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HOPF HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH $\mathfrak{D}$-PARALLEL SHAPE OPERATOR

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Abstract

In this paper we consider a generalized condition for shape operator of a real hypersurface M in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, namely, $\mathfrak{D}$-parallel shape operator of M. Using such a notion, we prove that there does not exist a real hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel shape operator.

Introduction

A real Grassmann manifold is known to be the set of all linear subspaces in $\mathbb{R}^n$ with the same dimension. As a kind of complex Grassmann manifold, we introduce the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ is the unique compact irreducible Riemannian manifold with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. For a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometric conditions that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M (see [2] and [3]).

The almost contact structure vector field ξ defined by $\xi = -JN$ is said to be the *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ spanning the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$ and $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

By using two invariant conditions mentioned above and the result in Alekseevskii [1], Berndt and Suh [2] proved the following:

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THEOREM A. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Furthermore, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be the *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in section 1 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

Using Theorem A, many geometers have given various characterizations of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with certain geometric objects, for example, shape operator, normal (or structure) Jacobi operator, Ricci tensor, and so on (see [3], [10], [11], [12] and [13]). From such a point of view, Lee and Suh [5] gave a characterization of Hopf hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

THEOREM B. *Let M be a connected orientable Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$, where the distribution $\mathfrak{D}$ denotes an orthogonal complement of* $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$.

For a real hypersurface M in quaternionic projective space $\mathbb{H}P^n$, Pérez [8] considered the notion of $\mathfrak{D}^\perp$-parallel shape operator, that is, $\nabla_{\xi_i} A = 0$, $i = 1, 2, 3$, where the three dimensional distribution $\mathfrak{D}^\perp$ is spanned by $\{\xi_1, \xi_2, \xi_3\}$. For real hypersurfaces M in complex projective space $\mathbb{C}P^n$, Pérez, Santos and Suh [9] studied a notion of Reeb parallel structure Jacobi operator with respect to the Lie derivatives, that is $\mathfrak{L}_\xi R_\xi = 0$.

In [12], Suh proved a non-existence property for all hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with *parallel shape operator*, that is, $(\nabla_X A)Y = 0$ for any tangent vector fields X and Y on M. As a generalization of this result, Suh [13] considered a new condition weaker than usual parallelism. When we restrict the shape operator to the distribution $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$, the shape operator A is said to be *$\mathfrak{F}$-parallel*. In such a case, Suh [13] could prove a non-existence theorem for a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{F}$-parallel shape operator.

Motivated by these results, we consider a new notion weaker than parallel shape operator, that is, $\mathfrak{D}$-*parallel shape operator* which is defined by

$$(*) \qquad (\nabla_X A)Y = 0$$

for all vector fields $X \in \mathfrak{D}$ and $Y \in TM$. This means that eigenspaces of the shape operator A are parallel along the geodesic curve γ with initial conditions $\gamma(0) = x \in M$ and $\dot{\gamma}(0) = X \in \mathfrak{D} \subset T_xM$. Here, the eigenspaces of the shape operator A are said to be *parallel along* γ if they are invariant with respect to any parallel displacement along γ. Related to the curvature function of a curve, we will give a more detailed geometric meaning of this notion in section 4. Using such a notion, we give a complete classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel shape operator as follows:

MAIN THEOREM. *There does not exist any Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel shape operator.*

1. Preliminaries

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2], [3] and [11]. The complex two-plane Grassmannian becomes a Riemannian homogeneous space, even a Riemannian symmetric space. Using Lie algebra, we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\tilde{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$(1.1) \qquad \tilde{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Furthermore, the Riemannian curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$(1.2) \qquad \begin{aligned} &\tilde{R}(X, Y)Z \\ &\quad = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ &\qquad - g(JX, Z)JY - 2g(JX, Y)JZ \\ &\qquad + \sum_{\nu=1}^{3}\{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z\} \\ &\qquad + \sum_{\nu=1}^{3}\{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\}, \end{aligned}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

Now, let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N. Let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{1.3}$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (ϕ, ξ, η, g) on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{1.4}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0,\\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\
&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned} \tag{1.5}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$, the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{aligned}
&\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu,\\
&\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi.
\end{aligned} \tag{1.6}$$

On the other hand, from the Kähler structure J, that is, $\tilde{\nabla} J = 0$ and the quaternionic Kähler structure J_ν (see (1.1)), together with Gauss and Weingarten formulas it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{1.7}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{1.8}$$

$$\begin{aligned}
(\nabla_X\phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y\\
&+ \eta_\nu(Y)AX - g(AX, Y)\xi_\nu.
\end{aligned} \tag{1.9}$$

Using the above expression for the curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi is given by

$$\begin{aligned}
&(\nabla_X A)Y - (\nabla_Y A)X \\
&\quad = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
&\qquad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\
&\qquad + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\
&\qquad + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu.
\end{aligned} \tag{1.10}$$

2. Key lemmas

From now on, we assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel shape operator, that is, the shape operator A of M is given by

$$(\nabla_X A)Y = 0 \tag{$*$}$$

for all vector fields $X \in \mathfrak{D}$ and $Y \in TM$.

Then from the equation of Codazzi (1.10), it implies that

$$\begin{aligned}
0 &= (\nabla_Y A)X + \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
&\quad + \sum_{\nu=1}^{3}\Big\{ - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\Big\} \\
&\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\Big\} \\
&\quad + \sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\Big\}\xi_\nu
\end{aligned} \tag{2.1}$$

for all vector fields $X \in \mathfrak{D}$ and $Y \in TM$.

In particular, since $(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX$, the condition $(*)$ implies

$$0 = (X\alpha)\xi + \alpha\phi AX - A\phi AX \tag{2.2}$$

for any vector field $X \in \mathfrak{D}$.

Taking the inner product of (2.2) with ξ, we have $X\alpha = 0$ for any vector field $X \in \mathfrak{D}$. From this, we obtain the following result:

LEMMA 2.1. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel shape operator. Then $X\alpha = 0$ for any vector field $X \in \mathfrak{D}$. Moreover, the vector ϕAX becomes a principal vector of A with the corresponding principal curvature α, that is, $A\phi AX = \alpha\phi AX$ for any vector $X \in \mathfrak{D} \subset T_xM$ for any point $x \in M$.*

In this section, we want to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or its orthogonal complement $\mathfrak{D}^{\perp}$, where $T_xM = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$, $x \in M$, in $G_2(\mathbb{C}^{m+2})$. In order to do this, without loss of generality, we may put the Reeb vector field ξ as follows:

$$(**) \qquad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$$

for some unit vector fields $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^{\perp}$.

On the other hand, using the notion of the geodesic Reeb flow, Berndt and Suh ([2]) proved the following:

LEMMA A. *If M is a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with geodesic Reeb flow, then we have the following equation*

$$(2.3) \qquad Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any tangent vector field Y on M.

Now, using these facts, we prove the following:

LEMMA 2.2. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel shape operator. Then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$.*

PROOF. Actually, when the smooth function $\alpha = g(A\xi, \xi)$ identically vanishes, this lemma can be verified directly from Pérez and Suh ([10, pp. 220–221]).

Thus, in this proof we consider only the case that the function α is non-vanishing. Moreover, under our assumptions, we have already proved that the principal curvature α is constant on $\mathfrak{D}$ in Lemma 2.1. So, if Y is restricted to $\mathfrak{D}$ in (2.3), then we get $(\xi\alpha)\eta(Y) - 4\eta_1(\xi)\eta_1(\phi Y) = 0$. Since $\phi\xi_1 = \eta(X_0)\phi_1X_0$, it follows

$$(2.4) \qquad (\xi\alpha)\eta(X_0)g(X_0, Y) + 4\eta(X_0)\eta_1(\xi)g(\phi_1X_0, Y) = 0,$$

for any $Y \in \mathfrak{D}$.

Substituting Y into X_0, the equation (2.4) becomes

$$\eta(X_0)(\xi\alpha) = 0,$$

because the structure tensor ϕ is skew-symmetric.

If $\xi\alpha \neq 0$, it gives $\eta(X_0) = 0$. From this, the Reeb vector field ξ becomes $\xi = \eta(\xi_1)\xi_1$. So, we conclude that ξ belongs to the distribution $\mathfrak{D}^\perp$.

Next, it remains to consider that $\xi\alpha = 0$. Since $\phi_1 X_0 \in \mathfrak{D}$, substituting Y into $\phi_1 X_0$ in (2.4), we get

$$\eta(X_0)\eta_1(\xi) = 0,$$

that is, $\eta(X_0) = 0$ or $\eta_1(\xi) = \eta(\xi_1) = 0$. Accordingly, we get a complete proof of our Lemma 2.2.

3. Proofs of the Main Theorem

Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel shape operator, that is, the shape operator A satisfies the following condition:

$$(\nabla_X A)Y = 0 \tag{$*$}$$

for all vector fields $X \in \mathfrak{D}$ and $Y \in TM$. Then by virtue of Lemma 2.2 we have the following two cases:

Case I: the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$,

Case II: the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.

Now, let us consider the first case, $\xi \in \mathfrak{D}^\perp$. For convenience's sake, we may put $\xi = \xi_1$.

LEMMA 3.1. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel shape operator. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then the distribution $\mathfrak{D}^\perp$ is invariant under the shape operator A of M.*

PROOF. Since we assume that the shape operator A of M is parallel on $\mathfrak{D}$, the equation of Codazzi (1.10) can be written as

$$2g(\phi X, Y) + 2\sum_{\nu=1}^{3} g(\phi_\nu X, Y)\eta(\xi_\nu) = 0$$

for all vector fields X and $Y \in \mathfrak{D}$.

From this, together with $\xi \in \mathfrak{D}^\perp$, it follows that

$$g(\phi X + \phi_1 X, Y) = 0 \tag{3.1}$$

for any tangent vector fields X and Y on $\mathfrak{D}$.

Let $\{e_1, e_2, \dots, e_{4m-1}\}$ be an orthonormal basis of $T_x M$, where x is any point of M. Without loss of generality, we may put $e_{4(m-1)+\nu} = \xi_\nu$, $\nu = 1, 2, 3$.

Then the tangent vector field $\phi X + \phi_1 X$ on M is given by

$$\begin{aligned}\phi X + \phi_1 X &= \sum_{i=1}^{4m-1} g(\phi X + \phi_1 X, e_i)e_i \\ &= \sum_{i=1}^{4m-4} g(\phi X + \phi_1 X, e_i)e_i + \sum_{\nu=1}^{3} g(\phi X + \phi_1 X, \xi_\nu)\xi_\nu \\ &= 0\end{aligned}$$

for any $X \in \mathfrak{D}$. The third equality holds from the equation (3.1) and the facts $\phi\xi_\nu, \phi_1\xi_\nu \in \mathfrak{D}^\perp$. Moreover, from our assumption $\xi = \xi_1$, we have naturally

$$\phi\xi_\nu + \phi_1\xi_\nu = 0, \quad \nu = 1, 2, 3.$$

Summing up these two facts, we assert

$$\phi X + \phi_1 X = 0 \tag{3.2}$$

for any tangent vector field X on M.

On the other hand, differentiating $\xi = \xi_1$ along any vector field $X \in TM$, we have

$$\phi AX = q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX, \tag{3.3}$$

where we have used (1.7) and (1.8).

Moreover, by taking the inner product with ξ_2 and ξ_3, we obtain

$$g(\phi AX, \xi_2) = q_3(X) + g(\phi_1 AX, \xi_2)$$

and

$$g(\phi AX, \xi_3) = -q_2(X) + g(\phi_1 AX, \xi_3),$$

respectively. It follows that

$$q_3(X) = 2g(AX, \xi_3) \quad \text{and} \quad q_2(X) = 2g(AX, \xi_2).$$

From these relations, the equation (3.3) can be written as

$$\phi AX = 2g(AX, \xi_3)\xi_2 - 2g(AX, \xi_2)\xi_3 + \phi_1 AX. \tag{3.4}$$

By applying ϕ to (3.4), we have

$$AX = \eta(AX)\xi + 2g(AX, \xi_2)\xi_2 + 2g(AX, \xi_3)\xi_3 - \phi\phi_1 AX \tag{3.5}$$

for any vector field X on M.

By the way, from (3.2) we know that $\phi_1 X = -\phi X$ for any X on M. Then equation (3.5) can be written as

$$AX = \eta(AX)\xi + 2g(AX, \xi_2)\xi_2 + 2g(AX, \xi_3)\xi_3 + \phi^2 AX,$$

that is,

$$AX = \eta(AX)\xi + g(AX, \xi_2)\xi_2 + g(AX, \xi_3)\xi_3$$

for any tangent vector filed X on M. Therefore we prove that the distribution $\mathfrak{D}^\perp$ is invariant under the shape operator A of M, that is, $AX \in \mathfrak{D}^\perp$ for $X \in \mathfrak{D}^\perp$.

From this Lemma and Theorem A, we assert the following:

PROPOSITION 3.2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel shape operator. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Now, let us check whether the shape operator A for a real hypersurface of Type (A) satisfies the condition $(*)$ for all vector fields $X \in \mathfrak{D}$ and $Y \in TM$.

In order to do this, we introduce one proposition due to Berndt and Suh [2]. They proved that a real hypersurface of Type (A) has three or four distinct constant principal curvatures as follows:

PROPOSITION A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\},\\
T_\beta &= \mathfrak{C}^\perp\xi = \mathfrak{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2, \xi_3\},\\
T_\lambda &= \{X \mid X \perp \mathfrak{H}\xi, JX = J_1 X\},\\
T_\mu &= \{X \mid X \perp \mathfrak{H}\xi, JX = -J_1 X\}
\end{aligned}$$

where $\mathsf{R}\xi$, $\mathsf{C}\xi$ *and* $\mathsf{H}\xi$ *respectively denotes real, complex and quaternionic span of the structure vector field* ξ *and* $\mathsf{C}^{\perp}\xi$ *denotes the orthogonal complement of* $\mathsf{C}\xi$ *in* $\mathsf{H}\xi$.

From now on, to check our question for a real hypersurface M of Type (A) in $G_2(\mathsf{C}^{m+2})$, let us assume M has the $\mathfrak{D}$-parallel shape operator. In particular, putting $X \in T_\lambda \subset \mathfrak{D}$ and $Y = \xi = \xi_1 \in T_\alpha$ in (2.1), it becomes

$$\begin{aligned} 0 &= (\nabla_\xi A)X + \eta(X)\phi\xi - \eta(\xi)\phi X - 2g(\phi X, \xi)\xi \\ &\quad + \sum_{\nu=1}^{3}\big\{-\eta_\nu(\xi)\phi_\nu X - 2g(\phi_\nu X, \xi)\xi_\nu\big\} \\ &\quad + \sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi\xi - \eta_\nu(\phi\xi)\phi_\nu\phi X\big\} \\ &\quad + \sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi\xi) - \eta(\xi)\eta_\nu(\phi X)\big\}\xi_\nu \\ &= (\nabla_\xi A)X - \phi X - \phi_1 X \\ &= \alpha\phi AX - A\phi AX + \phi X + \phi_1 X - \phi X - \phi_1 X \\ &= \alpha\lambda\phi X - \lambda^2\phi X, \end{aligned}$$

where we have used the equation of Codazzi (1.10) and $A\xi = \alpha\xi$.

Taking the inner product with ϕX in the above equation, we get

$$\lambda^2 - \alpha\lambda = 0.$$

Since $\alpha = \sqrt{8}\cot(\sqrt{8}r)$ and $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$, this gives a contradiction. So we have given a proof of our main Theorem for $\xi \in \mathfrak{D}^\perp$.

Next, let us consider the case $\xi \in \mathfrak{D}$. From Theorem B, we have the following:

PROPOSITION 3.3. *Let* M *be a Hopf hypersurface in* $G_2(\mathsf{C}^{m+2})$ *with* $\mathfrak{D}$*-parallel shape operator. If the Reeb vector field* ξ *belongs to the distribution* $\mathfrak{D}$*, then* M *is locally congruent to an open part of a tube around a totally geodesic* $\mathsf{H}P^n$ *in* $G_2(\mathsf{C}^{m+2})$*,* $m = 2n$.

Now, let us check whether the shape operator A of a real hypersurface M of Type (B) satisfies the condition $(*)$ for all vector fields $X \in \mathfrak{D}$ and $Y \in TM$. As it is well known, a real hypersurface M of Type (B) has five distinct constant principal curvatures as follows [2]:

PROPOSITION B. *Let* M *be a connected real hypersurface of* $G_2(\mathsf{C}^{m+2})$*. Suppose that* $A\mathfrak{D} \subset \mathfrak{D}$*,* $A\xi = \alpha\xi$*, and* ξ *is tangent to* $\mathfrak{D}$*. Then the quaternionic* *dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \mathrm{Span}\{\xi\}, \\ &T_\beta = \mathfrak{J}J\xi = \mathrm{Span}\{\xi_\nu \mid \nu = 1, 2, 3\}, \\ &T_\gamma = \mathfrak{J}\xi = \mathrm{Span}\{\phi_\nu\xi \mid \nu = 1, 2, 3\}, \\ &T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^\perp$ is the orthogonal complement of $\mathbb{H}\mathbb{C}\xi$ where

$$\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

Putting $X = \xi \in \mathfrak{D}$ and $Y = \xi_2 \in T_\beta$ in (2.1), we obtain

$$0 = \alpha\beta\phi\xi_2,$$

because $A\phi_2\xi = \gamma\phi_2\xi$ and $\gamma = 0$. From this, it follows that

$$\alpha\beta = 0.$$

However, from Proposition B, we see that $\alpha\beta = -4$ for some radius $r \in (0, \pi/4)$. This gives a contradiction. So this case can not occur.

Hence summing up two cases mentioned above, we give a complete proof of our main theorem in the introduction.

4. Geometric meaning of $\mathfrak{D}$-parallel shape operator

Let $\bar{M}$ be a Kähler manifold with the Riemannian metric G and Riemannian connection $\bar{\nabla}$. Let M be a real hypersurface in $\bar{M}$ with the induced metric g and the induced Riemannian connection ∇. Since M is a real hypersurface in

$\bar{M}$, there only exists one normal vector field N on M in $\bar{M}$. Thus we have the following two formulas:

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + g(AX, Y)N \quad &\text{(Gauss formula)} \\ \bar{\nabla}_X N &= -AX &\text{(Weingarten formula)} \end{aligned} \tag{4.1}$$

for arbitrary tangent vector fields X, Y on M.

Now, we introduce some notions for parallelism of the shape operator:

A real hypersurface M in $\bar{M}$ is called *cyclic parallel* (or *cyclic $\mathfrak{T}$-parallel*, resp.) if it satisfies

$$\mathfrak{S}_{X,Y,Z}\, g((\nabla_X A)Y, Z) = g((\nabla_X A)Y, Z) \\ + g((\nabla_Y A)Z, X) + g((\nabla_Z A)X, Y) = 0$$

for any tangent vector fields X, Y, Z on M (or $X, Y, Z \in \mathfrak{T}$, resp.). Here $\mathfrak{T}$ denotes a certain distribution defined on M. In particular, when it holds on $\mathfrak{T} = \mathfrak{h}$ where the distribution $\mathfrak{h}$ is given by $\mathfrak{h} = \{X \in TM \mid X \perp \xi\}$, the shape operator A of M is said to be *cyclic η-parallel* (see [4]).

Under these situations, for arbitrary geodesic γ on M in $\bar{M}$, we assert:

LEMMA 4.1. *The shape operator A of M in $\bar{M}$ is cyclic parallel if and only if*

(C_1) *the first curvature function of γ as a curve in the ambient space $\bar{M}$ is a constant function.*

PROOF. Assume that the first curvature function for an arbitrary geodesic curve $\gamma : I \to \bar{M}$ is constant. By definition it means that $\bar{\nabla}_{\dot\gamma}\dot\gamma$ has constant length in $\bar{M}$, that is, $G(\bar{\nabla}_{\dot\gamma}\dot\gamma, \bar{\nabla}_{\dot\gamma}\dot\gamma)$ is constant on the interval I. From the Gauss formula in (4.1), we have $G(\bar{\nabla}_{\dot\gamma}\dot\gamma, \bar{\nabla}_{\dot\gamma}\dot\gamma) = g(A\dot\gamma, \dot\gamma)^2$. Hence our assumption is equivalent to the constancy of $g(A\dot\gamma, \dot\gamma)$ on I.

By differentiation and using $\nabla_{\dot\gamma}\dot\gamma = 0$, we obtain $g((\nabla_{\dot\gamma} A)\dot\gamma, \dot\gamma) = 0$ on I. Therefore our assumption is equivalent to

$$g((\nabla_X A)X, X) = 0 \tag{4.2}$$

for any tangent vector X of M.

Using the linearity of the Riemannian connection, it follows that

$$g\big((\nabla_{X+Y+Z} A)(X+Y+Z), X+Y+Z\big) = 2\mathfrak{S}_{X,Y,Z}\, g((\nabla_X A)Y, Z) = 0, \tag{4.3}$$

where we have used

$$\begin{aligned} &g\big((\nabla_{X+Y} A)(X+Y), X+Y\big) \\ &\quad = g((\nabla_X A)X, Y) + g((\nabla_X A)Y, X) + g((\nabla_Y A)Y, X) \\ &\qquad + g((\nabla_Y A)X, X) + g((\nabla_Y A)X, Y) + g((\nabla_X A)Y, Y) \end{aligned}$$

for tangent vector fields X, Y, Z on M. Therefore, we can assert M is cyclic parallel under our assumption.

The converse is trivial if we put $X = Y = Z$ for arbitrary tangent vector fields $X, Y, Z \in T_pM$.

REMARK 4.2. The contents in Lemma 4.1 above were remarked by S. Maeda [7]. But in this section we have proved the statement in detail.

Motivated by Lemma 4.1, we can assert the following

LEMMA 4.3. *The shape operator A of M in $\bar{M}$ is cyclic $\mathfrak{X}$-parallel if and only if*

(C$_2$) *every geodesic curve γ with $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = X \in \mathfrak{X} \subset T_pM$ has constant first curvature.*

Now let us consider our case for $\bar{M} = G_2(\mathbb{C}^{m+2})$. That is, we want to give a geometric meaning of $\mathfrak{D}$-parallel shape operator for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. It means that the shape operator A of M satisfies

$$(\nabla_X A)Y = 0,$$

for any tangent vector field $X \in \mathfrak{D}$ and $Y \in TM$ where the distribution $\mathfrak{D}$ denotes an orthogonal complement of $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_\nu \mid \nu = 1, 2, 3\}$. From this, we know that the shape operator A naturally becomes cyclic $\mathfrak{D}$-parallel. Therefore by virtue of Lemma 4.3, we can give a geometric meaning of $\mathfrak{D}$-parallel as follows:

LEMMA 4.4. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel shape operator, $m \geq 3$. Then every geodesic γ with initial conditions $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = X \in \mathfrak{D}$ has constant first curvature.*

REMARK 4.5. By the Codazzi equation (1.10), we know that any cyclic $\mathfrak{D}$-parallel hypersurface in $G_2(\mathbb{C}^{m+2})$ can not be $\mathfrak{D}$-parallel. Therefore, the converse of Lemma 4.4 does not hold.

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