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A COMPLEX OF MODULES AND ITS APPLICATIONS TO LOCAL COHOMOLOGY AND EXTENSION FUNCTORS

KAMAL BAHMANPOUR*

Abstract

Let (R, m) be a commutative Noetherian complete local ring and let M be a non-zero Cohen-Macaulay R-module of dimension n. It is shown that,

(i) if $\operatorname{projdim}_R(M) < \infty$, then $\operatorname{injdim}_R(D(H^n_{\mathfrak{m}}(M))) < \infty$, and

(ii) if $\operatorname{injdim}_R(M) < \infty$, then $\operatorname{projdim}_R(D(H^n_{\mathfrak{m}}(M))) < \infty$,

where $D(-) := \operatorname{Hom}_R(-, E)$ denotes the Matlis dual functor and $E := E_R(R/\mathfrak{m})$ is the injective hull of the residue field $R/\mathfrak{m}$.

Also, it is shown that if $(R, \mathfrak{m})$ is a Noetherian complete local ring, M is a non-zero finitely generated R-module and $x_1, \ldots, x_k$, $(k \geq 1)$, is an M-regular sequence, then

$$D(H^k_{(x_1,\ldots,x_k)}(D(H^k_{(x_1,\ldots,x_k)}(M)))) \simeq M.$$

In particular, $\operatorname{Ann} H^k_{(x_1,\ldots,x_k)}(M) = \operatorname{Ann} M$. Moreover, it is shown that if R is a Noetherian ring, M is a finitely generated R-module and $x_1, \ldots, x_k$ is an M-regular sequence, then

$$\operatorname{Ext}^{k+1}_R(R/(x_1, \ldots, x_k), M) = 0.$$

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I an ideal of R. For an R-module M, the i^{th} local cohomology module of M with respect to I is defined as

$$H^i_I(M) = \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [4] or [2] for more details about local cohomology.

In this paper we introduce a new complex of modules. Then we present some of its applications to the local cohomology and extension functors.

Recall that an ordered sequence $a_1, \ldots, a_n \in R$ is said to be an M-regular sequence if for all $1 \leq i \leq n$, $a_i \notin Z_R(M/(a_1, \ldots, a_{i-1})M)$ and

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$(a_1, \dots, a_n)M \neq M$, where $Z_R(M/(a_1, \dots, a_{i-1})M)$ denotes the set of all zero-divisors of $M/(a_1, \dots, a_{i-1})M$ in R. In the sequel let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Recall that a finitely generated R-module is called *Cohen-Macaulay* if $\operatorname{depth}(M) = \dim(M)$ and is called *maximal Cohen-Macaulay*, when $\operatorname{depth}(M) = \dim(R)$.

Throughout this paper, for any R-module M we denote the injective dimension of M by $\operatorname{injdim}_R(M)$. Also, we denote the flat dimension and the projective dimension of M by $\operatorname{flatdim}_R(M)$ and $\operatorname{projdim}_R(M)$, respectively. For any R-module M, the Matlis dual functor of M is denoted by $D(M)$. For any unexplained notation and terminology we refer the reader to [2] and [5].

2. A complex of modules

For technical reason we need the following new definition.

Definition 2.1. Let R be a ring (not necessary Noetherian) and n be a positive integer. Let $x_1, \dots, x_n \in R$ and M, N be R-modules. Then we write $M \overset{[x_1,\dots,x_n]}{\simeq} N$ if and only if there exists an exact sequence;

$$0 \longrightarrow M \xrightarrow{\varepsilon} K_1 \xrightarrow{h_1} K_2 \longrightarrow \cdots \longrightarrow K_{n-1} \xrightarrow{h_{n-1}} K_n \xrightarrow{\theta} N \longrightarrow 0,$$

such that for each $1 \leq i \leq n$, the R-homomorphism $K_i \xrightarrow{x_i} K_i$ is an isomorphism.

The following result will be useful in this paper.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero finitely generated R-module and let $x_1, \dots, x_t \in \mathfrak{m}$, $(t \geq 1)$. Then the following statements are equivalent:*

(i) $M \overset{[x_1,\dots,x_t]}{\simeq} H^t_{(x_1,\dots,x_t)}(M)$.

(ii) *$x_1, \dots, x_t$ is an M-regular sequence.*

Proof. (i) $\Rightarrow$ (ii) It follows from the definition that $H^i_{(x_1,\dots,x_j)}(M) = 0$ for all $0 \leq i \leq j - 1$ and all $1 \leq j \leq t$ and so the assertion follows from [2, Exercise 6.2.14].

(ii) $\Rightarrow$ (i) Since by the hypothesis $x_1, \dots, x_t$ is an M-regular sequence, for each $1 \leq i \leq t$, we have $\operatorname{Ass}_R(H^i_{(x_1,\dots,x_i)}(M)) = \operatorname{Ass}_R(M/(x_1, \dots, x_i)M)$. Therefore, for each $1 \leq i \leq t - 1$ we have $\Gamma_{Rx_{i+1}}(H^i_{(x_1,\dots,x_i)}(M)) = 0$. Now let $K_1 := M_{x_1}$ and $K_i := (H^{i-1}_{(x_1,\dots,x_{i-1})}(M))_{x_i}$, for each $2 \leq i \leq t$. Then, in view of [2, Remark 2.2.17] we have the exact sequence

$$0 \longrightarrow M \xrightarrow{\varepsilon} K_1 \xrightarrow{f_1} H^1_{Rx_1}(M) \longrightarrow 0$$

and the exact sequences

$$0 \longrightarrow H^{i-1}_{(x_1,\dots,x_{i-1})}(M) \xrightarrow{g_{i-1}} K_i \longrightarrow H^1_{Rx_i}(H^{i-1}_{(x_1,\dots,x_{i-1})}(M)) \longrightarrow 0, \tag{2.1}$$

for $i = 2, \dots, t$. On the other hand, for $i = 2, \dots, t$, by [6, Corollary 3.5] there exists an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1_{Rx_i}(H^{i-1}_{(x_1,\dots,x_{i-1})}(M)) \longrightarrow H^i_{(x_1,\dots,x_i)}(M) \\ \longrightarrow H^0_{Rx_i}(H^i_{(x_1,\dots,x_{i-1})}(M)) \longrightarrow 0. \end{aligned} \tag{2.2}$$

Also, in view of [2, Theorem 3.3.1] we have $H^i_{(x_1,\dots,x_{i-1})}(M) = 0$ and hence the exact sequence (2.2) yields the isomorphism

$$H^i_{(x_1,\dots,x_i)}(M) \simeq H^1_{Rx_i}(H^{i-1}_{(x_1,\dots,x_{i-1})}(M)). \tag{2.3}$$

Now, using the isomorphism (2.3), the exact sequence (2.1) yields the exact sequence

$$0 \longrightarrow H^{i-1}_{(x_1,\dots,x_{i-1})}(M) \xrightarrow{g_{i-1}} K_i \xrightarrow{f_i} H^i_{(x_1,\dots,x_i)}(M) \longrightarrow 0,$$

for $i = 2, \dots, t$. Using these exact sequences we can construct the following exact sequence

$$\begin{aligned} 0 \longrightarrow M \xrightarrow{\varepsilon} K_1 \xrightarrow{h_1} K_2 \longrightarrow \cdots \\ \longrightarrow K_{t-1} \xrightarrow{h_{t-1}} K_t \xrightarrow{\theta} H^t_{(x_1,\dots,x_t)}(M) \longrightarrow 0, \end{aligned}$$

where $h_i = g_i \circ f_i$, for $i = 1, \dots, t-1$ and $\theta = f_t$. Also it is clear that for each $1 \le i \le t$, the R-homomorphism $K_i \xrightarrow{x_i} K_i$ is an isomorphism.

The following corollary gives a new characterization of Cohen-Macaulay modules.

COROLLARY 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero finitely generated R-module of dimension $n \ge 1$. Let $x_1, \dots, x_n$ be a system of parameters for M. Then the following statements are equivalent:*

(i) $M \overset{[x_1,\dots,x_n]}{\simeq} H^n_{\mathfrak{m}}(M)$.

(ii) *M is a Cohen-Macaulay R-module.*

PROOF. The assertion follows from Lemma 2.2 using the fact that

$$H^n_{(x_1,\dots,x_n)}(M) \simeq H^n_{\mathfrak{m}}(M).$$

The following theorem is the main result of this section.

THEOREM 2.4. *Let R be a ring (not necessary Noetherian) and L, M, N be R-modules. Let $x_1, \dots, x_n$ be a sequence of elements in* $\operatorname{Ann} L$. *If $M \overset{[x_1,\dots,x_n]}{\simeq} N$, then for each integer $i \geq 0$ there are the following isomorphisms:*

(i) $\operatorname{Ext}_R^{i+n}(L, M) \simeq \operatorname{Ext}_R^i(L, N)$,

(ii) $\operatorname{Ext}_R^{i+n}(N, L) \simeq \operatorname{Ext}_R^i(M, L)$,

(iii) $\operatorname{Tor}^R_{i+n}(N, L) \simeq \operatorname{Tor}^R_i(M, L)$.

PROOF. By hypothesis, there exists an exact sequence of the R-modules as;

$$(2.4)\quad 0 \longrightarrow M \xrightarrow{\varepsilon} K_1 \xrightarrow{h_1} K_2 \longrightarrow \cdots \longrightarrow K_{n-1} \xrightarrow{h_{n-1}} K_n \xrightarrow{\theta} N \longrightarrow 0,$$

such that for each $1 \leq i \leq n$, the R-homomorphism $K_i \xrightarrow{x_i} K_i$ is an isomorphism. So, for each $j \geq 0$, each of the R-homomorphisms

$$\begin{aligned} &\operatorname{Ext}_R^j(L, K_i) \xrightarrow{x_i} \operatorname{Ext}_R^j(L, K_i) \\ \text{and}\quad &\operatorname{Ext}_R^j(K_i, L) \xrightarrow{x_i} \operatorname{Ext}_R^j(K_i, L) \\ \text{and}\quad &\operatorname{Tor}_j^R(K_i, L) \xrightarrow{x_i} \operatorname{Tor}_j^R(K_i, L), \end{aligned}$$

is an isomorphism, for each $1 \leq i \leq n$. Hence, it follows from the hypothesis $x_1, \dots, x_n \in \operatorname{Ann} L$, that

$$\operatorname{Ext}_R^j(L, K_i) = 0, \qquad \operatorname{Ext}_R^j(K_i, L) = 0 \qquad \text{and} \qquad \operatorname{Tor}_j^R(K_i, L) = 0,$$

for each $j \geq 0$ and each $1 \leq i \leq n$. Now, using these isomorphisms and by splitting the exact sequence (2.4) to some short exact sequences, the assertion easily follows.

We shall use the following consequences of Theorem 2.4 in the next section.

COROLLARY 2.5. *Let R be a ring (not necessary Noetherian) and M be an R-module. If $x_1, \dots, x_n \in R$ is a regular sequence on M, then for each integer $i \geq 0$ there are the following isomorphisms:*

(i) $\operatorname{Ext}_R^{i+n}(R/(x_1, \dots, x_n), M) \simeq \operatorname{Ext}_R^i(R/(x_1, \dots, x_n), H^n_{(x_1,\dots,x_n)}(M))$,

(ii) $\operatorname{Ext}_R^{i+n}(H^n_{(x_1,\dots,x_n)}(M), R/(x_1, \dots, x_n)) \simeq \operatorname{Ext}_R^i(M, R/(x_1, \dots, x_n))$,

(iii) $\operatorname{Tor}^R_{i+n}(H^n_{(x_1,\dots,x_n)}(M), R/(x_1, \dots, x_n)) \simeq \operatorname{Tor}^R_i(M, R/(x_1, \dots, x_n))$.

PROOF. By the proof of Lemma 2.2 we have $M \overset{[x_1,\dots,x_n]}{\simeq} H^n_{(x_1,\dots,x_n)}(M)$, so the assertion follows from Theorem 2.4.

COROLLARY 2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a non-zero finitely generated Cohen-Macaulay R-module of dimension n. Let L be a finitely generated R-module such that $M \otimes_R L$ is of finite length. Then for each integer $i \geq 0$ there are the following isomorphisms:*

(i) $\operatorname{Ext}_R^{i+n}(L, M) \simeq \operatorname{Ext}_R^i(L, H_{\mathfrak{m}}^n(M))$,

(ii) $\operatorname{Ext}_R^{i+n}(H_{\mathfrak{m}}^n(M), L) \simeq \operatorname{Ext}_R^i(M, L)$,

(iii) $\operatorname{Tor}_{i+n}^R(H_{\mathfrak{m}}^n(M), L) \simeq \operatorname{Tor}_i^R(M, L)$.

PROOF. Since by hypothesis the R-module $M \otimes_R L$ is of finite length, it follows that the ideal $\operatorname{Ann} L$ contains a system of parameters for M as $x_1, \dots, x_n$. Also by Corollary 2.3 we have $M \overset{[x_1,\dots,x_n]}{\simeq} H_{\mathfrak{m}}^n(M)$. Therefore, the assertion follows from Theorem 2.4.

3. Vanishing of the extension and torsion functors

The following lemmata are needed in the proof of some results of this paper.

LEMMA 3.1 (See [3, Exercise 1.1.12]). *Let R be a Noetherian ring and M be an R-module. Let $x_1, \dots, x_k$ be an M-regular sequence. Then*

$$\operatorname{Tor}_1^R(R/(x_1, \dots, x_k), M) = 0.$$

LEMMA 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module of dimension $n \geq 1$. Let $x_1, \dots, x_k$ be an M-regular sequence. Then $x_1, \dots, x_k$ is an $D(H_{(x_1,\dots,x_k)}^k(M))$-regular sequence.*

PROOF. We argue, using induction on k. For $k = 1$, the exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0,$$

yields the exact sequence

$$0 \longrightarrow M/x_1M \longrightarrow H_{(x_1)}^1(M) \xrightarrow{x_1} H_{(x_1)}^1(M) \longrightarrow H_{(x_1)}^1(M/x_1M).$$

But, since M/x_1M is (x_1)-torsion it follows that $H_{(x_1)}^1(M/x_1M) = 0$. Hence, we have the following exact sequence

$$0 \longrightarrow M/x_1M \longrightarrow H_{(x_1)}^1(M) \xrightarrow{x_1} H_{(x_1)}^1(M) \longrightarrow 0,$$

and applying the Matils dual functor to this exact sequence we get the exact sequence

$$0 \longrightarrow D(H_{(x_1)}^1(M)) \xrightarrow{x_1} D(H_{(x_1)}^1(M)) \longrightarrow D(M/x_1M) \longrightarrow 0.$$

Now, since $M/x_1M \neq 0$, it follows that

$$D(H^1_{(x_1)}(M))/x_1D(H^1_{(x_1)}(M)) \cong D(M/x_1M) \neq 0$$

and so x_1 is an $D(H^1_{(x_1)}(M))$-regular sequence. Now, let $k \geq 2$ and assume that the result has been proved for all regular sequences of length smaller than k. Then, since $x_2, \dots, x_k$ is an M/x_1M-regular sequence, it follows from the inductive hypothesis that, $x_2, \dots, x_k$ is an $D(H^{k-1}_{(x_2,\dots,x_k)}(M/x_1M))$-regular sequence. On the other hand, the exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0,$$

using [2, Theorem 6.2.7], induces the exact sequence

$$\begin{aligned} 0 \longrightarrow H^{k-1}_{(x_1,\dots,x_k)}(M/x_1M) &\longrightarrow H^k_{(x_1,\dots,x_k)}(M) \\ &\xrightarrow{x_1} H^k_{(x_1,\dots,x_k)}(M) \longrightarrow H^k_{(x_1,\dots,x_k)}(M/x_1M). \end{aligned}$$

Moreover, in view of [2, Exercise 2.1.9] and [2, Theorem 3.3.1] we have

$$H^k_{(x_1,\dots,x_k)}(M/x_1M) \cong H^k_{(x_2,\dots,x_k)}(M/x_1M) = 0.$$

Now, the exact sequence

$$0 \longrightarrow H^{k-1}_{(x_1,\dots,x_k)}(M/x_1M) \longrightarrow H^k_{(x_1,\dots,x_k)}(M) \xrightarrow{x_1} H^k_{(x_1,\dots,x_k)}(M) \longrightarrow 0$$

yields the exact sequence

$$\begin{aligned} 0 \longrightarrow D(H^k_{(x_1,\dots,x_k)}(M)) &\xrightarrow{x_1} D(H^k_{(x_1,\dots,x_k)}(M)) \\ &\longrightarrow D(H^{k-1}_{(x_1,\dots,x_k)}(M/x_1M)) \longrightarrow 0. \end{aligned}$$

But, in view of [2, Exercise 2.1.9] we have

$$H^{k-1}_{(x_1,\dots,x_k)}(M/x_1M) \cong H^{k-1}_{(x_2,\dots,x_k)}(M/x_1M).$$

So, $x_2, \dots, x_k$ is an $D(H^{k-1}_{(x_1,\dots,x_k)}(M/x_1M))$-regular sequence. Now, it is clear that $x_1, \dots, x_k$ is an $D(H^k_{(x_1,\dots,x_k)}(M))$-regular sequence. This completes the inductive step and the proof of the lemma.

The following result is needed in the proof of Theorem 3.4.

LEMMA 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module. Let $x_1, \dots, x_k$ be an M-regular sequence. Then*

$$\operatorname{Ext}^{k+1}_R(R/(x_1, \dots, x_k), M) = 0.$$

PROOF. By Corollary 2.5 we have

$$\operatorname{Ext}^{k+1}_R(R/(x_1,\dots,x_k),M)\simeq \operatorname{Ext}^1_R(R/(x_1,\dots,x_k),H^k_{(x_1,\dots,x_k)}(M)).$$

Moreover, by the adjointness and using Lemmata 3.1 and 3.2 we have

$$\begin{aligned}&D(\operatorname{Ext}^1_R(R/(x_1,\dots,x_k),H^k_{(x_1,\dots,x_k)}(M)))\\ &\qquad\simeq \operatorname{Tor}^R_1(R/(x_1,\dots,x_k),D(H^k_{(x_1,\dots,x_k)}(M)))=0.\end{aligned}$$

So, $\operatorname{Ext}^1_R(R/(x_1,\dots,x_k),H^k_{(x_1,\dots,x_k)}(M))=0$ and hence

$$\operatorname{Ext}^{k+1}_R(R/(x_1,\dots,x_k),M)=0.$$

THEOREM 3.4. *Let $(R,\mathfrak{m})$ is a Noetherian complete local ring, M is a non-zero finitely generated R-module and $x_1,\dots,x_k$, $(k\geq 1)$, is an M-regular sequence, then the following statements hold:*

(i) $D(H^k_{(x_1,\dots,x_k)}(D(H^k_{(x_1,\dots,x_k)}(M))))\simeq M$.

(ii) $\operatorname{Ann} H^k_{(x_1,\dots,x_k)}(M)=\operatorname{Ann} M$.

PROOF. (i) Since $x_1,\dots,x_k$ is an M-regular sequence, from [5, Theorem 16.1] it follows that for each positive integer n, the sequence $x_1^n,\dots,x_k^n$ is M-regular. Moreover, since M is $\mathfrak{m}$-adically complete, the exercise [5, Exercise 8.2] implies that M is also $(x_1,\dots,x_k)$-adically complete. Therefore, we have

$$\varprojlim_{n\geq 1} M/(x_1^n,\dots,x_k^n)M\simeq M.$$

Now, using adjointness and Corollary 2.5 we have

$$\begin{aligned}&D(H^k_{(x_1,\dots,x_k)}(D(H^k_{(x_1,\dots,x_k)}(M))))\\ &\quad\simeq D\Big(\varinjlim_{n\geq 1}\operatorname{Ext}^k_R\big(R/(x_1^n,\dots,x_k^n),D(H^k_{(x_1,\dots,x_k)}(M))\big)\Big)\\ &\quad\simeq D\Big(\varinjlim_{n\geq 1} D(\operatorname{Tor}^R_k\big(R/(x_1^n,\dots,x_k^n),H^k_{(x_1,\dots,x_k)}(M))\big)\Big)\\ &\quad\simeq \varprojlim_{n\geq 1} D(D(\operatorname{Tor}^R_0(R/(x_1^n,\dots,x_k^n),M)))\\ &\quad\simeq \varprojlim_{n\geq 1} D(D(M/(x_1^n,\dots,x_k^n)M))\\ &\quad\simeq \varprojlim_{n\geq 1} M/(x_1^n,\dots,x_k^n)M\simeq M.\end{aligned}$$

(ii) Follows from (i).

The following theorem is the main result of this section.

THEOREM 3.5. *Let R be a Noetherian ring and M be a finitely generated R-module. Let $x_1, \ldots, x_k$ be an M-regular sequence. Then*

$$\operatorname{Ext}_R^{k+1}(R/(x_1, \ldots, x_k), M) = 0.$$

PROOF. Suppose that $\operatorname{Ext}_R^{k+1}(R/(x_1, \ldots, x_k), M) \neq 0$. Then there exists

$$\mathfrak{p} \in \operatorname{Supp}(\operatorname{Ext}_R^{k+1}(R/(x_1, \ldots, x_k), M)).$$

Then

$$\operatorname{Ext}_{R_\mathfrak{p}}^{k+1}(R_\mathfrak{p}/(x_1, \ldots, x_k)R_\mathfrak{p}, M_\mathfrak{p}) \neq 0.$$

But, in the situation it is easy to see that $x/1, \ldots, x_k/1 \in \mathfrak{p}R_\mathfrak{p}$ is an $M_\mathfrak{p}$-regular sequence and hence by Lemma 3.3 we have $\operatorname{Ext}_{R_\mathfrak{p}}^{k+1}(R_\mathfrak{p}/(x_1, \ldots, x_k)R_\mathfrak{p}, M_\mathfrak{p}) = 0$, which is a contradiction.

COROLLARY 3.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and M be a maximal Cohen-Macaulay R-module. Let $x_1, \ldots, x_d$ be a system of parameters for R. Then the following statements hold:*

(i) $\operatorname{Tor}_{d+1}^R(R/(x_1, \ldots, x_d), H_\mathfrak{m}^d(M)) = 0$,

(ii) $\operatorname{Tor}_{d+1}^R(H_\mathfrak{m}^d(R), H_\mathfrak{m}^d(M)) = 0$,

(iii) $\operatorname{Ext}_R^{d+1}(R/(x_1, \ldots, x_d), M) = 0$.

PROOF. The first assertion follows from Corollary 2.6 together with Lemma 3.1. The second assertion follows from part (i) using the isomorphism

$$H_\mathfrak{m}^d(R) = H_{(x_1,\ldots,x_d)}^d(R) \simeq \varinjlim_{n \geq 1} R/(x_1^n, \ldots, x_d^n),$$

and the fact that the torsion functor $\operatorname{Tor}_{d+1}^R(-, H_\mathfrak{m}^d(M))$ commutes with direct limits. The third assertion follows from Theorem 3.5.

4. Top local cohomology modules of Cohen-Macaulay modules

We need the following well known result and its corollary.

LEMMA 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let A be a non-zero Artinian R-module and M be a non-zero finitely generated R-module. Then*

$$\operatorname{flatdim}_R(A) = \sup\{\, n \in \mathsf{N}_0 : \operatorname{Tor}_n^R(R/\mathfrak{m}, A) \neq 0 \,\},$$

and

$$\operatorname{projdim}_R(M) = \sup\{\, n \in \mathsf{N}_0 : \operatorname{Tor}_n^R(R/\mathfrak{m}, M) \neq 0\,\},$$

PROOF. See [1, Corollary 2.9] and [5, §19 Lemma 1].

COROLLARY 4.2. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* M *be a non-zero finitely generated Cohen-Macaulay* R*-module of dimension* n*. Then for the following statements hold:*

(i) $\operatorname{projdim}_R(M) < \infty$ *if and only if* $\operatorname{flatdim}_R(H^n_{\mathfrak{m}}(M)) < \infty$.

(ii) *If* $\operatorname{projdim}_R(M) < \infty$*, then* $\operatorname{flatdim}_R(H^n_{\mathfrak{m}}(M)) = n + \operatorname{projdim}_R(M) = \operatorname{depth}(R)$.

PROOF. The assertion follows from Corollary 2.6, Lemma 4.1, and [5, Theorem 19.1].

Recall that, in view of the New Intersection Theorem, over a Noetherian local ring R, the existence of a non-zero Cohen-Macaulay module with finite projective dimension or the existence of a non-zero finitely generated module with finite injective dimension is equivalent to the fact that R is a Cohen-Macaulay ring. The following result shows that, when R is complete then for a given non-zero Cohen-Macaulay module M with finite projective dimension, it is easy to find a non-zero finitely generated module $N = D(H^n_{\mathfrak{m}}(M))$ with finite injective dimension for which $\operatorname{Ann} N = \operatorname{Ann} M$.

THEOREM 4.3. *Let* $(R, \mathfrak{m})$ *be a Noetherian Cohen-Macaulay local ring and* M *be a non-zero finitely generated Cohen-Macaulay* R*-module of dimension* n*, such that* $\operatorname{projdim}_R(M) < \infty$*. Then* $\operatorname{injdim}_R(D(H^n_{\mathfrak{m}}(M))) < \infty$.

PROOF. By Corollary 4.2 we have $\operatorname{flatdim}_R(H^n_{\mathfrak{m}}(M)) = \operatorname{depth}(R)$. Let $\operatorname{depth}(R) = t$. Then there is a finite flat resolution

$$0 \longrightarrow Q_t \longrightarrow Q_{t-1} \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow H^n_{\mathfrak{m}}(M) \longrightarrow 0,$$

for the R-module $H^n_{\mathfrak{m}}(M)$. Then the exact sequence

$$\begin{aligned} 0 \longrightarrow D(H^n_{\mathfrak{m}}(M)) \longrightarrow D(Q_0) \longrightarrow D(Q_1) \\ \longrightarrow \cdots \longrightarrow D(Q_{t-1}) \longrightarrow D(Q_t) \longrightarrow 0, \end{aligned}$$

is an injective resolution for $D(H^n_{\mathfrak{m}}(M))$. So, the R-module $D(H^n_{\mathfrak{m}}(M))$ is of finite injective dimension.

PROPOSITION 4.4. *Let* $(R, \mathfrak{m})$ *be a Noetherian Cohen-Macaulay complete local ring and* M *be a non-zero finitely generated Cohen-Macaulay* R*-module of dimension* n*, such that* $\operatorname{injdim}_R(M) < \infty$*. Then* $\operatorname{projdim}_R(D(H^n_{\mathfrak{m}}(M))) < \infty$.

PROOF. Since $\operatorname{injdim}_R(M) < \infty$ it follows that $\operatorname{Ext}^i_R(R/m, M) = 0$, for each $i > \dim(R)$. Therefore, by Theorem 2.4 it follows that $\operatorname{Ext}^i_R(R/m, H^n_{\mathfrak{m}}(M)) = 0$, for each $i > \dim(R)-n$. So, as $H^n_{\mathfrak{m}}(M)$ is an Artinian R-module we can deduce that $\operatorname{injdim}_R(H^n_{\mathfrak{m}}(M)) < \infty$. Now, as R is complete, applying the functor $D(-)$ to a minimal injective resolution of $H^n_{\mathfrak{m}}(M)$ we get a finite free resolution for $D(H^n_{\mathfrak{m}}(M))$, which implies that $\operatorname{projdim}_R(D(H^n_{\mathfrak{m}}(M))) < \infty$.

The following result is the main result of this section.

THEOREM 4.5. *Let $(R, \mathfrak{m})$ be a complete Noetherian Cohen-Macaulay local ring and I a proper ideal of R. Then the following statements are equivalent:*

(i) *There exists a non-zero Cohen-Macaulay R-module M with* $\operatorname{injdim}_R(M) < \infty$ *such that* $\operatorname{Ann} M = I$.

(ii) *There exists a non-zero Cohen-Macaulay R-module N with* $\operatorname{projdim}_R(N) < \infty$ *such that* $\operatorname{Ann} N = I$.

PROOF. Using Lemma 3.2, Theorem 4.3 and Proposition 4.4 we can take

$$N := D(H^{\dim(M)}_{\mathfrak{m}}(M)) \qquad \text{and} \qquad M := D(H^{\dim(N)}_{\mathfrak{m}}(N))$$

with the desired properties. (Note that the conditions on the annihilators follow from Theorem 3.4.)

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FACULTY OF MATHEMATICAL SCIENCES
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MOHAGHEGH ARDABILI
56199-11367
ARDABIL
IRAN
and
SCHOOL OF MATHEMATICS
INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM)
P.O. BOX 19395-5746
TEHRAN
IRAN
E-mail: bahmanpour.k@gmail.com