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CUNTZ-KRIEGER ALGEBRAS ASSOCIATED WITH HILBERT C^*-QUAD MODULES OF COMMUTING MATRICES

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Abstract

Let $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ be the C^*-algebra associated with the Hilbert C^*-quad module arising from commuting matrices A, B with entries in $\{0, 1\}$. We will show that if the associated tiling space $X_{A,B}^\kappa$ is transitive, the C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ is simple and purely infinite. In particular, for two positive integers N, M, the K-groups of the simple purely infinite C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}$ are computed by using the Euclidean algorithm.

1. Introduction

In [9], the author has introduced a notion of C^*-symbolic dynamical system, which is a generalization of a finite labeled graph, a λ-graph system and an automorphism of a unital C^*-algebra (cf. [10]). It is denoted by $(\mathcal{A}, \rho, \Sigma)$ and consists of a finite family $\{\rho_\alpha\}_{\alpha\in\Sigma}$ of endomorphisms of a unital C^*-algebra $\mathcal{A}$ such that $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$ and $\sum_{\alpha\in\Sigma} \rho_\alpha(1) \geq 1$ where $Z_{\mathcal{A}}$ denotes the center of $\mathcal{A}$, and endomorphisms are not necessarily unital. It provides a subshift Λ_ρ over Σ and a Hilbert C^*-bimodule $\mathcal{H}_{\mathcal{A}}^\rho$ over $\mathcal{A}$ which gives rise to a C^*-algebra $\mathcal{O}_\rho$ as a Cuntz-Pimsner algebra ([9], cf. [5], [16]). In [11] and [12], the author has extended the notion of C^*-symbolic dynamical system to C^*-textile dynamical system which is a higher dimensional analogue of C^*-symbolic dynamical system. The C^*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ consists of two C^*-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with a common unital C^*-algebra $\mathcal{A}$ and a commutation relation between the endomorphisms ρ and η through a map κ stated below. Set

$$\Sigma^{\rho\eta} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_\alpha \neq 0\},$$

$$\Sigma^{\eta\rho} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \neq 0\}.$$

We assume that there exists a bijection $\kappa : \Sigma^{\rho\eta} \to \Sigma^{\eta\rho}$, which we fix and call

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a specification. Then the required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \qquad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \tag{1.1}$$

A C^*-textile dynamical system provides a two-dimensional subshift and a multi-structure of Hilbert C^*-bimodules that has multi-right actions and multi-left actions and multi-inner products. Such a multi-structure of Hilbert C^*-bimodules is called a Hilbert C^*-quad module, denoted by $\mathscr{H}_\kappa^{\rho,\eta}$. In [12], the author has introduced a C^*-algebra associated with the Hilbert C^*-quad module defined by a C^*-textile dynamical system. The C^*-algebra $\mathcal{O}_{\mathscr{H}_\kappa^{\rho,\eta}}$ has been constructed in a concrete way from the structure of the Hilbert C^*-quad module $\mathscr{H}_\kappa^{\rho,\eta}$ by a two-dimensional analogue of Pimsner's construction of C^*-algebras from Hilbert C^*-bimodules. It is generated by the quotient images of the creation operators on two-dimensional analogue of Fock Hilbert module by module maps of compact operators. As a result, the C^*-algebra has been proved to have a universal property subject to certain operator relations of generators encoded by structure of the Hilbert C^*-quad module of C^*-textile dynamical system ([12], cf. [13]).

Let A, B be two $N \times N$ matrices with entries in nonnegative integers. We assume that both A and B are essential, which means that they have no rows or columns identically to zero vector. They yield directed graphs $G_A = (V, E_A)$ and $G_B = (V, E_B)$ with a common vertex set $V = \{v_1, \dots, v_N\}$ and edge sets E_A and E_B respectively, where the edge set E_A consists of $A(i, j)$-edges from the vertex v_i to the vertex v_j and E_B consists of $B(i, j)$-edges from the vertex v_i to the vertex v_j. Denote by $s(e), r(e)$ the source vertex and the range vertex of an edge e. We set $\mathscr{A}_N = \mathbb{C}^N$. Denote by $E_1, \dots, E_N$ the set of minimal projections of $\mathscr{A}_N$ defined by the standard basis of $\mathbb{C}^N$ which correspond to the vertex set $v_1, \dots, v_N$ respectively, so that $\sum_{i=1}^N E_i = 1$. For $\alpha \in E_A$, define ρ_α^A an endomorphism of $\mathscr{A}_N$ by $\rho_\alpha^A(E_i) = E_j$ if $s(\alpha) = v_i, r(\alpha) = v_j$, otherwise $\rho_\alpha^A(E_i) = 0$. Similarly we have an endomorphism ρ_a^B of $\mathscr{A}_N$ for $a \in E_B$. We then have two C^*-symbolic dynamical systems $(\mathscr{A}_N, \rho^A, E_A)$ and $(\mathscr{A}_N, \rho^B, E_B)$ with $\mathscr{A}_N = \mathbb{C}^N$. Put

$$\Sigma^{AB} = \{(\alpha, b) \in E_A \times E_B \mid r(\alpha) = s(b)\},$$

$$\Sigma^{BA} = \{(a, \beta) \in E_B \times E_A \mid r(a) = s(\beta)\}.$$

Assume that the commutation relation

$$AB = BA \tag{1.2}$$

holds. We may take a bijection $\kappa : \Sigma^{AB} \to \Sigma^{BA}$ such that $s(\alpha) = s(a), r(b) = r(\beta)$ for $\kappa(\alpha, b) = (a, \beta)$, which we fix and call a specification by following

Nasu's terminology in [14]. This situation is called an LR-textile system introduced by Nasu ([14]). We then have a C^*-textile dynamical system (see [12])

$$(\mathcal{A}_N, \rho^A, \rho^B, E_A, E_B, \kappa).$$

Let us denote by $\mathscr{H}_\kappa^{A,B}$ the associated Hilbert C^*-quad module defined in [12]. We set

$$E_\kappa = \{(\alpha, b, a, \beta) \in E_A \times E_B \times E_B \times E_A \mid \kappa(\alpha, b) = (a, \beta)\}. \tag{1.3}$$

Each element of E_κ is called a tile. Let $X_{A,B}^\kappa \subset (E_\kappa)^{\mathbb{Z}^2}$ be the two-dimensional subshift of the Wang tilings of E_κ (cf. [19]). It consists of the two-dimensional configurations $x : \mathbb{Z}^2 \to E_\kappa$ compatible to their boundary edges on each tile, and is called the subshift of the tiling space for the specification $\kappa : \Sigma^{AB} \to \Sigma^{BA}$. We say that $X_{A,B}^\kappa$ is transitive if for two tiles $\omega, \omega' \in E_\kappa$, there exists $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X_{A,B}^\kappa$ such that $\omega_{0,0} = \omega$, $\omega_{i,j} = \omega'$ for some $(i, j) \in \mathbb{Z}^2$ with $j < 0 < i$. We set

$$\Omega_\kappa = \{(\alpha, a) \in E_A \times E_B \mid s(\alpha) = s(a), \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A, b \in E_B\} \tag{1.4}$$

and define two $|\Omega_\kappa| \times |\Omega_\kappa|$-matrices A_κ and B_κ with entries in $\{0, 1\}$ by

$$A_\kappa((\alpha, a), (\delta, b)) = \begin{cases} 1 & \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A, \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

for $(\alpha, a), (\delta, b) \in \Omega_\kappa$,

$$B_\kappa((\alpha, a), (\beta, d)) = \begin{cases} 1 & \kappa(\alpha, b) = (a, \beta) \text{ for some } b \in E_B, \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

for $(\alpha, a), (\beta, d) \in \Omega_\kappa$ respectively. Put the block matrix

$$H_\kappa = \begin{bmatrix} A_\kappa & A_\kappa \\ B_\kappa & B_\kappa \end{bmatrix}. \tag{1.7}$$

It has been proved in [12] that the C^*-algebra $\mathcal{O}_{\mathscr{H}_\kappa^{A,B}}$ associated with the Hilbert C^*-quad module $\mathscr{H}_\kappa^{A,B}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{H_\kappa}$ for the matrix H_κ (cf. [2]). In this paper, we first show the following theorem.

THEOREM 1.1 (Theorem 2.10). *The subshift $X_{A,B}^\kappa$ of the tiling space is transitive if and only if the matrix H_κ is irreducible. In this case, H_κ satisfies condition* (I) *in the sense of* [2]. *Hence if the subshift $X_{A,B}^\kappa$ of the tiling space is transitive, the C^*-algebra $\mathcal{O}_{\mathscr{H}_\kappa^{A,B}}$ is simple and purely infinite.*

We then see the following theorem.

THEOREM 1.2 (Theorem 2.11). *If the matrix A or B is irreducible, the matrix H_κ is irreducible and satisfies condition (I), so that the C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ is simple and purely infinite.*

Let N, M be positive integers with $N, M > 1$. They give 1×1 commuting matrices $A = [N]$, $B = [M]$. The directed graph G_A associated to the matrix $A = [N]$ is a graph consists of N-self directed loops denoted by E_A with a vertex denoted by v. Similarly the directed graph G_B consists of M-self directed loops denoted by E_B with the vertex v. We fix a specification $\kappa : E_A \times E_B \to E_B \times E_A$ defined by exchanging $\kappa(\alpha, a) = (a, \alpha)$ for $(\alpha, a) \in E_A \times E_B$. The specification is called the exchanging specification between E_A and E_B. We present the following K-theory formulae for the C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}$. In its computation, the Euclidean algorithm is used. For integers $1 < N \le M \in \mathbb{N}$, let $d = (N-1, M-1)$ be the greatest common divisor of $N-1$ and $M-1$. Let $k_0, k_1, \dots, k_{j+1}$ be the successive integral quotients of $M-1$ by $N-1$ by the Euclidean algorithm such as

$$
\begin{aligned}
M - 1 &= (N-1)k_0 + r_0 && \text{for some} \quad k_0 \in \mathbb{Z}_+, \ \ 0 < r_0 < N-1, \\
N - 1 &= r_0 k_1 + r_1 && \text{for some} \quad k_1 \in \mathbb{Z}_+, \ \ 0 < r_1 < r_0, \\
&\ \ \vdots \\
r_{j-2} &= r_{j-1}k_j + r_j && \text{for some} \quad k_j \in \mathbb{Z}_+, \ \ 0 < r_j < r_{j-1}, \\
r_{j-1} &= d k_{j+1}.
\end{aligned}
$$

THEOREM 1.3 (Theorem 3.5). *For integers $1 < N \le M \in \mathbb{N}$ and the exchanging specification κ between directed N-loops and M-loops, the C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}$ is a simple purely infinite Cuntz-Krieger algebra whose K-groups are*

$$
\begin{aligned}
K_1(\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}) &\cong 0, \\
K_0(\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}) &\cong \overbrace{\mathbb{Z}/(N-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(N-1)\mathbb{Z}}^{M-2} \\
&\quad \oplus \ \overbrace{\mathbb{Z}/(M-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(M-1)\mathbb{Z}}^{N-2} \\
&\quad \oplus \ \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/[k_1, k_2, \dots, k_{j+1}](M-1)(M+N-1)\mathbb{Z}
\end{aligned}
$$

where $d = (N-1, M-1)$ the greatest common divisor of $N-1$ and $M-1$, and the sequence $k_0, k_1, \dots, k_{j+1}$ is the successive integral quotients of $M-1$ by $N-1$ by the Euclidean algorithm above, and the integer $[k_1, k_2, \dots, k_{j+1}]$

is defined by inductively

$$[k_0] = 1, \quad [k_1] = k_1, \quad [k_1, k_2] = 1 + k_1 k_2, \\ \dots, \quad [k_1, k_2, \dots, k_{j+1}] = [k_1, k_2, \dots, k_j] k_{j+1} + [k_1, \dots, k_{j-1}].$$

We remark that the C^*-algebras studied in this paper are different from the higher rank graph algebras studied by G. Robertson-T. Steger [18], A. Kumjian-D. Pask [6], V. Deaconu [3], etc., (cf. [4], [17], [15], etc.). Throughout the paper, we denote by N and by Z_+ the set of positive integers and the set of nonnegative integers respectively.

2. Transitivity of tilings $X^{\kappa}_{A,B}$ and simplicity of $\mathcal{O}_{\mathcal{H}^{A,B}_{\kappa}}$

Let Σ be a finite set. The two-dimensional full shift over Σ is defined to be

$$\Sigma^{\mathsf{Z}^2} = \{(x_{i,j})_{(i,j)\in\mathsf{Z}^2} \mid x_{i,j} \in \Sigma\}.$$

An element $x \in \Sigma^{\mathsf{Z}^2}$ is regarded as a function $x : \mathsf{Z}^2 \to \Sigma$ which is called a configuration on Z^2. For a vector $m = (m_1, m_2) \in \mathsf{Z}^2$, let $\sigma^m : \Sigma^{\mathsf{Z}^2} \to \Sigma^{\mathsf{Z}^2}$ be the translation along vector m defined by

$$\sigma^m((x_{i,j})_{(i,j)\in\mathsf{Z}^2}) = (x_{i+m_1, j+m_2})_{(i,j)\in\mathsf{Z}^2}.$$

A subset $X \subset \Sigma^{\mathsf{Z}^2}$ is said to be translation invariant if $\sigma^m(X) = X$ for all $m \in \mathsf{Z}^2$. It is obvious to see that a subset $X \subset \Sigma^{\mathsf{Z}^2}$ is translation invariant if and only if X is invariant only both horizontally and vertically, that is, $\sigma^{(1,0)}(X) = X$ and $\sigma^{(0,1)}(X) = X$. For $k \in \mathsf{Z}_+$, put

$$[-k, k]^2 = \{(i, j) \in \mathsf{Z}^2 \mid -k \le i, j \le k\} = [-k, k] \times [-k, k].$$

A metric d on Σ^{Z^2} is defined by for $x, y \in \Sigma^{\mathsf{Z}^2}$ with $x \ne y$

$$d(x, y) = \frac{1}{2^k} \qquad \text{if} \quad x_{(0,0)} = y_{(0,0)},$$

where $k = \max\{k \in \mathsf{Z}_+ \mid x_{[-k,k]^2} = y_{[-k,k]^2}\}$. If $x_{(0,0)} \ne y_{(0,0)}$, put $k = -1$ on the above definition. If $x = y$, we set $d(x, y) = 0$. A two-dimensional subshift X is defined to be a closed, translation invariant subset of Σ^{Z^2} (cf. [8, p. 467]). A two-dimensional subshift X is said to have the *diagonal property* if for $(x_{i,j})_{(i,j)\in\mathsf{Z}^2}, (y_{i,j})_{(i,j)\in\mathsf{Z}^2} \in X$, the conditions $x_{i,j} = y_{i,j}, x_{i+1,j-1} = y_{i+1,j-1}$ imply $x_{i,j-1} = y_{i,j-1}, x_{i+1,j} = y_{i+1,j}$ (see [11]). The diagonal property has the following property: for $x \in X$ and $(i, j) \in \mathsf{Z}^2$, the configuration x is determined by the diagonal line $(x_{i+n,j-n})_{n\in\mathsf{Z}}$ through (i, j).

We henceforth go back to our previous situation of C^*-textile dynamical system $(\mathscr{A}_N, \rho^A, \rho^B, E_A, E_B, \kappa)$ coming from $N \times N$ commuting matrices A and B with specification κ as in Section 1. We always assume that both matrices A and B are essential. Recall that the matrices A and B give rise to directed graphs $G_A = (V, E_A)$ and $G_B = (V, E_B)$ with a common vertex set $V = \{v_1, \dots, v_N\}$ and edge sets E_A and E_B respectively, where the edge set E_A consists of $A(i, j)$-edges from the vertex v_i to the vertex v_j and E_B consists of $B(i, j)$-edges from the vertex v_i to the vertex v_j. A two-dimensional subshift $X^\kappa_{A,B}$ is defined as in the following way. Let Σ be the set E_κ of tiles defined in (1.3). For $\omega = (\alpha, b, a, \beta) \in E_\kappa$, define maps $t(=\text{top})$, $b(=\text{bottom}) : E_\kappa \to E_A$ and $l(=\text{left})$, $r(=\text{right}) : E_\kappa \to E_B$ by setting

$$t(\omega) = \alpha, \quad b(\omega) = \beta, \quad l(\omega) = a, \quad r(\omega) = b$$

as in the following figure:

$$\begin{array}{ccc} \circ & \xrightarrow{\alpha = t(\omega)} & \circ \\ {\scriptstyle a = l(\omega)}\big\downarrow & & \big\downarrow{\scriptstyle b = r(\omega)} \\ \circ & \xrightarrow[\beta = b(\omega)]{} & \circ \end{array}$$

A configuration $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in E_\kappa^{\mathbb{Z}^2}$ is said to be *paved* if the conditions

$$\begin{aligned} t(\omega_{i,j}) &= b(\omega_{i,j+1}), & r(\omega_{i,j}) &= l(\omega_{i+1,j}), \\ l(\omega_{i,j}) &= r(\omega_{i-1,j}), & b(\omega_{i,j}) &= t(\omega_{i,j-1}) \end{aligned}$$

hold for all $(i, j) \in \mathbb{Z}^2$. Let $X^\kappa_{A,B}$ be the set of all paved configurations $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in E_\kappa^{\mathbb{Z}^2}$. It consists of the Wang tilings of the tiles of E_κ (see [19]). The following proposition is easy.

PROPOSITION 2.1. *$X^\kappa_{A,B}$ is a two-dimensional subshift having the diagonal property.*

We write $\mathscr{A}_N = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_N$ for the minimal projections $E_i, i = 1, \dots, N$ of $\mathscr{A}_N$ such that $\sum_{i=1}^N E_i = 1$. Let us define the matrices $\widehat{A}, \widehat{B}$ by setting for $\alpha \in E_A, a \in E_B, i, j = 1, \dots, N$,

$$\widehat{A}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(\alpha) = i, r(\alpha) = j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\widehat{B}(i, a, j) = \begin{cases} 1 & \text{if } s(a) = i, r(a) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that the endomorphisms ρ^A_α, ρ^B_a of $\mathscr{A}_N$ for $\alpha \in E_A, a \in E_B$ are defined by

$$\rho^A_\alpha(E_i) = \sum_{j=1}^{N} \widehat{A}(i, \alpha, j) E_j, \qquad \rho^B_a(E_i) = \sum_{j=1}^{N} \widehat{B}(i, a, j) E_j$$

for $i = 1, \dots, N$. They yield the C^*-textile dynamical system

$$(\mathscr{A}_N, \rho^A, \rho^B, E_A, E_B, \kappa)$$

with specification κ ([12]). Let e_ω, $\omega \in E_\kappa$ be the standard basis of $\mathbb{C}^{|E_\kappa|}$. Put the projection $E_\omega = \rho^B_b \circ \rho^A_\alpha(1) (= \rho^A_\beta \circ \rho^B_a(1)) \in \mathscr{A}_N$ for $\omega = (\alpha, b, a, \beta) \in E_\kappa$. We set

$$\mathscr{H}^{A,B}_\kappa = \sum_{\omega \in E_\kappa} e_\omega \otimes E_\omega \mathscr{A}_N.$$

Then $\mathscr{H}^{A,B}_\kappa$ has a natural structure of not only Hilbert C^*-right module over $\mathscr{A}_N$ but also two other Hilbert C^*-bimodule structure, called Hilbert C^*-quad module. By two-dimensional analogue of Pimsner's construction of Hilbert C^*-bimodule algebra ([16]), we have introduced a C^*-algebra $\mathcal{O}_{\mathscr{H}^{A,B}_\kappa}$ (see [12] and [13] for detail construction). Let Ω_κ be the subset of $E_A \times E_B$ defined in (1.4). We define two $|\Omega_\kappa| \times |\Omega_\kappa|$-matrcies A_κ and B_κ with entries in $\{0, 1\}$ as in (1.5) and (1.6). The matrices A_κ and B_κ represent the concatenations of edges as in the following figures respectively:

$$\begin{array}{ccccc} \circ & \xrightarrow{\alpha} & \circ & \xrightarrow{\delta} & \circ \\ {\scriptstyle a}\Big\downarrow & & {\scriptstyle b}\Big\downarrow & & \\ \circ & \longrightarrow & \circ & & \end{array} \qquad \text{if } A_\kappa((\alpha, a), (\delta, b)) = 1,$$

and

$$\begin{array}{ccc} \circ & \xrightarrow{\alpha} & \circ \\ {\scriptstyle a}\Big\downarrow & & \Big\downarrow \\ \circ & \xrightarrow{\beta} & \circ \\ {\scriptstyle d}\Big\downarrow & & \end{array} \qquad \text{if } B_\kappa((\alpha, a), (\beta, d)) = 1.$$

Let H_κ be the $2|\Omega_\kappa| \times 2|\Omega_\kappa|$ matrix defined in (1.7). We have proved the following result in [12].

THEOREM 2.2. *The C^*-algebra $\mathcal{O}_{\mathscr{H}^{A,B}_\kappa}$ associated with Hilbert C^*-quad module $\mathscr{H}^{A,B}_\kappa$ defined by commuting matrices A, B and a specification κ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{H_\kappa}$ for the matrix H_κ. Its K-groups $K_*(\mathcal{O}_{H_\kappa})$ are computed as*

$$K_0(\mathcal{O}_{H_\kappa}) = \mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n, \quad K_1(\mathcal{O}_{H_\kappa}) = \mathrm{Ker}(A_\kappa + B_\kappa - I_n) \ \textit{in} \ \mathbb{Z}^n,$$

where $n = |\Omega_\kappa|$.

We will study a relationship between transitivity of the tiling space $X^{\kappa}_{A,B}$ and simplicity of the C^*-algebra $\mathcal{O}_{\mathcal{H}^{A,B}_{\kappa}}$. An essential matrix with entries in $\{0, 1\}$ is said to satisfy condition (I) (in the sense of [2]) if the shift space defined by the topological Markov chain for the matrix is homeomorphic to a Cantor discontinuum. The condition is equivalent to the condition that every loop in the associated directed graph has an exit ([7]). It is a fundamental result that a Cuntz-Krieger algebra is simple and purely infinite if the underlying matrix is irreducible and satisfies condition (I) ([2]). We will find a condition of the two-dimensional subshift $X^{\kappa}_{A,B}$ of the tiling space under which the matrix H_{κ} is irreducible and satisfies condition (I). Hence the condition on $X^{\kappa}_{A,B}$ yields the simplicity and purely infiniteness of the algebra $\mathcal{O}_{\mathcal{H}^{A,B}_{\kappa}}$.

We are assuming that both of the matrices A and B are essential. Then we have

Lemma 2.3. *Both of the matrices A_{κ} and B_{κ} are essential.*

Proof. For $(\alpha, a) \in \Omega_{\kappa}$, by definition of Ω_{κ}, there exist $\beta \in E_A$ and $b \in E_B$ such that $\kappa(\alpha, b) = (a, \beta)$. Since A is essential, one may take $\beta_1 \in E_A$ such that $s(\beta_1) = r(b)(= r(\beta))$. Hence $(b, \beta_1) \in \Sigma^{BA}$. Put $(\alpha_1, b_1) = \kappa^{-1}(b, \beta_1) \in \Sigma^{AB}$ so that $(\alpha_1, b) \in \Omega_{\kappa}$ and $A_{\kappa}((\alpha, a), (\alpha_1, b)) = 1$ as in the following figure:

$$\begin{array}{ccccc} \circ & \xrightarrow{\alpha} & \circ & \xrightarrow{\alpha_1} & \circ \\ {\scriptstyle a}\Big\downarrow & & {\scriptstyle b}\Big\downarrow & & {\scriptstyle b_1}\Big\downarrow \\ \circ & \xrightarrow{\beta} & \circ & \xrightarrow{\beta_1} & \end{array}$$

For $(\delta, b) \in \Omega_{\kappa}$ there exists $\alpha \in E_A$ such that $r(\alpha) = s(\delta)(= s(b))$ because A is essential. Hence $(\alpha, b) \in \Sigma^{AB}$. Put $(a, \beta) = \kappa(\alpha, b)$ so that $(\alpha, a) \in \Omega_{\kappa}$ and $A_{\kappa}((\alpha, a), (\delta, b)) = 1$ as in the following figure:

$$\begin{array}{ccccc} \circ & \xrightarrow{\alpha} & \circ & \xrightarrow{\delta} & \circ \\ {\scriptstyle a}\Big\downarrow & & {\scriptstyle b}\Big\downarrow & & \\ \circ & \xrightarrow{\beta} & \circ & & \end{array}$$

Therefore one sees that A_{κ} is essential, and similarly that B_{κ} is essential.

Hence we have

Proposition 2.4. *The matrix H_{κ} is essential and satisfies condition* (I).

Proof. By the previous lemma, both of the matrices A_{κ} and B_{κ} are essential. Hence every row of A_{κ} and of B_{κ} has at least one 1. Since

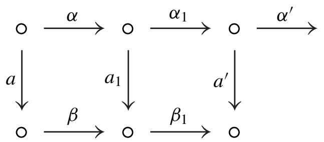
$$H_{\kappa} = \begin{bmatrix} A_{\kappa} & A_{\kappa} \\ B_{\kappa} & B_{\kappa} \end{bmatrix},$$

every row of H_κ has at least two $1'$s. This implies that a loop in the directed graph associated to H_κ must has an exit so that H_κ satisfies condition (I).

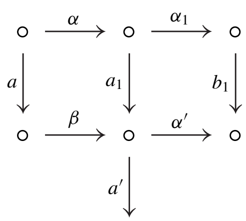
For $(\alpha, a), (\alpha', a') \in \Omega_\kappa$, and $C, D = A$ or B, we have

$$[C_\kappa D_\kappa]((\alpha, a), (\alpha', a')) = \sum_{(\alpha_1, a_1) \in \Omega_\kappa} C_\kappa((\alpha, a), (\alpha_1, a_1)) D_\kappa((\alpha_1, a_1), (\alpha', a')).$$

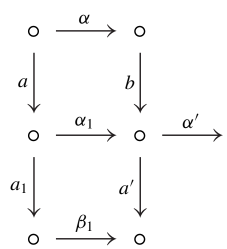
Hence $[A_\kappa A_\kappa]((\alpha, a), (\alpha', a')) \neq 0$ if and only if there exists $(\alpha_1, a_1) \in \Omega_\kappa$ such that $\kappa(\alpha, a_1) = (a, \beta)$ for some $\beta \in E_A$ and $\kappa(\alpha_1, a') = (a_1, \beta_1)$ for some $\beta_1 \in E_A$ as in the following figure:



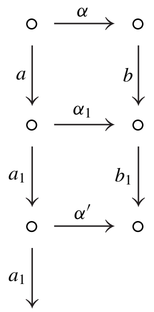
And also $[A_\kappa B_\kappa]((\alpha, a), (\alpha', a')) \neq 0$ if and only if there exists $(\alpha_1, a_1) \in \Omega_\kappa$ such that $\kappa(\alpha, a_1) = (a, \beta)$ for some $\beta \in E_A$ and $\kappa(\alpha_1, b_1) = (a_1, \alpha')$ for some $b_1 \in E_B$ as in the following figure:



Similarly $[B_\kappa A_\kappa]((\alpha, a), (\alpha', a')) \neq 0$ if and only if there exists $(\alpha_1, a_1) \in \Omega_\kappa$ such that $\kappa(\alpha, b) = (a, \alpha_1)$ for some $b \in E_B$ and $\kappa(\alpha_1, a') = (a_1, \beta_1)$ for some $\beta_1 \in E_A$ as in the following figure:

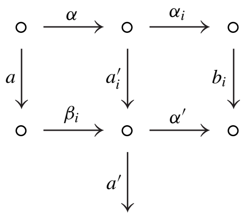


And also $[B_\kappa B_\kappa]((\alpha, a), (\alpha', a')) \neq 0$ if and only if there exists $(\alpha_1, a_1) \in \Omega_\kappa$ such that $\kappa(\alpha, b) = (a, \alpha_1)$ for some $b \in E_B$ and $\kappa(\alpha_1, b_1) = (a_1, \alpha')$ for some $b_1 \in E_B$ as in the following figure:

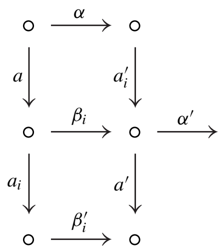


LEMMA 2.5. $A_\kappa B_\kappa = B_\kappa A_\kappa$.

PROOF. For $(\alpha, a), (\alpha', a') \in \Omega_\kappa$, we have $[A_\kappa B_\kappa]((\alpha, a), (\alpha', a')) = m$ if and only if there exist $(\alpha_i, a'_i) \in \Omega_\kappa$, $i = 1, \dots, m$ such that $\kappa(\alpha, a'_i) = (a, \beta_i)$ for some $\beta_i \in E_A$ and $\kappa(\alpha_i, b_i) = (a'_i, \alpha')$ for some $b_i \in E_B$ as in the following figure:



Put $(a_i, \beta'_i) = \kappa(\beta_i, a')$. We then have $(\beta_i, a_i) \in \Omega_\kappa$ as in the following figure:



If $(\beta_i, a_i) = (\beta_j, a_j)$ in Ω_κ, then we have $\beta_i = \beta_j$ so that $a'_i = a'_j$ and hence $\alpha_i = \alpha_j$. Therefore we have $[B_\kappa A_\kappa]((\alpha, a), (\alpha', a')) = m$.

LEMMA 2.6. *The following four conditions are equivalent.*

(i) *The matrix H_κ is irreducible.*

(ii) *For $(\alpha, a), (\alpha', a') \in \Omega_\kappa$, there exist $n, m \in \mathsf{Z}_+$ such that*

$$A_\kappa(A_\kappa + B_\kappa)^n((\alpha, a), (\alpha', a')) > 0,$$

$$B_\kappa(A_\kappa + B_\kappa)^m((\alpha, a), (\alpha', a')) > 0.$$

(iii) *The matrix* $A_\kappa + B_\kappa$ *is irreducible.*

(iv) *For* $(\alpha, a), (\alpha', a') \in \Omega_\kappa$*, there exists a paved configuration* $(\omega_{i,j})_{(i,j)\in \mathsf{Z}^2} \in X^\kappa_{A,B}$ *such that*

$$t(\omega_{0,0}) = \alpha, \quad l(\omega_{0,0}) = a, \quad t(\omega_{i,j}) = \alpha', \quad l(\omega_{i,j}) = a'$$

for some $(i, j) \in \mathsf{Z}^2$ *with* $j < 0 < i$.

PROOF. (i) $\Leftrightarrow$ (ii): The identity

$$H^n_\kappa = \begin{bmatrix} A_\kappa(A_\kappa + B_\kappa)^n & A_\kappa(A_\kappa + B_\kappa)^n \\ B_\kappa(A_\kappa + B_\kappa)^n & B_\kappa(A_\kappa + B_\kappa)^n \end{bmatrix} \tag{2.1}$$

implies the equivalence between (i) and (ii).

(ii) $\Rightarrow$ (iii): Suppose that for $(\alpha, a), (\alpha', a') \in \Omega_\kappa$, there exists $n \in \mathsf{Z}_+$ such that $A_\kappa(A_\kappa + B_\kappa)^n((\alpha, a), (\alpha', a')) > 0$ so that

$$(A_\kappa + B_\kappa)^{n+1}((\alpha, a), (\alpha', a')) > 0.$$

Hence the matrix $A_\kappa + B_\kappa$ is irreducible.

(iii) $\Rightarrow$ (ii): As A_κ and B_κ are both essential, for $(\alpha, a), (\alpha', a') \in \Omega_\kappa$ there exists $(\alpha_1, a_1), (\alpha_2, a_2) \in \Omega_\kappa$ such that

$$A_\kappa((\alpha, a), (\alpha_1, a_1)) = 1,$$
$$B_\kappa((\alpha, a), (\alpha_2, a_2)) = 1.$$

Since $A_\kappa + B_\kappa$ is irreducible, there exist $n, m \in \mathsf{Z}_+$ such that

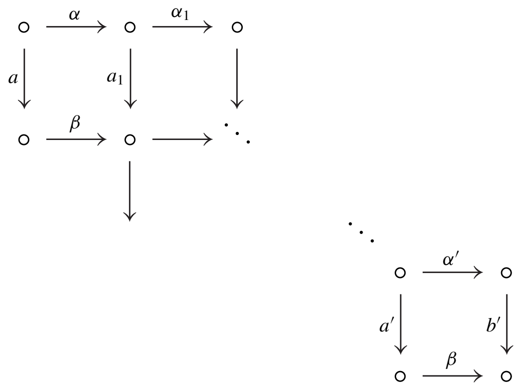
$$(A_\kappa + B_\kappa)^n((\alpha_1, a_1), (\alpha', a')) > 0,$$
$$(A_\kappa + B_\kappa)^m((\alpha_2, a_2), (\alpha', a')) > 0.$$

Hence we have

$$A_\kappa(A_\kappa + B_\kappa)^n((\alpha, a), (\alpha', a')) > 0,$$
$$B_\kappa(A_\kappa + B_\kappa)^m((\alpha, a), (\alpha', a')) > 0.$$

(ii) $\Rightarrow$ (iv): For $(\alpha, a), (\alpha', a') \in \Omega_\kappa$, take $(\alpha_1, a_1) \in \Omega_\kappa$ and $\beta \in E_A$ such that $\kappa(\alpha, a_1) = (a, \beta)$. By (ii), there exists $m \in \mathsf{Z}_+$ with $B_\kappa(A_\kappa + B_\kappa)^m((\alpha, a), (\alpha', a')) > 0$. One may take $b' \in E_B$ and $\beta' \in E_A$ satisfying $\kappa(\alpha', b') = (a', \beta')$, so that there exists a paved configuration $(\omega_{i,j})_{(i,j)\in\mathsf{Z}^2} \in X^\kappa_{A,B}$ such that $\omega_{0,0} = (\alpha, a_1, a, \beta)$ and $\omega_{i,j} = (\alpha', b', a', \beta')$ for some $(i, j) \in$

Z^2 with $j < 0 < i$ as in the following figure:

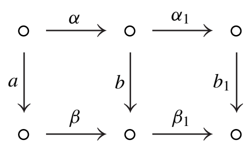


(iv) $\Rightarrow$ (ii): The assertion is clear.

DEFINITION 2.7. A two-dimensional subshift $X^{\kappa}_{A,B}$ is said to be *transitive* if for two tiles $\omega, \omega' \in E_\kappa$ there exists a paved configuration $(\omega_{i,j})_{(i,j)\in\mathsf{Z}^2} \in X^{\kappa}_{A,B}$ such that $\omega_{0,0} = \omega$ and $\omega_{i,j} = \omega'$ for some $(i, j) \in \mathsf{Z}^2$ with $j < 0 < i$.

THEOREM 2.8. *The subshift $X^{\kappa}_{A,B}$ of the tiling space is transitive if and only if the matrix H_κ is irreducible.*

PROOF. Assume that the matrix H_κ is irreducible. Hence the condition (iv) in Lemma 2.6 holds. Let $\omega = (\alpha, b, a, \beta), \omega' = (\alpha', b', a', \beta') \in E_\kappa$ be two tiles. Since A is essential, there exists $\beta_1 \in E_A$ such that $r(\beta)(= r(b)) = s(\beta_1)$, so that $(b, \beta_1) \in \Sigma^{BA}$. One may take $(\alpha_1, b_1) \in \Sigma^{AB}$ such that $\kappa(\alpha_1, b_1) = (b, \beta_1)$ and hence $(\alpha_1, b) \in \Omega_\kappa$ as in the following figure:



For $(\alpha_1, b), (\alpha', a') \in \Omega_\kappa$, by (iv) in Lemma 2.6, there exists $(\omega_{i,j})_{(i,j)\in\mathsf{Z}^2} \in X^{\kappa}_{A,B}$ such that $t(\omega_{0,0}) = \alpha_1, l(\omega_{0,0}) = b, t(\omega_{i,j}) = \alpha', l(\omega_{i,j}) = a'$ for some $(i, j) \in \mathsf{Z}^2$ with $j < 0 < i$. Since $X^{\kappa}_{A,B}$ has the diagonal property, there exists a paved configuration $(\omega'_{i,j})_{(i,j)\in\mathsf{Z}^2} \in X^{\kappa}_{A,B}$ such that $\omega'_{0,0} = \omega, \omega'_{i,j} = \omega'$. Hence $X^{\kappa}_{A,B}$ is transitive.

Conversely assume that $X^{\kappa}_{A,B}$ is transitive. For $(\alpha, a), (\alpha', a') \in \Omega_\kappa$, there exist $b, b' \in E_B$ and $\beta, \beta' \in E_A$ such that $\omega = (\alpha, b, a, \beta), \omega' = (\alpha', b', a', \beta') \in E_\kappa$. It is clear that the transitivity of $X^{\kappa}_{A,B}$ implies the condition (iv) in Lemma 2.6, so that H_κ is irreducible.

LEMMA 2.9. *If A or B is irreducible, $X^{\kappa}_{A,B}$ is transitive.*

PROOF. Suppose that the matrix A is irreducible. For two tiles $\omega = (\alpha, b, a, \beta), \omega' = (\alpha', b', a', \beta') \in E_\kappa$, there exist concatenated edges $(\beta, \beta_1, \ldots, \beta_n, \alpha')$ in the graph G_A for some edges $\beta_1, \ldots, \beta_n \in E_A$. Since $X^{\kappa}_{A,B}$ has the diagonal property, there exists a configuration $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2} \in X^{\kappa}_{A,B}$ such that $\omega' = \omega_{i,j}$ for some $i > 0$, $j = -1$. Hence $X^{\kappa}_{A,B}$ is transitive.

Since the C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{H_\kappa}$ by [12], we see the following theorems.

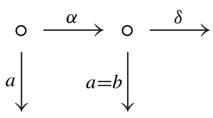
THEOREM 2.10. *The subshift $X^{\kappa}_{A,B}$ of the tiling space is transitive if and if the matrix H_κ is irreducible. In this case, H_κ satisfies condition* (I). *Hence if the subshift $X^{\kappa}_{A,B}$ of the tiling space is transitive, the C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ is simple and purely infinite.*

By Lemma 2.9, we have

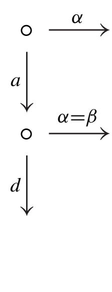
THEOREM 2.11. *If the matrix A or B is irreducible, the matrix H_κ is irreducible and satisfies condition (I), so that the C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ is simple and purely infinite.*

3. The algebra $\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}$ for two positive integers N, M

Let N, M be positive integers with $N, M > 1$. They give 1×1 commuting matrices $A = [N]$, $B = [M]$. We will present K-theory formulae for the C^*-algebras $\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}$ with the exchanging specification κ. In the computations below, we will use the Euclidean algorithm to find order of the torsion part of the K_0-group. The directed graph G_A for the matrix $A = [N]$ is a graph consisting of N-self directed loops with a vertex denoted by v. The N-self directed loops are denoted by E_A. Similarly the directed graph G_B for $B = [M]$ consists of M-self directed loops denoted by E_B with the vertex v. We fix a specification $\kappa : E_A \times E_B \to E_B \times E_A$ defined by exchanging $\kappa(\alpha, a) = (a, \alpha)$ for $(\alpha, a) \in E_A \times E_B$. Hence $\Omega_\kappa = E_A \times E_B$ so that $|\Omega_\kappa| = |E_A| \times |E_B| = N \times M$. We then know $A_\kappa((\alpha, a), (\delta, b)) = 1$ if and only if $b = a$, and $B_\kappa((\alpha, a), (\beta, d)) = 1$ if and only if $\beta = \alpha$ as in the following figures respectively.



and



In [12], the K-groups for the case $N = 2$ and $M = 3$ have been computed such that

$$K_0(\mathcal{O}_{\mathcal{H}_\kappa^{[2],[3]}}) \cong \mathbb{Z}/8\mathbb{Z}, \qquad K_1(\mathcal{O}_{\mathcal{H}_\kappa^{[2],[3]}}) \cong 0.$$

Hence $\mathcal{O}_{\mathcal{H}_\kappa^{[2],[3]}}$ is stably isomorphic to the Cuntz algebra $\mathcal{O}_9$ of order 9 ([1]). We will generalize the above computations.

Let I_n be the $n \times n$ identity matrix and E_n the $n \times n$ matrix whose entries are all $1'$s. For an $N \times N$-matrix $C = [c_{i,j}]_{i,j=1}^N$ and an $M \times M$-matrix $D = [d_{k,l}]_{k,l=1}^M$, denote by $C \otimes D$ the $NM \times NM$ matrix

$$C \otimes D = \begin{bmatrix} c_{11}D & c_{12}D & \dots & c_{1N}D \\ c_{21}D & c_{22}D & \dots & c_{2N}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1}D & c_{N2}D & \dots & c_{NN}D \end{bmatrix}.$$

Hence we have

$$E_N \otimes I_M = \begin{bmatrix} I_M & I_M & \dots & I_M \\ I_M & I_M & \dots & I_M \\ \vdots & \vdots & \ddots & \vdots \\ I_M & I_M & \dots & I_M \end{bmatrix},$$

$$I_N \otimes E_M = \begin{bmatrix} E_M & 0 & \dots & 0 \\ 0 & E_M & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & E_M \end{bmatrix}.$$

We put $E_{[N]} = \{\alpha_1, \dots, \alpha_N\}$, $E_{[M]} = \{a_1, \dots, a_M\}$. As $\Omega_\kappa = E_{[N]} \times E_{[M]}$, the basis of $\mathbb{C}^N \otimes \mathbb{C}^M$ are ordered lexicographically from left as in the following way:

(3.1) $$\begin{aligned} &(\alpha_1, a_1), \dots, (\alpha_1, a_M), (\alpha_2, a_1), \dots, (\alpha_2, a_M), \\ &\qquad \dots, (\alpha_N, a_1), \dots, (\alpha_N, a_M). \end{aligned}$$

Let A_κ and B_κ be the matrices defined in the previous section for the matrices $A = [N]$, $B = [M]$ with the exchanging specification κ. The following lemma is direct.

LEMMA 3.1. *The matrices A_κ, B_κ are written as*

$$A_\kappa = E_N \otimes I_M, \qquad B_\kappa = I_N \otimes E_M$$

along the ordered basis (3.1). *Hence we have*

$$A_\kappa + B_\kappa - I_{NM} = \begin{bmatrix} E_M & I_M & \dots & I_M \\ I_M & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_M \\ I_M & \dots & I_M & E_M \end{bmatrix}. \tag{3.2}$$

We denote by $H(0)$ the matrix $A_\kappa + B_\kappa - I_{NM}$. By Theorem 2.2, the K-groups of the algebra $\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}$ are given by the kernel $\mathrm{Ker}(H(0))$ and the cokernel $\mathrm{Coker}(H(0))$ of the matrix $H(0)$ in Z^{NM}. For an $M \times M$ matrix C and $i, j = 1, 2, \dots, N$ with $i \neq j$, define an $N \times N$ block matrix $\mathsf{E}_{i,j}(C) = [\mathsf{E}_{i,j}(C)(k,l)]_{k,l=1}^N$, whose entries $\mathsf{E}_{i,j}(C)(k,l)$, $k, l = 1, 2, \dots, N$ are $M \times M$ matrices, by setting

$$\mathsf{E}_{i,j}(C)(k,l) = \begin{cases} I_M & (k = l), \\ C & (k = i,\ j = l), \\ 0 & \text{else.} \end{cases}$$

The multiplication of the matrix $\mathsf{E}_{i,j}(C)$ from the left (resp. right) corresponds to the operation of adding the C-multiplication of the jth row (resp. ith column) to the ith row (resp. jth column). We will transform $H(0)$ preserving isomorphism classes of the groups $\mathrm{Ker}(H(0))$ and $\mathrm{Coker}(H(0))$ in Z^{NM} by multiplying the matrices $\mathsf{E}_{i,j}(C)$, $i, j = 1, 2, \dots, N$.

We first consider row operations and set

$$\begin{aligned} H(1) &= \mathsf{E}_{N-1,N}(-I_M)\mathsf{E}_{N-2,N-1}(-I_M)\cdots\mathsf{E}_{1,2}(-I_M)H(0), \\ H(2) &= \mathsf{E}_{N,N-1}(I_M)\mathsf{E}_{N-1,N-2}(I_M)\cdots\mathsf{E}_{2,1}(I_M)H(1), \\ H(3) &= \mathsf{E}_{1,2}(I_M)\mathsf{E}_{2,3}(I_M)\cdots\mathsf{E}_{N-2,N-1}(I_M)\mathsf{E}_{N-1,N}(E_M - I_M)H(2). \end{aligned}$$

It is straightforward to see that the matrix $H(3)$ goes to

$$H(3) = \begin{bmatrix} p_M(N-1) & 0 & & \dots & & 0 \\ p_M(N-2) & E_M - I_M & & & & \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ p_M(2) & \vdots & & \ddots & & \\ p_M(1) & E_M - I_M & \dots & \dots & E_M - I_M & 0 \\ E_M & I_M & \dots & \dots & I_M & I_M \end{bmatrix}$$

where $p_M(i) = E_M^2 + (i-1)E_M - iI_M = (E_M + iI_M)(E_M - I_M)$ for $i = 1, \dots N-1$.

We second consider column operations and set

$$H(4) = H(3)\mathsf{E}_{N,N-1}(-I_M)\mathsf{E}_{N,N-2}(-I_M)\cdots\mathsf{E}_{N,2}(-I_M)\mathsf{E}_{N,1}(-E_M)$$

which goes to

$$H(4) = \begin{bmatrix} p_M(N-1) & 0 & \dots & & & 0 \\ p_M(N-2) & E_M - I_M & & & & \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ p_M(2) & \vdots & & \ddots & & \\ p_M(1) & E_M - I_M & \dots & \dots & E_M - I_M & 0 \\ 0 & 0 & \dots & \dots & 0 & I_M \end{bmatrix}.$$

By successive multiplications of the matrices

$$\mathsf{E}_{N-1,N-2}(-I_M)\mathsf{E}_{N-1,N-3}(-I_M)\cdots\mathsf{E}_{N-1,2}(-I_M)\mathsf{E}_{N-1,1}(-(E_M + I_M))$$
$$\mathsf{E}_{N-2,N-3}(-I_M)\mathsf{E}_{N-2,N-4}(-I_M)\cdots\mathsf{E}_{N-2,2}(-I_M)\mathsf{E}_{N-2,1}(-(E_M + 2I_M))$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\mathsf{E}_{3,2}(-I_M)\mathsf{E}_{3,1}(-(E_M + (N-2)I_M))$$
$$\mathsf{E}_{2,1}(-(E_M + (N-1)I_M)),$$

from the right side of $H(4)$, we obtain the diagonal matrix

$$\tilde{H} = \begin{bmatrix} p_M(N-1) & 0 & & \dots & & 0 \\ 0 & E_M - I_M & & & & \\ & & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & \\ & & & & E_M - I_M & 0 \\ 0 & & \dots & & 0 & I_M \end{bmatrix}.$$

As $E_M^2 = ME_M$, we have $p_M(N-1) = (M+N-2)E_M - (N-1)I_M$. We thus have

LEMMA 3.2.
$$\mathrm{Ker}(A_\kappa + B_\kappa - I_{NM}) \ \textit{in} \ \mathsf{Z}^{NM} \cong 0$$

and

$$\operatorname{Coker}(A_\kappa + B_\kappa - I_{NM}) \ \textit{in} \ \mathsf{Z}^{NM}$$
$$\cong \overbrace{\mathsf{Z}^M/(E_M - I_M)\mathsf{Z}^M \oplus \cdots \oplus \mathsf{Z}^M/(E_M - I_M)\mathsf{Z}^M}^{(N-2)}$$
$$\oplus\, \mathsf{Z}^M/((M+N-2)E_M - (N-1)I_M)\mathsf{Z}^M.$$

PROOF. It is straightforward to see that the matrix $A_\kappa + B_\kappa - I_{NM}$ is invertible by the formula (3.2). Since

$$\operatorname{Coker}(A_\kappa + B_\kappa - I_{NM}) \text{ in } \mathsf{Z}^{NM} \cong \mathsf{Z}^{NM}/\tilde{H}\mathsf{Z}^{NM},$$

the formula for the cokernel is obvious.

We will next compute the following groups to compute $\operatorname{Coker}(A_\kappa + B_\kappa - I_{NM})$ in Z^{NM}.

(i) $\mathsf{Z}^M/(E_M - I_M)\mathsf{Z}^M$,

(ii) $\mathsf{Z}^M/((M+N-2)E_M - (N-1)I_M)\mathsf{Z}^M$

For an integer c and $i, j = 1, 2, \dots, M$ with $i \neq j$, define an $M \times M$ matrix $E_{i,j}(c) = [E_{i,j}(c)(k,l)]_{k,l=1}^M$ by setting

$$E_{i,j}(c)(k,l) = \begin{cases} 1 & (k = l), \\ c & (k = i,\ j = l), \\ 0 & \text{else.} \end{cases} \tag{3.3}$$

(i) By successive multiplications of the matrices

$$E_{M-1,M}(-1)E_{M-2,M-1}(-1)\cdots E_{1,2}(-1),$$
$$E_{M,M-1}(1)E_{M-1,M-2}(1)\cdots E_{2,1}(1),$$
$$E_{M,M-1}(-1)E_{M,M-2}(-1)\cdots E_{M,1}(-1)$$

from the left side of the matrix

$$E_M - I_M = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{bmatrix},$$

we get the matrix

$$\begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ -1 & 0 & \dots & 0 & 1 \\ M-1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

which goes to the diagonal matrix with diagonal entries $[1, 1, \dots, 1, M-1]$ by elementary column operations. Hence we see that

$$\mathsf{Z}^M/(E_M - I_M)\mathsf{Z}^M \cong \mathsf{Z}/(M-1)\mathsf{Z}. \tag{3.4}$$

(ii) Put $e = (M+N-2) - (N-1) = M-1$ and $f = M+N-2$. Then we have

$$(M+N-2)E_M - (N-1)I_M = \begin{bmatrix} e & f & \dots & f \\ f & e & \ddots & \vdots \\ \vdots & \ddots & \ddots & f \\ f & \dots & f & e \end{bmatrix}. \tag{3.5}$$

By a similar manner to the preceding matrix operations from $H(1)$ to $H(4)$, one obtains the following matrix denoted by $L(1)$ from the matrix (3.5)

$$L(1) = \begin{bmatrix} e-f & f-e & 0 & \dots & 0 \\ e-f & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & f-e & 0 \\ e-f & 0 & \dots & 0 & f-e \\ e & f & \dots & f & f \end{bmatrix}.$$

By exchanging columns in the matrix

$$L(1)E_{2,1}(1)E_{3,1}(1)\cdots E_{M,1}(1),$$

we have

$$L(2) = \begin{bmatrix} f-e & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & f-e & 0 & 0 \\ 0 & \dots & 0 & f-e & 0 \\ f & \dots & f & f & e+(M-1)f \end{bmatrix}.$$

It is easy to see that the matrix

$$E_{M-1,M-2}(1)\cdots E_{3,2}(1)E_{2,1}(1)L(2)E_{2,1}(-1)E_{3,2}(-1)\cdots E_{M-1,M-2}(-1)$$

goes to

$$\tilde{L} = \begin{bmatrix} f-e & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & f-e & 0 \\ 0 & \dots & 0 & f & e+(M-1)f \end{bmatrix}.$$

Put the 2×2 matrix $L_{(N,M)}$ by setting

$$L_{(N,M)} = \begin{bmatrix} f-e & 0 \\ f & e+(M-1)f \end{bmatrix}.$$

As $f - e = N - 1$, we have the following lemma with (3.4).

LEMMA 3.3.

(i) $\mathsf{Z}^M/(E_M - I_M)\mathsf{Z}^M \cong \mathsf{Z}/(M-1)\mathsf{Z}$.

(ii) $\mathsf{Z}^M/((M+N-2)E_M - (N-1)I_M)\mathsf{Z}^M$

$$\cong \overbrace{\mathsf{Z}/(N-1)\mathsf{Z} \oplus \cdots \oplus \mathsf{Z}/(N-1)\mathsf{Z}}^{M-2} \oplus \mathsf{Z}^2/L_{(N,M)}\mathsf{Z}^2.$$

It remains to compute the group $\mathsf{Z}^2/L_{(N,M)}\mathsf{Z}^2$. Put $n = N-1$, $m = M-1$. As $f-e = n$ and $f = m+n$, we have $e+(M-1)f = (M-1)(M+N-1) = m(m+n+1)$ so that

$$L_{(N,M)} = \begin{bmatrix} n & 0 \\ n+m & m(m+n+1) \end{bmatrix}.$$

For an integer c and $i, j = 1, 2$ with $i \neq j$, define an 2×2 matrix $E_{i,j}(c) = [E_{i,j}(c)(k,l)]_{k,l=1}^2$ in a similar way to (3.3). Put $L_{n,m} = E_{2,1}(-1)L_{(N,M)}$ so that

$$L_{n,m} = \begin{bmatrix} n & 0 \\ m & m(m+n+1) \end{bmatrix}.$$

We may assume that $M \geq N$ and hence $m \geq n$.

If m is divided by n so that $m = nk$ for some $k \in \mathsf{N}$, the matrix $E_{2,1}(-k)L_{n,m}$ goes to the diagonal matrix:

$$\begin{bmatrix} n & 0 \\ 0 & m(m+n+1) \end{bmatrix} = \begin{bmatrix} N-1 & 0 \\ 0 & (M-1)(M+N-1) \end{bmatrix}.$$

Hence we have

$$\mathsf{Z}^2/L_{(N,M)}\mathsf{Z}^2 \cong \mathsf{Z}/(N-1)\mathsf{Z} \oplus \mathsf{Z}/(M-1)(M+N-1)\mathsf{Z}.$$

Otherwise, by the Euclidean algorithm, we have lists of integers $r_0, r_1, \dots, r_j$ and $k_0, k_1, \dots, k_{j+1}$ for some $j \in \mathsf{N}$ such that

$$\begin{aligned} m &= nk_0 + r_0, && 0 < r_0 < n, \\ n &= r_0 k_1 + r_1, && 0 < r_1 < r_0, \\ r_0 &= r_1 k_2 + r_2, && 0 < r_2 < r_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots && \\ r_{j-2} &= r_{j-1}k_j + r_j, && 0 < r_j < r_{j-1}, \\ r_{j-1} &= r_j k_{j+1}, && 0 = r_{j+1} \end{aligned}$$

where $r_j = (m, n)$ the greatest common divisor of m and n. Put $g = m(m + n + 1)$. We set

$$L_{n,m}(0) = E_{2,1}(-k_0)L_{n,m} = \begin{bmatrix} n & 0 \\ r_0 & g \end{bmatrix}.$$

We define a finite sequence of matrices $L_{n,m}(l), l = 1, 2, \dots$ by

$$L_{n,m}(1) = E_{1,2}(-k_1)L_{n,m}(0), \qquad L_{n,m}(2) = E_{2,1}(-k_2)L_{n,m}(1)$$

and inductively

$$\begin{aligned} L_{n,m}(2i-1) &= E_{1,2}(-k_{2i-1})L_{n,m}(2i-2), \\ L_{n,m}(2i) &= E_{2,1}(-k_{2i})L_{n,m}(2i-1). \end{aligned}$$

The Euclidean algorithm stops at $j+1 = 2i-1$ or $j+1 = 2i$ for some $i \in \mathsf{N}$. We set

$$[k_0] = 1, \quad [k_1] = k_1, \quad [k_1, k_2] = 1 + k_1 k_2, \quad [k_1, k_2, k_3] = [k_1, k_2]k_3 + [k_1],$$
$$\dots, \quad [k_1, k_2, \dots, k_{j+1}] = [k_1, k_2, \dots, k_j]k_{j+1} + [k_1, \dots, k_{j-1}].$$

Then we have

$$L_{n,m}(1) = \begin{bmatrix} r_1 & -[k_1]g \\ r_0 & g \end{bmatrix}, \qquad L_{n,m}(2) = \begin{bmatrix} r_1 & -[k_1]g \\ r_2 & [k_1, k_2]g \end{bmatrix},$$

and inductively

$$L_{n,m}(2i-1) = \begin{bmatrix} r_{2i-1} & -[k_1, k_2, \dots, k_{2i-1}]g \\ r_{2i-2} & [k_1, k_2, \dots, k_{2i-2}]g \end{bmatrix},$$

$$L_{n,m}(2i) = \begin{bmatrix} r_{2i-1} & -[k_1, k_2, \dots, k_{2i-1}]g \\ r_{2i} & [k_1, k_2, \dots, k_{2i}]g \end{bmatrix}$$

for $i = 1, 2, \dots$. We denote by d the greatest common divisor (m, n) of m and n, so that $d = r_j$. Take $m_0 \in \mathsf{Z}$ such that $m = m_0 d$. Put $g_0 = m_0(m + n + 1)$ so that $g = g_0 d$. We have two cases.

Case 1: $j + 1 = 2i - 1$ for some $i \in \mathsf{N}$. We have

$$L_{n,m}(j+1) = \begin{bmatrix} r_{j+1} & -[k_1, k_2, \dots, k_{j+1}]g \\ r_j & [k_1, k_2, \dots, k_j]g \end{bmatrix} = \begin{bmatrix} 0 & -[k_1, k_2, \dots, k_{j+1}]g \\ d & [k_1, k_2, \dots, k_j]g_0 d \end{bmatrix}$$

and hence

$$L_{n,m}(j+1)E_{1,2}(-[k_1, k_2, \dots, k_j]g_0) = \begin{bmatrix} 0 & -[k_1, k_2, \dots, k_{j+1}]g \\ d & 0 \end{bmatrix}.$$

Case 2: $j + 1 = 2i$ for some $i \in \mathsf{N}$. We have

$$L_{n,m}(j+1) = \begin{bmatrix} r_j & -[k_1, k_2, \dots, k_j]g \\ r_{j+1} & [k_1, k_2, \dots, k_{j+1}]g \end{bmatrix} = \begin{bmatrix} d & -[k_1, k_2, \dots, k_j]g_0 d \\ 0 & [k_1, k_2, \dots, k_{j+1}]g \end{bmatrix}$$

and hence

$$L_{n,m}(j+1)E_{1,2}([k_1, k_2, \dots, k_j]g_0) = \begin{bmatrix} d & 0 \\ 0 & [k_1, k_2, \dots, k_{j+1}]g \end{bmatrix}.$$

We reach the following lemma.

LEMMA 3.4.

$$\mathsf{Z}^2/L_{(N,M)}\mathsf{Z}^2 \cong \mathsf{Z}/d\mathsf{Z} \oplus \mathsf{Z}/[k_1, k_2, \dots, k_{j+1}]g\mathsf{Z}.$$

Therefore we have

THEOREM 3.5. *For positive integers $1 < N \le M \in \mathsf{N}$ and the exchanging specification κ between N-loops and M-loops in a graph with one vertex, the*

C^*-algebra $\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}$ is a simple purely infinite Cuntz-Krieger algebra whose K-groups are

$$K_1(\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}) \cong 0,$$

$$K_0(\mathcal{O}_{\mathcal{H}_\kappa^{[N],[M]}}) \cong \overbrace{\mathbb{Z}/(N-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(N-1)\mathbb{Z}}^{M-2} \oplus \overbrace{\mathbb{Z}/(M-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(M-1)\mathbb{Z}}^{N-2} \oplus \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/[k_1, k_2, \dots, k_{j+1}](M-1)(M+N-1)\mathbb{Z}$$

where $d = (N-1, M-1)$ is the greatest common divisor of $N-1$ and $M-1$, the sequence $k_0, k_1, \dots, k_{j+1}$ of integers is the list of the successive integral quotients of $M-1$ by $N-1$ in the Euclidean algorithm such as

$$\begin{aligned} M-1 &= (N-1)k_0 + r_0 && \textit{for some} \quad k_0 \in \mathbb{Z}_+, \ 0 < r_0 < N-1, \\ N-1 &= r_0 k_1 + r_1 && \textit{for some} \quad k_1 \in \mathbb{Z}_+, \ 0 < r_1 < r_0, \\ &\vdots \\ r_{j-2} &= r_{j-1}k_j + r_j && \textit{for some} \quad k_j \in \mathbb{Z}_+, \ 0 < r_j < r_{j-1}, \\ r_{j-1} &= dk_{j+1}, \end{aligned}$$

and the integer $[k_1, k_2, \dots, k_{j+1}]$ is defined by inductively

$$[k_0] = 1, \quad [k_1] = k_1, \quad [k_1, k_2] = 1 + k_1 k_2, \quad \dots, \quad [k_1, k_2, \dots, k_{j+1}] = [k_1, k_2, \dots, k_j]k_{j+1} + [k_1, \dots, k_{j-1}].$$

We finally present examples.

EXAMPLES 3.6. 1. For the case $1 < N = M$, we have $d = N-1, k_0 = 1, r_0 = 0$. As we see $[k_1, \dots, k_{j+1}] = 1$, we have

$$[k_1, \dots, k_{j+1}](M-1)(M+N-1) = (N-1)(2N-1).$$

Hence

$$K_0(\mathcal{O}_{\mathcal{H}_\kappa^{[N],[N]}}) \cong \overbrace{\mathbb{Z}/(N-1)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/(N-1)\mathbb{Z}}^{2N-3} \oplus \mathbb{Z}/(N-1)(2N-1)\mathbb{Z}.$$

2. For the case $N = 2$ and $M \geq 2$, we have $d = 1, r_0 = 0$. As we see $[k_1, \dots, k_{j+1}] = 1$, we have

$$[k_1, \dots, k_{j+1}](M-1)(M+N-1) = 1 \times (M-1)(M+1) = M^2 - 1.$$

Hence

$$K_0(\mathcal{O}_{\mathcal{H}_\kappa^{[2],[M]}}) \cong \mathsf{Z}/(M^2-1)\mathsf{Z}.$$

The formula for $N = 2$, $M = 3$ is already seen in [12].

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