$ that interset $\Sigma_{\varepsilon'}$. For $\xi \in \mathscr{R}^{(2)}$ the R-orbit through ξ contains $|r_2\xi_2|/2$ Z-orbits. Now we turn to orbits contained in $R^{(3)}$. For a given number $k \in \mathsf{Z} \setminus \{0\}$ let $p, q \in \mathsf{Z}$, $q > 0$ be such that

$$\frac{|r_2|}{|r_1 k|} = \frac{p}{q}, \qquad (p, q) = 1$$

and put $q(k) := q$. Moreover, for $l, q \in \mathsf{N}$, $q > 0$ we define

$$M(l, q) := \{(m_1, m_2) \mid m_1, m_2 \in \mathsf{N} \setminus \{0\}, m_1 + m_2 = l, q | m_1 m_2\}.$$

We will show:

(1) The set

$$\mathscr{R}^{(3)} := \left\{\xi \in \Sigma_{\varepsilon'} \mid 0 \le \xi_2 < |r_1 \xi_1|, M(\xi_2, q(\xi_1)) = \emptyset\right\}$$

is a set of representatives of R-orbits in $R^{(3)}$ that intersect $\Sigma_{\varepsilon'}$.

(2) For $\xi \in \mathscr{R}^{(3)}$ the number of Z-orbits contained in the R-orbit of ξ equals

$$m(\xi_1, \xi_2) := \#\left\{k \in \mathsf{N} \mid \xi_2 + 2k < |r_1 \xi_1|, q(\xi_1) | k(k + \xi_2)\right\}.$$

Take $\xi \in R^{(3)} \cap \Sigma_{\varepsilon'}$ and denote by θ the R-orbit of ξ. Using (29) we see that $A^\top(t)\xi$ is in $\Sigma_{\varepsilon'}$ if and only if $t r_1 \xi_1$ and $t^2 r_1 r_2 \xi_1 / 2 + t r_2 \xi_2$ are in $2\mathsf{Z}$. The latter condition is equivalent to

$$t = \frac{2k}{r_1 \xi_1}, \qquad q(\xi_1) | k(k + \xi_2) \tag{31}$$

for some $k \in \mathsf{Z}$. Obviously, we may choose $\hat{\xi} = (\xi_1, \hat{\xi}_2, \hat{\xi}_3) \in \theta$ such that $0 \le \hat{\xi}_2 < |r_1 \xi_1|$. Now we want to choose $\hat{\xi}$ is such a way that $\hat{\xi}_2 \ge 0$ is minimal, which ensures the uniqueness of the representative. By (31), $\hat{\xi}_2$ is minimal if and only if there does not exist an integer k, $-[\hat{\xi}_2/2] \le k \le -1$, such that $q(\xi_1) | k(k + \hat{\xi}_2)$. The latter condition is equivalent to $q(\xi_1) | (-k)(k + \hat{\xi}_2)$. Hence $\hat{\xi}_2$ is minimal if and only if $\hat{\xi}_2$ does not decompose as a sum $\hat{\xi}_2 = m_1 + m_2$ with $m_1, m_2 \in \mathsf{N} \setminus \{0\}$ and $q(\xi_1) | m_1 m_2$. This proves the first assertion. The second one follows from (31).

Now we can give an expression for $m(D)$. In the following sums are taken over $\xi_i \in \varepsilon_1 + 2\mathsf{Z}$, $i = 1, 2, 3$. Moreover, we will take another index of summation, namely $\xi_4 \in \dot{\varepsilon}(1) + 2\mathsf{Z}$. Furthermore, $\kappa \in \{1, -1\}$. Then

$$
\begin{aligned}
&m(D)\\
&\quad = \sum_{\xi_3,\xi_4} \sum_{\kappa} \delta\big(\kappa\pi(\xi_3^2+\xi_4^2)^{1/2}\big)\\
&\qquad + \sum_{\xi_2>0} |r_2\xi_2| \left(\delta(0) + \sum_{k=1}^{\infty}\sum_{\kappa} \delta(\kappa(2\pi k|r_2\xi_2|)^{1/2})\right)\\
&\qquad + \sum_{\xi_1\neq 0,\xi_2,\xi_3} m(\xi_1,\xi_2)\\
&\qquad \cdot \sum_{k=0}^{\infty}\sum_{\kappa} \delta\left(\kappa\left(\pi\xi_1\frac{r_1r_2}{2}\right)^{1/3}\lambda_k\left(\left(\left(\pi\xi_1\frac{r_1r_2}{2}\right)^{-1/3}\pi\left(\xi_3-\frac{8\xi_2^2r_2}{\xi_1r_1}\right)\right)^{1/2}\right)\right).
\end{aligned}
$$

4.5. Non-hypoellipticity of sub-Dirac operators

The aim of this subsection is to combine the results of the preceding two subsections. In particular, we discuss the consequences for the hypoellipticity of the sub-Dirac operator D acting on sections of the spinor bundle $S(\dot{\mathscr{H}})$ over $\Gamma \setminus G$.

Let $S(\dot{\mathscr{H}})^*$ denote the dual vector bundle of $S(\dot{\mathscr{H}})$. Elements of the locally convex space $\Gamma(S(\dot{\mathscr{H}})^*)'$ of all continuous linear functionals on $\Gamma(S(\dot{\mathscr{H}})^*)$ are called distributions with values in $S(\dot{\mathscr{H}})$. Smooth sections of $S(\dot{\mathscr{H}})$ can be considered as elements of $\Gamma(S(\dot{\mathscr{H}})^*)'$ in a natural way.

We say that D is hypoelliptic if the following condition is satisfied for every open subset W of $\Gamma \setminus G$: If $u \in \Gamma(S(\dot{\mathscr{H}})^*)'$ such that $(Du)|_W$ is given by a smooth section of $S(\dot{\mathscr{H}})|_W$, then $u|_W$ is smooth.

Identifying $\Gamma(S(\dot{\mathscr{H}}))$ with $C^\infty(G, \varepsilon, \Delta)$ as before, the above property translates to the following: If W is an ε-invariant open subset of G and $u \in C^\infty(G, \varepsilon, \Delta^*)'$ such that $(Du)|_W \in C^\infty(W, \varepsilon, \Delta)$, then $u|_W \in C^\infty(W, \varepsilon, \Delta)$.

A sub-Riemannian structure $(\mathscr{H}, g, P_{\mathrm{Spin}}, \nabla, \mu)$ endowed with a spin structure P_{Spin}, a connection ∇ and volume form μ on Lie group G is called left-invariant if $\mathscr{H}, g, \nabla$ are left-invariant in the obvious sense and μ is the volume form defining the Haar measure of G.

Now we can formulate the following theorem, which comprises the results of the preceding sections.

THEOREM 4.4. *There exists a uniform discrete subgroup Γ and a left-invariant sub-Riemannian structure $(\mathscr{H}, g, P_{\mathrm{Spin}}, \nabla, \mu)$ on a simply connected two-step nilpotent Lie group G with a distribution of codimension two in the tangent bundle of G such that the associated sub-Dirac operator D on $\Gamma \setminus G$ is symmetric but neither hypoelliptic nor Fredholm.*

PROOF. As in the end of Section 4.3, we choose $G = \mathsf{R}^4 \rtimes_A \mathsf{R}$, $\mathfrak{g} = \mathsf{R}^4 \rtimes_B \mathsf{R}$ where $A(t) = \exp(tB)$ and $B \in \mathrm{End}(\mathsf{R}^4)$ is given by $Be_1 = B_2 = 0$, $Be_3 = e_1$ and $Be_4 = e_2$. Put $s_1 = e_3$, $s_2 = e_4$ and $s_3 = b$. The corresponding left-invariant vector fields span the distribution $\mathscr{H} := \mathrm{span}\{s_1, s_2, s_3\}$ of TG which is endowed with the orientation and the Riemannian metric g such that s_1, s_2, s_3 is a positively oriented, orthonormal frame. We choose the left-invariant connection ∇ such that $\Gamma_{12}^3 = \Gamma_{23}^1 = \Gamma_{31}^2 = 0$ and a spin structure given by a homomorphism $\varepsilon : \mathsf{Z}^4 \ltimes_A \mathsf{Z} \to \mathsf{Z}_2$ satisfying $\varepsilon_3 = \varepsilon_4 = 0$.

We consider the sequence $\eta_m \in \mathscr{R}_{\varepsilon'}$ given by $\eta_{m1} = 2m + \varepsilon_1$, $\eta_{m2} = 2 + \varepsilon_2$ and $\eta_{m3} = \eta_{m4} = 0$ for $m \geq 1$. Note that η_m and η_n lie on different R-orbits provided that $m \neq n$. Clearly $\lambda_0(\eta_m) = 0$ for all m. This means that the kernel of D is of infinite dimension. In particular, D is not a Fredholm operator.

Suppose that the operator D acting on sections of $S(\dot{\mathscr{H}})$ is hypoelliptic. As before, the extension of D to distributions is again denoted by D. By the hypoellipticity of D, it follows that $X := \ker D$ is contained in the subspace $C^\infty(G, \varepsilon, \Delta)$ of $L^2(G, \varepsilon, \Delta)$.

The C^∞-topology τ_{C^∞} of $C^\infty(G, \varepsilon, \Delta)$ is given by the sequence of norms $p_N(u) = \sum_{|\mu| \leq N} |E_1^{\mu_1} \dots E_5^{\mu_5} u|_\infty$ where $E_j = \sum_{i=1}^n A_{ij}(t)\partial_{x_i}$ and $E_5 = \partial_t$ are the left-invariant vector fields corresponding to $e_1, \dots, e_5$. By the Arzelà-Ascoli theorem and a diagonal sequence argument it follows that $(C^\infty(G,\varepsilon,\Delta), \tau_{C^\infty})$ has the Heine-Borel property. Since X is complete in both topologies τ_{L^2} and τ_{C^∞}, the open mapping theorem implies $(X, \tau_{L^2}) = (X, \tau_{C^\infty})$. Thus X is locally compact and hence finite-dimensional. This contradiction proves the theorem. Note that this argument can be applied to arbitrary hypoelliptic operators on compact manifolds.

Since hypoellipticity is a local property, Theorem 4.4 yields:

COROLLARY 4.5. *In the situation of the preceding theorem, the sub-Dirac operator D_0 acting on sections of the spinor bundle $S(\mathscr{H})$ over G is not hypoelliptic either.*

REFERENCES

1. Agrachev, A., Boscain, U., Gauthier, J.-P., and Rossi, F., *The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups*, J. Funct. Anal. 256 (2009), 2621–2655.
2. Ammann, B., and Bär, C., *The Dirac operator on nilmanifolds and collapsing circle bundles*, Ann. Global Anal. Geom. 16 (1998), no. 3, 221–253.
3. Bauer, W., Furutani, K., and Iwasaki, C., *Spectral zeta function of the sub-Laplacian on two step nilmanifolds*, J. Math. Pures Appl. 97 (2012), no. 3, 242–261.
4. Bauer, W., and Furutani, K., *Spectral zeta function of a sub-Laplacian on product sub-Riemannian manifolds and zeta-regularized determinant*, J. Geom. Phys. 60 (2010), no. 9, 1209–1234.

5. Bauer, W., and Furutani, K., *Spectral analysis and geometry of a sub-Riemannian structure on S^3 and S^7*, J. Geom. Phys. 58 (2008), no. 12, 1693–1738.
6. Brüning, J., Kamber, F. W., and Richardson, K., *Index theory for basic Dirac operators on Riemannian foliations*, Contemp. Math. 546 (2011), 39–81.
7. Eremenko, A., Gabrielov, A., and Shapiro, B., *High energy eigenfunctions of one-dimensional Schrödinger operators with polynomial potentials*, Comput. Methods Funct. Theory 8 (2008), no. 1-2, 513–529.
8. van Erp, E., *The AtiyahSinger index formula for subelliptic operators on contact manifolds. Part I*, Ann. of Math. (2) 171 (2010), no. 3, 1647–1681.
9. Friedrich, T., and Sulanke, S., *Ein Kriterium für die formale Selbstadjungiertheit des Dirac-Operators*, Colloq. Math. 40 (1978/79), no. 2, 239–247.
10. Ginoux, N., *The Dirac spectrum*, Lecture Notes in Mathematics, Vol. 1976, Springer-Verlag, 2009.
11. Gordon, C. S., and Wilson, E. N., *The spectrum of the Laplacian on Riemannian Heisenberg manifolds*, Michigan Math. J. 33, 1986, no. 2, 253–271.
12. Hislop, P. D., and Sigal, I. M., *Introduction to spectral theory* (*With applications to Schrödinger operators*), Applied Mathematical Sciences, Vol. 113, Springer-Verlag, New York, 1996.
13. Hörmander, L., *Hypoelliptic second order differential equations*, Acta Math. 119 (1967), 147–171.
14. Jammes, P., *Sur le spectre des fibrés en tore qui s'effondrent*, Manuscripta Math. 110 (2003), no. 1, 13–31.
15. Lawson, H. B., and Michelsohn, M.-L., *Spin geometry*, Princeton Mathematical Series, 38, Princeton University Press, 1989.
16. Leptin, H., and Ludwig, J., *Unitary representation theory of exponential Lie groups*, De Gruyter Expositions in Mathematics, Vol. 18, Walter de Gruyter & Co., Berlin, 1994.
17. Montgomery, R., *A Tour of Sub-Riemannian Geometries, Their Geodesics and Applications*, Math. Surveys Monogr., Vol. 91, Amer. Math. Soc., Providence, RI, 2002.
18. Petit, R., Spin^c*-structures and Dirac operators on contact manifolds*, Differential Geom. Appl. 22 (2005), no. 2, 229–252.
19. Ponge, R. S., *Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds*, Mem. Amer. Math. Soc. 194 (2008), no. 906.
20. Prokhorenkov, I., and Richardson, K., *Natural equivariant transversally elliptic Dirac operators*, Geom. Dedicata 151 (2011), 411–429.
21. Rockland, C., *Hypoellipticity on the Heisenberg group-representation-theoretic criteria*, Trans. Amer. Math. Soc. 240 (1978), 1–52.
22. Titchmarsh, E. C., *Eigenfunction expansions associated with second-order differential equations. Part I*, Clarendon Press, Oxford, 1962.
23. Wolf, J. A., *Harmonic analysis on commutative spaces*, Mathematical Surveys and Monographs, Vol. 142, AMS, 2007.

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