MATH. SCAND. 117 (2015), 304–319

THE CONVERGENCE OF SOME PRODUCTS IN THE ADAMS SPECTRAL SEQUENCE

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Abstract

In this paper, we will use the family of homotopy elements $\zeta_n \in \pi_* S$, represented by $h_0 b_n \in \mathrm{Ext}_A^{3,p^{n+1}q+q}(\mathsf{Z}_p, \mathsf{Z}_p)$ in the Adams spectral sequence, to detect a ζ_n-related family $\gamma_{s+3}\beta_2\zeta_{n-1}$ in $\pi_* S$. Our main methods are the Adams spectral sequence and the May spectral sequence, here prime $p \geq 7$, $n > 3$, $q = 2(p-1)$.

1. Introduction

The problem of understanding the stable homotopy groups of sphere $\pi_* S$ has long been one of the important problem of algebraic topology. We are interested in the detection of nontrivial elements in the stable homotopy groups of sphere.

After the detection of $\eta_j \in \pi_{p^j q + pq - 2} S$, for $p = 2$, $j \neq 2$, by Mahowald, in [9], which is represented by $h_1 h_j \in \mathrm{Ext}_A^{2,p^j q + pq}(\mathsf{Z}_p, \mathsf{Z}_p)$, many infinite families in $\pi_* S$ were found. In this paper, $\mathsf{Z}_p = \mathsf{Z}/p\mathsf{Z}$. In [3], for $p > 2$, R. L. Cohen proved that a family $\zeta_n \in \pi_{p^n q + q - 3} S$ in the Adams spectral sequence (Adams SS) is represented by $h_0 b_n \in \mathrm{Ext}_A^{3,p^{n+1}q+q}(\mathsf{Z}_p, \mathsf{Z}_p)$. Zhou and Lee proved that $\beta_1 \xi_j$, $\beta_1 \zeta_n$ and $\beta_1^{p-1}\zeta_n$ are all nontrivial, see [5] and [15]. Furthermore, Lin proved in [4] that $b_0 h_n - h_1 b_{n-1}$ survives to E_∞ in the Adams SS. Liu also detected some new families of homotopy elements, see [7], [8]. Wang and Zhong established the convergence of $\widetilde{\beta}_s h_0 h_n$ under the condition of $p + 1 < s < 2p - 1$ and $n > 4$ ([14]).

In this paper, we show that the product with the R. L. Cohen's ζ-element is nontrivial. The main result is obtained as follows:

THEOREM 1.1. *Let $p \geq 7$, $0 \leq s < p - 4$, $n > 3$, then $\gamma_{s+3}\beta_2\zeta_{n-1} \neq 0$ in $\pi_* S$.*

For the convenience of the reader, let us briefly indicate the necessary preliminaries in the proof of the above theorem. Let S be the sphere spectrum, M be the Moore spectrum modulo an odd prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i_1} M \xrightarrow{j_1} \Sigma S.$$

Received 2 July 2013.

Let $\alpha : \Sigma^q M \to M$ be the Adams map and $V(1)$ is its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i_2} V(1) \xrightarrow{j_2} \Sigma^{q+1} M.$$

Let $\beta : \Sigma^{(p+1)q} V(1) \to V(1)$ be the v_2-mapping. It is well known that, in the Adams SS, the β-element $\beta_s = j_1 j_2 \beta^s i_2 i_1$ is a nontrivial element in $\pi_{spq+(s-1)q-2}S$, where $p \geq 5$ [12]. $V(2)$ is the cofibre of $\beta : \Sigma^{(p+1)q} V(1) \to V(1)$ sitting in the cofibration sequence

$$\Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_3} V(2) \xrightarrow{j_3} \Sigma^{(p+1)q+1} V(1).$$

Let $\gamma : \Sigma^{(p^2+p+1)q} V(2) \to V(2)$ be the v_3-mapping and the γ-element $\gamma_s = j_1 j_2 j_3 \gamma^s i_3 i_2 i_1$ is also a nontrivial element in $\pi_{sp^2q+(s-1)pq+(s-2)q-3}S$, where $p \geq 7$ [13].

Furthermore,

$$\beta_s \in \pi_{spq+(s-1)q-2}S,$$
$$\gamma_s \in \pi_{sp^2q+(s-1)pq+(s-2)q-3}S$$

is represented by the second, third Greek letter family element

$$\widetilde{\beta}_s \in \mathrm{Ext}_A^{s,spq+(s-1)q+s-2,*}(\mathsf{Z}_p, \mathsf{Z}_p),$$
$$\widetilde{\gamma}_s \in \mathrm{Ext}_A^{s,sp^2q+(s-1)pq+(s-2)q+s-3,*}(\mathsf{Z}_p, \mathsf{Z}_p)$$

in the Adams SS and $\widetilde{\beta}_s$, $\widetilde{\gamma}_s$ are represented by the elements $s(s-1)a_2^{s-2}h_{2,0}h_{1,1}$ and $s(s-1)(s-2)a_3^s h_{3,0}h_{2,1}h_{1,2}$ in the May spectral sequence (May SS).

Several methods have been found to determine $\pi_* S$. For example, we have the Adams SS based on the Eilenberg-Maclane spectrum $K\mathsf{Z}_p$,

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathsf{Z}_p, \mathsf{Z}_p), \qquad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1},$$

where A denotes the mod p Steenrod algebra, see [1]. Furthermore, we also have the Adams-Novikov spectral sequence (Adams-Novikov SS), see [10], [11].

From [6], $\mathrm{Ext}_A^{1,*}(\mathsf{Z}_p, \mathsf{Z}_p)$ has a Z_p-basis consisting of

$$a_0 \in \mathrm{Ext}_A^{1,1}(\mathsf{Z}_p, \mathsf{Z}_p), \qquad h_i \in \mathrm{Ext}_A^{1,p^i q}(\mathsf{Z}_p, \mathsf{Z}_p)$$

for all $i \geq 0$. $\mathrm{Ext}_A^{2,*}(\mathsf{Z}_p, \mathsf{Z}_p)$ has a Z_p-basis consisting of

$$\alpha_2,\ a_0^2,\ a_0 h_i\ (i > 0),\ g_i,\ k_i,\ b_i,\ h_i h_j\ (i \geq 0, j \geq i+2),$$

whose internal degrees are $2q+1$, 2, $p^i q + 1$, $q(p^{i+1} + 2p^i)$, $q(2p^{i+1} + p^i)$, $p^{i+1}q$ and $q(p^i + p^j)$ respectively. $\mathrm{Ext}_A^{3,*}(\mathsf{Z}_p, \mathsf{Z}_p)$ for $p > 2$ has been computed by Aikawa [2].

The Adams SS and May SS play very important roles in the proof of the main results, especially the May SS. In this paper, three problems must be resolved: Calculation of the E_2-terms $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$, computation of the Adams differentials, and the extensions from E_∞ to $\pi_* S$.

REMARK 1.1. Note that in the Adams SS, when $s \neq 0, 1, 2 \pmod p$, $\widetilde{\gamma}_s$, $\widetilde{\beta}_2$, $h_0 b_{n-1}$ are all permanent cycles, so $\widetilde{\gamma}_s \widetilde{\beta}_2 h_0 b_{n-1}$ is a permanent cycle, that is $d_r(\widetilde{\gamma}_s \widetilde{\beta}_2 h_0 b_{n-1}) = 0$ $(r \geq 2)$.

The paper is organized as follows: after giving some useful knowledge about the May SS in Section 2, we will make use of the May SS to prove some important results on Ext groups. The proof of Theorem 1.1 will be given in the last section.

2. The May spectral sequence

To compute $\pi_* S$ with the Adams SS, we must compute the E_2-term of the Adams SS, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$. The most successful method for computing it is the May SS.

From [11], there is a May SS $\{E_r^{s,t,*}, d_r\}$, which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ with E_1-term

$$(2.1)\quad E_1^{*,*,*} = E(h_{i,j} | i > 0, j \geq 0) \otimes P(b_{i,j} | i > 0, j \geq 0) \otimes P(a_i | i \geq 0),$$

where $E(\,)$ denotes the exterior algebra, $P(\,)$ denotes the polynomial algebra, and

$$h_{i,j} \in E_1^{1,2(p^i-1)p^j,2i-1}, \quad b_{i,j} \in E_1^{2,2(p^i-1)p^{j+1},p(2i-1)}, \quad a_i \in E_1^{1,2p^i-1,2i+1}.$$

One has $d_r : E_r^{s,t,M} \to E_r^{s+1,t,M-r}$, for $r \geq 1$, and if $x \in E_r^{s,t,*}$, $y \in E_r^{s',t',*}$, then

$$(2.2)\qquad d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x d_r(y).$$

Furthermore, the May E_1-term is graded commutative in the sense that:

$$\begin{cases} a_m h_{n,j} = h_{n,j} a_m, & h_{m,k} h_{n,j} = -h_{n,j} h_{m,k}, \\ a_m b_{n,j} = b_{n,j} a_m, & h_{m,k} b_{n,j} = b_{n,j} h_{m,k}, \\ a_m a_n = a_n a_m, & b_{m,n} b_{i,j} = b_{i,j} b_{m,n}. \end{cases}$$

The first May differential d_1 is given by

$$(2.3)\qquad \begin{cases} d_1(h_{i,j}) = -\sum\limits_{0<k<i} h_{i-k,k+j}h_{k,j} \\ d_1(a_i) = -\sum\limits_{0<k<i} h_{i-k,k}a_k \\ d_1(b_{i,j}) = 0. \end{cases}$$

For each element $x \in E_1^{s,t,*}$, if we denote $\dim x = s$, $\deg x = t$, then we have

$$(2.4)\qquad \begin{cases} \dim h_{i,j} = \dim a_i = 1, \dim b_{i,j} = 2 \\ \deg h_{i,j} = 2(p^i-1)p^j = q(p^{i+j-1}+\cdots+p^j), \\ \deg b_{i,j} = 2(p^i-1)p^{j+1} = q(p^{i+j}+\cdots+p^{j+1}) \\ \deg a_i = 2p^i - 1 = q(p^{i-1}+\cdots+1)+1, \\ \deg a_0 = 1, \end{cases}$$

where $i \geq 1$, $j \geq 0$.

REMARK 2.1. Any integer $t \geq 0$ can be expressed uniquely as

$$t = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e,$$

where $0 \leq c_i < p (0 \leq i < n)$, $p > c_n > 0, 0 \leq e < q$.

3. Some preliminaries on Ext groups

In this section, we will prove some results on Ext groups which will be used in the proof of the main Theorem 1.1.

LEMMA 3.1. *Let $p \geq 7$, $n > 3$, $0 \leq s < p-4$ and $r \geq 1$. The May E_1-term satisfies*

$$E_1^{s+8-r,t(s,n)+1-r,*} = \begin{cases} G_1, & r = 1 \text{ and } s = p-5, \\ G_2, & r = 1 \text{ and } s = p-6, \\ G_3, & r = 1 \text{ and } s = p-7, \\ 0, & \text{others.} \end{cases}$$

where $t(s,n) = q[p^n + (s+3)p^2 + (s+4)p + (s+3)] + s$.

(1) *G_1 is the Z_p-module generated by the following elements*

$$\begin{cases} g_1 = a_3^{p-5}h_{1,0}h_{2,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n}, \\ g_2 = a_3^{p-5}h_{1,0}h_{2,0}h_{3,0}h_{2,1}h_{1,2}h_{1,1}h_{1,n}, \\ g_3 = a_n^{p-5}h_{n,0}h_{n-1,1}h_{n-3,3}h_{n-k,k}h_{k,0}h_{2,0}h_{1,3} \qquad (4 \leq k < n-1). \end{cases}$$

(2) *G_2 is generated by two elements*

$$\begin{cases} g_4 = a_3^{p-6} h_{1,0} h_{2,0} h_{3,0} h_{2,1} b_{2,0} h_{1,n}, \\ g_5 = a_3^{p-6} h_{1,0} h_{2,0} h_{3,0} h_{2,1} h_{1,2} h_{1,1} h_{1,n}. \end{cases}$$

(3) *G_3 is generated by two elements*

$$\begin{cases} g_6 = a_3^{p-7} h_{1,0} h_{2,0} h_{3,0} h_{2,1} b_{2,0} h_{1,n}, \\ g_7 = a_3^{p-7} h_{1,0} h_{2,0} h_{3,0} h_{2,1} h_{1,2} h_{1,1} h_{1,n}. \end{cases}$$

PROOF. It is easy to show that $E_1^{s+8-r,t(s,n)+1-r,*} = 0$ $(r \geq s+2)$. Thus, in the rest of the proof, we assume that $1 \leq r < s+2$.

In the May SS, let $g = \omega_1 \omega_2 \ldots \omega_l \in E_1^{s+8-r,t(s,n)+1-r,*}$, where ω_i is one of a_k, $h_{r,j}$ or $b_{u,z}$, $0 \leq k, r+j \leq n+1$, $0 \leq u+z \leq n$, and $r, j, z \geq 0$, $u > 0$.

Assume that

$$\deg \omega_i = q(c_{i,n} p^n + c_{i,n-1} p^{n-1} + \cdots + c_{i,1} p + c_{i,0}) + e_i,$$

where $c_{i,j} = 0$ or 1, $e_i = 1$ if $\omega_i = a_{k_i}$, or $e_i = 0$. It follows that

$$\begin{aligned} \dim g &= \sum_{i=1}^{l} \dim \omega_i = s+8-r, \\ \deg g &= \sum_{i=1}^{l} \deg \omega_i \\ &= q\Big[\Big(\sum_{i=1}^{l} c_{i,n}\Big) p^n + \cdots + \Big(\sum_{i=1}^{l} c_{i,2}\Big) p^2 \\ &\qquad + \Big(\sum_{i=1}^{l} c_{i,1}\Big) p + \sum_{i=1}^{l} c_{i,0}\Big] + \sum_{i=1}^{l} e_i \\ &= q[p^n + (s+3)p^2 + (s+4)p + (s+3)] + (s+1-r). \end{aligned}$$

Note that $\dim h_{i,j} = \dim a_i = 1$, $\dim b_{i,j} = 2$, $1 \leq r < s+2$ and $0 \leq s < p-4$. From $\dim g = \sum_{i=1}^{l} \dim \omega_i = s+8-r$, we have $l \leq s+8-r < p+4-r \leq p+3$. Using $0 \leq s+4$, $s+1-r < p$, and the knowledge on

p-adic expression (Remark 2.1), we have

(3.1)
$$\begin{cases} \sum_{i=1}^{l} e_i = s+1-r; \\ \sum_{i=1}^{l} c_{i,0} = s+3; \\ \sum_{i=1}^{l} c_{i,1} = s+4; \\ \sum_{i=1}^{l} c_{i,2} = s+3; \\ \sum_{i=1}^{l} c_{i,3} = 0+\lambda_3 p, \lambda_3 \geq 0; \end{cases} \quad \begin{cases} \sum_{i=1}^{l} c_{i,4} + \lambda_3 = 0+\lambda_4 p, \lambda_4 \geq 0; \\ \sum_{i=1}^{l} c_{i,5} + \lambda_4 = 0+\lambda_5 p, \lambda_5 \geq 0; \\ \vdots \\ \sum_{i=1}^{l} c_{i,n-1} + \lambda_{n-2} = 0+\lambda_{n-1} p, \lambda_{n-1} \geq 0; \\ \sum_{i=1}^{l} c_{i,n} + \lambda_{n-1} = 1. \end{cases}$$

Consider the fifth equation of (3.1), $\sum_{i=1}^{l} c_{i,3} = 0 + \lambda_3 p$. Since $c_{i,3} = 0$ or 1 and $l \leq p+1$, we see that $\lambda_3 = 0$ or $\lambda_3 = 1$.

Case 1: $\lambda_3 = 0$. We claim that $\lambda_4 = 0$. If $\lambda_4 = 1$, we would have the following equations,

$$\sum_{i=1}^{l} c_{i,2} = s+3, \qquad \sum_{i=1}^{l} c_{i,3} = 0, \qquad \sum_{i=1}^{l} c_{i,4} = p.$$

By $\sum_{i=1}^{l} c_{i,2} = s+3$ and (2.4), there exist $s+3$ factors among g such that

$$\deg x_i = q(\text{higher terms on } p + p^2 + \text{lower terms on } p) + \delta_i,$$

where δ_i may equal 0 or 1. Similarly, according to $\sum_{i=1}^{l} c_{i,4} = p$, there would be p factors among g such that

$$\deg \omega_i = q(\text{higher terms on } p + p^4 + \text{lower terms on } p) + \delta_i.$$

Thus, by $l \leq p+1$ and by (2.4), there would be at least $p+3+s-(p+1) = s+1$ factors in g such that

$$\deg \omega_i = q(\text{higher terms on } p + p^4 + p^3 + \text{lower terms on } p) + \delta_i.$$

Thus we would have $\sum_{i=1}^{l} c_{i,3} \geq s+2$, which contradicts $\sum_{i=1}^{l} c_{i,3} = 0$. The claim that $\lambda_4 = 0$ is proved.

By induction on j, we have that $\lambda_j = 0$ $(4 \leq j \leq n-1)$. Hence, we have the following two cases.

Case 1.1: If there is a factor $h_{1,n}$ in g, up to sign $g = h_{1,n}\widetilde{g}$ with $\widetilde{g} \in E_1^{s+7-r,q[(s+3)p^2+(s+4)p+(s+3)]+(s+1-r),*}$. By (2.4), for $r = 1$, we have that

$$E_1^{s+6,q[(s+3)p^2+(s+4)p+(s+3)]+s} = \mathsf{Z}_p\{a_3^s h_{1,0}h_{2,0}h_{3,0}h_{2,1}b_{2,0},\ a_3^s h_{1,0}h_{2,0}h_{3,0}h_{2,1}h_{1,2}h_{1,1}\}.$$

When $r \geq 2$, we can make use of (2.4) to get

$$E_1^{s+7-r,q[(s+3)p^2+(s+4)p+(s+3)]+(s+1-r),*} = 0.$$

Case 1.2: If there is a factor $b_{1,n-1}$ in g, then up to sign $g = b_{1,n-1}\widetilde{g}$ with $\widetilde{g} \in E_1^{s+6-r,q[(s+3)p^2+(s+4)p+(s+3)]+(s+1-r),*} = 0$.

Thus, in this case, the generator g exists, and up to sign g can equal one of the following

$$a_3^s h_{1,0}h_{2,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n} \qquad \text{or} \qquad a_3^s h_{1,0}h_{2,0}h_{3,0}h_{2,1}h_{1,2}h_{1,1}h_{1,n}.$$

Case 2: $\lambda_3 = 1$. If $r \geq 4$, then we would have $l \leq s+8-r < p+4-r \leq p$. It is easy to see that λ_3 can not be equal to 1. Thus, in the rest of this case, we always assume $r \leq 3$.

By the sixth equation of (3.1), $\sum_{i=1}^{l} c_{i,4} + 1 = \lambda_4 p$ and as also $0 \leq \sum_{i=1}^{l} c_{i,4} \leq l < p+1$, we can deduce $\lambda_4 = 1$. By induction on j, $\lambda_j = 1 (4 \leq j \leq n-1)$, thus, the equations of (3.1) turn into

$$(3.2)\qquad \begin{cases} \sum_{i=1}^{l} e_i = s+1-r; \\ \sum_{i=1}^{l} c_{i,0} = s+3; \\ \sum_{i=1}^{l} c_{i,1} = s+4; \\ \sum_{i=1}^{l} c_{i,2} = s+3; \end{cases} \qquad \begin{cases} \sum_{i=1}^{l} c_{i,3} = p; \\ \sum_{i=1}^{l} c_{i,4} = p-1; \\ \vdots \\ \sum_{i=1}^{l} c_{i,n-1} = p-1; \\ \sum_{i=1}^{l} c_{i,n} = 0. \end{cases}$$

From the fifth equation of (3.2), $\sum_{i=1}^{l} c_{i,3} = p$, using $c_{i,3} = 0$ or 1, we must have that $l \geq p$. Note that $l \leq s+7$, thus $s \geq p-7$. By $0 \leq s < p-4$, s may equal $p-7$, $p-6$ or $p-5$.

Case 2.1: When $s = p-7$, $g = \omega_1\omega_2\ldots\omega_l \in E_1^{p+1-r,t(p-7,n)+1-r,*}$, in this case, $l = p$. From the following two equations: $\sum_{i=1}^{l} e_i = p-6-r$ and

$\sum_{i=1}^{l} c_{i,n-1} = p-1$, we have that up to sign the generator g must be of the form $g = a_n^{p-7-r} x_{p-6-r} \dots x_p$. In this case, r must equal 1, then, we have that up to sign $g = a_n^{p-8} x_{p-7} \dots x_p$, where

$$x_{p-7} \dots x_p \in E_1^{8,q(6p^{n-1}+6p^{n-2}+\dots+6p^4+8p^3+4p^2+5p+4)+1,*} = 0,$$

which is trivial by (2.4). Thus, the generator g doesn't exist.

Case 2.2: When $s = p-6$, $g = \omega_1\omega_2\dots\omega_l \in E_1^{p+2-r,t(p-6,n)+1-r,*}$.

Case 2.2.1: $l = p$. From the following two equations: $\sum_{i=1}^{l} e_i = p-5-r$ and $\sum_{i=1}^{l} c_{i,n-1} = p-1$, we have that up to sign the generator g must be of the form $g = a_n^{p-6-r} x_{p-5-r} \dots x_p$.

If $r = 1$, then we have that up to sign $g = a_n^{p-7} x_{p-6} \dots x_p$, and by (2.4),

$$x_{p-6} \dots x_p \in E_1^{8,q(6p^{n-1}+6p^{n-2}+\dots+6p^4+7p^3+4p^2+5p+4)+1,*} = 0.$$

If $r = 2$, then $g = a_n^{p-8} x_{p-7} \dots x_p$, by (2.4),

$$x_{p-7} \dots x_p \in E_1^{8,q(7p^{n-1}+7p^{n-2}+\dots+7p^4+8p^3+5p^2+6p+5)+1,*} = 0.$$

Case 2.2.2: $l = p+1$. In this case, it is easy to see that r must equal 1. From the following two equations: $\sum_{i=1}^{l} e_i = p-5-r$ and $\sum_{i=1}^{l} c_{i,n-1} = p-1$, we have that up to sign the generator g must be of the form $g = a_n^{p-7-r} x_{p-6-r} \dots x_{p+1}$. Then we have that up to sign $g = a_n^{p-8} x_{p-7} \dots x_{p+1}$, and by (2.4),

$$x_{p-7} \dots x_{p+1} \in E_1^{9,q(7p^{n-1}+7p^{n-2}+\dots+7p^4+8p^3+5p^2+6p+5)+2,*} = 0.$$

Thus, the generator g doesn't exist.

Case 2.3: When $s = p-5$, $g = \omega_1\omega_2\dots\omega_l \in E_1^{p+3-r,t(p-5,n)+1-r,*}$.

Case 2.3.1: $l = p$. From the following two equations: $\sum_{i=1}^{l} e_i = p-4-r$ and $\sum_{i=1}^{l} c_{i,n-1} = p-1$, we have that up to sign the generator g must be of the form $g = a_n^{p-5-r} x_{p-4-r} \dots x_p$.

If $r = 1$, we have that up to sign $g = a_n^{p-6} x_{p-5} \dots x_p$, and by (2.4),

$$x_{p-5} \dots x_p \in E_1^{8,q(5p^{n-1}+5p^{n-2}+\dots+5p^4+6p^3+4p^2+5p+4)+1,*} = \mathsf{Z}_p\{a_n h_{n,0} h_{n-1,1} h_{n-3,3} h_{n-k,k} h_{k,0} h_{2,0} h_{1,3}\}$$

$(4 \le k < n-1)$.

If $r = 2$, we have that up to sign $g = a_n^{p-7} x_{p-6} \dots x_p$, and by (2.4),

$$x_{p-6} \dots x_p \in E_1^{8,q(6p^{n-1}+6p^{n-2}+\cdots+6p^4+7p^3+5p^2+6p+5)+1,*} = 0.$$

If $r = 3$, we have that up to sign $g = a_n^{p-8} x_{p-7} \dots x_p$, and by (2.4),

$$x_{p-7} \dots x_p \in E_1^{8,q(7p^{n-1}+7p^{n-2}+\cdots+7p^4+8p^3+6p^2+7p+6)+1,*} = 0.$$

Case 2.3.2: $l = p + 1$. From the following two equations: $\sum_{i=1}^{l} e_i = p - 4 - r$ and $\sum_{i=1}^{l} c_{i,n-1} = p - 1$, we have that up to sign the generator g must be of the form $g = a_n^{p-6-r} x_{p-5-r} \dots x_{p+1}$.

If $r = 1$, we have that up to sign $g = a_n^{p-7} x_{p-6} \dots x_{p+1}$, and by (2.4),

$$\begin{aligned} x_{p-6} \dots x_{p+1} \in E_1^{9,q(6p^{n-1}+6p^{n-2}+\cdots+6p^4+7p^3+5p^2+6p+5)+2,*} \\ = \mathsf{Z}_p\{a_n^2 h_{n,0} h_{n-1,1} h_{n-3,3} h_{n-k,k} h_{k,0} h_{2,0} h_{1,3}\} \end{aligned}$$

$(4 \le k < n - 1)$.

If $r = 2$, we have that up to sign $g = a_n^{p-8} x_{p-7} \dots x_{p+1}$, and by (2.4),

$$x_{p-7} \dots x_{p+1} \in E_1^{9,q(7p^{n-1}+7p^{n-2}+\cdots+7p^4+8p^3+6p^2+7p+6)+2,*} = 0.$$

Case 2.3.3: $l = p + 2$. From the following two equations: $\sum_{i=1}^{l} e_i = p - 4 - r$ and $\sum_{i=1}^{l} c_{i,n-1} = p - 1$, we have that up to sign the generator g must be of the form $g = a_n^{p-7-r} x_{p-6-r} \dots x_{p+2}$. In this case, r must equal 1, and then we have that up to sign $g = a_n^{p-8} x_{p-7} \dots x_{p+2}$, and by (2.4),

$$x_{p-7} \dots x_{p+2} \in E_1^{10,q(7p^{n-1}+7p^{n-2}+\cdots+7p^4+8p^3+6p^2+7p+6)+3,*} = 0.$$

From the above discussion,

$$g = a_n^{p-5} h_{n,0} h_{n-1,1} h_{n-3,3} h_{n-k,k} h_{k,0} h_{2,0} h_{1,3} \qquad (4 \le k < n - 1).$$

Summing up Case 1 and Case 2, the Lemma follows.

LEMMA 3.2. (1) *For the generator of $E_1^{p+2,t(p-5,n),*}$, we have that*

$$\begin{cases} M(g_1) = 10p - 20, \\ M(g_2) = 7p - 18, \\ M(g_3) = (2n + 1)p - 2n - 14. \end{cases}$$

For the generator of $E_1^{p+1,t(p-6,n),*}$*, we have that*

$$\begin{cases} M(g_4) = 10p - 27, \\ M(g_5) = 7p - 25. \end{cases}$$

For the generator of $E_1^{p,t(p-7,n),*}$*, we have that*

$$\begin{cases} M(g_6) = 10p - 34, \\ M(g_7) = 7p - 31. \end{cases}$$

(2) *For the May* E_1*-module* G_1 *in Lemma 3.1, we have*

$$G_1 = E_1^{p+2,t(p-5,n),10p-20} \oplus E_1^{p+2,t(p-5,n),7p-18} \oplus E_1^{p+2,t(p-5,n),(2n+1)p-2n-14},$$

where

$$\begin{cases} E_1^{p+2,t(p-5,n),10p-20} = \mathsf{Z}_p\{g_1\}, \\ E_1^{p+2,t(p-5,n),7p-18} = \mathsf{Z}_p\{g_2\}, \\ E_1^{p+2,t(p-5,n),(2n+1)p-2n-14} = \mathsf{Z}_p\{g_3\}; \end{cases}$$

$G_2 = E_1^{p+1,t(p-6,n),10p-27} \oplus E_1^{p+1,t(p-6,n),7p-25}$*, where*

$$\begin{cases} E_1^{p+1,t(p-6,n),10p-27} = \mathsf{Z}_p\{g_4\}, \\ E_1^{p+1,t(p-6,n),7p-25} = \mathsf{Z}_p\{g_5\}; \end{cases}$$

and $G_3 = E_1^{p,t(p-7,n),10p-34} \oplus E_1^{p,t(p-7,n),7p-31}$*, where*

$$\begin{cases} E_1^{p,t(p-7,n),10p-34} = \mathsf{Z}_p\{g_6\}, \\ E_1^{p,t(p-7,n),7p-31} = \mathsf{Z}_p\{g_7\}. \end{cases}$$

PROOF. (1) It is an easy calculation.

(2) By Lemma 3.1 and the above result (1), it is an easy conclusion.

LEMMA 3.3. *For* $r \geq 2$*, about the May* E_r*-module, we have the following results:*

$$(1) \quad \begin{cases} E_r^{p+2,t(p-5,n),10p-20} = 0, \\ E_r^{p+1,t(p-6,n),10p-27} = 0, \\ E_r^{p+1,t(p-6,n),7p-25} = 0, \\ E_1^{p,t(p-7,n),10p-34} = 0, \\ E_r^{p,t(p-7,n),7p-31} = 0. \end{cases}$$

(2) $E_r^{p+2,t(p-5,n),(2n+1)p-2n-14} = 0$.

(3) $E_r^{p+2,t(p-5,n),7p-18}$ *has an unique generator* $a_3^{p-5}h_{1,0}h_{2,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n}$.

PROOF. (1) From Lemma 3.2 (2),

$$E_1^{p+2,t(p-5,n),10p-20} = \mathsf{Z}_p\{g_1\}.$$

By using of (2.2), we have that up to sign

$$d_1(g_1) = a_n^{p-5}h_{1,0}h_{2,0}h_{3,0}h_{1,2}h_{1,1}b_{2,0}h_{1,n} + \cdots \neq 0.$$

That is $E_2^{p+2,t(p-5,n),10p-20} = 0$. Thus,

$$E_r^{p+2,t(p-5,n),10p-20} = 0 \qquad (r \geq 2).$$

Similarly, we can get the other results in (1).

(2) From Lemma 3.2 (2),

$$E_1^{p+2,t(p-5,n),(2n+1)p-2n-14} = \mathsf{Z}_p\{g_3\}.$$

By using of (2.2), we have that up to sign

$$d_1(g_3) = a_n^{p-5}h_{n,0}h_{n-1,1}h_{n-3,3}h_{n-k,k}h_{1,0}h_{k-1,1}h_{2,0}h_{1,3} + \cdots \neq 0.$$

So $E_2^{p+2,t(p-5,n),(2n+1)p-2n-14} = 0$, and also

$$E_r^{p+2,t(p-5,n),(2n+1)p-2n-14} = 0 \qquad (r \geq 2).$$

(3) By Lemma 3.2 (2),

$$E_1^{p+2,t(p-5,n),7p-18} = \mathsf{Z}_p\{a_3^{p-5}h_{1,0}h_{2,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n}\}.$$

By using of (2.2),

$$d_1(a_3^{p-5}h_{1,0}h_{2,0}h_{3,0}h_{2,1}b_{2,0}h_{1,n}) = 0.$$

This shows that the May E_r-module $E_r^{p+2,t(p-5,n),7p-18}$ has only one permanent cycle for $r \geq 2$.

By using the above Lemmas, we will next prove some results on Ext groups, which will be used in the proof of the main theorem.

THEOREM 3.1. *Let* $p \geq 7, n > 3, 0 \leq s < p-4$. *There exists nontrivial product*

$$0 \neq \widetilde{\gamma}_{s+3}k_0h_0b_{n-1} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathsf{Z}_p, \mathsf{Z}_p),$$

where $t(s,n) = p^n q + (s+3)p^2 q + (s+4)pq + (s+3)q + s$.

PROOF. It is known that

$$h_{1,0},\ b_{1,j},\ h_{2,0}h_{1,1},\ a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_1^{*,*,*}$$

are all permanent cycle in the May SS converging nontrivially to

$$h_0,\ b_j,\ k_0,\ \widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{*,*}(\mathsf{Z}_p, \mathsf{Z}_p),\ j \geq 0.$$

So

$$h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_1^{s+8,t(s,n),*}$$

is a permanent cycle in the May SS that converges nontrivially to

$$\widetilde{\gamma}_{s+3}k_0h_0b_{n-1} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathsf{Z}_p, \mathsf{Z}_p).$$

Now we need to show that the permanent cycle

$$h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^s h_{3,0}h_{2,1}h_{1,2}$$

is not hit by any of the May differentials d_r $(r \geq 1)$. Firstly, let us consider the structure of $E_1^{s+7,t(s,n),*}$ in the May SS.

Case 1: When $0 \leq s < p-7$, by Lemma 3.1, we know that, in the May SS, $E_1^{s+7,t(s,n),*} = 0$. Then $E_r^{s+7,t(s,n),*} = 0$ $(r \geq 1)$. Thus in the May SS, the permanent cycle

$$h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^s h_{3,0}h_{2,1}h_{1,2}$$

doesn't bound and converges nontrivially to

$$\widetilde{\gamma}_{s+3}k_0h_0b_{n-1} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathsf{Z}_p, \mathsf{Z}_p),$$

then $\widetilde{\gamma}_{s+3}k_0h_0b_{n-1} \neq 0$.

Case 2: When $s = p-7$, by Lemma 3.1 and 3.2 (2), we have that $G_3 = E_1^{p,t(p-7,n),10p-34} \oplus E_1^{p,t(p-7,n),7p-31}$, thus

$$\begin{cases} M(E_1^{p,t(p-7,n),10p-34}) = 10p-34, \\ M(E_1^{p,t(p-7,n),7p-31}) = 7p-31. \end{cases}$$

Furthermore,

$$M(h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^{p-7}h_{3,0}h_{2,1}h_{1,2}) = 8p-35,$$

and because $d_1 : E_1^{s,t,M} \to E_1^{s+1,t,M-1}$, we know that

$$\begin{cases} h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^{p-7}h_{3,0}h_{2,1}h_{1,2} \notin d_1(E_1^{p,t(p-7,n),7p-31}), \\ h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^{p-7}h_{3,0}h_{2,1}h_{1,2} \notin d_1(E_1^{p,t(p-7,n),10p-34}). \end{cases}$$

Moreover, by Lemma 3.3, one has $E_r^{p,t(p-7,n),10p-34} = 0$ $(r \geq 2)$ and $E_r^{p,t(p-7,n),7p-31} = 0$ $(r \geq 2)$. Thus, from the above discussion, the permanent cycle

$$h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^{p-7}h_{3,0}h_{2,1}h_{1,2}$$

doesn't bound and converges nontrivially to

$$\widetilde{\gamma}_{p-4}k_0h_0b_{n-1} \in \mathrm{Ext}_A^{p+1,t(p-7,n)}(\mathsf{Z}_p, \mathsf{Z}_p)$$

in the May SS. Consequently, $\widetilde{\gamma}_{p-4}k_0h_0b_{n-1} \neq 0$.

Case 3: When $s = p - 6$, the proof is the same as Case 2.

Case 4: When $s = p - 5$, from Lemma 3.1 and 3.2 (2), we have that

$$G_1 = E_1^{p+2,t(p-5,n),10p-20} \oplus E_1^{p+2,t(p-5,n),7p-18} \oplus E_1^{p+2,t(p-5,n),(2n+1)p-2n-14}.$$

Thus,

$$\begin{cases} M(E_1^{p+2,t(p-5,n),10p-20}) = 10p - 20, \\ M(E_1^{p+2,t(p-5,n),7p-18}) = 7p - 18, \\ M(E_1^{p+2,t(p-5,n),(2n+1)p-2n-14}) = (2n+1)p - 2n - 14. \end{cases}$$

Furthermore, $M(h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^{p-5}h_{3,0}h_{2,1}h_{1,2}) = 8p - 21$ and $d_1 : E_1^{s,t,M} \to E_1^{s+1,t,M-1}$, we know that

$$\begin{cases} h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^{p-5}h_{3,0}h_{2,1}h_{1,2} \notin d_1(E_1^{p+2,t(p-5,n),10p-20}), \\ h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^{p-5}h_{3,0}h_{2,1}h_{1,2} \notin d_1(E_1^{p+2,t(p-5,n),7p-18}), \\ h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^{p-5}h_{3,0}h_{2,1}h_{1,2} \notin d_1(E_1^{p+2,t(p-5,n),(2n+1)p-2n-14}). \end{cases}$$

Moreover, by Lemma 3.3, when $r \geq 2$, one has

$$\begin{cases} E_r^{p+2,t(p-5,n),10p-20} = 0, \\ E_r^{p+2,t(p-5,n),7p-18} = 0, \\ E_r^{p+2,t(p-5,n),(2n+1)p-2n-14} = 0. \end{cases}$$

Thus, from the above discussion, the permanent cycle

$$h_{1,0}b_{1,n-1}h_{2,0}h_{1,1}a_3^{p-5}h_{3,0}h_{2,1}h_{1,2}$$

doesn't bound and converges nontrivially to

$$\widetilde{\gamma}_{p-2}k_0h_0b_{n-1} \in \mathrm{Ext}_A^{p+3,t(p-5,n)}(\mathsf{Z}_p, \mathsf{Z}_p)$$

in the May SS. This means that $\widetilde{\gamma}_{p-2}k_0h_0b_{n-1} \neq 0$.

From Case 1 to 4, the Theorem follows.

THEOREM 3.2. *Let $p \geq 7, n > 3, 0 \leq s < p-4, 2 \leq r \leq s+8$. Then*

$$\mathrm{Ext}_A^{s+8-r,t(s,n)+1-r}(\mathsf{Z}_p, \mathsf{Z}_p) = 0,$$

where $t(s,n) = p^nq + (s+3)p^2q + (s+4)pq + (s+3)q + s$.

PROOF. We only need to prove that $E_1^{s+8-r,t(s,n)+1-r,*} = 0$ in the May SS. From Lemma 3.1, the desired result follows.

4. The proof of Theorem 1.1

PROOF. From [3], $h_0b_{n-1} \in \mathrm{Ext}_A^{3,q(p^n+1)}(\mathsf{Z}_p, \mathsf{Z}_p)$ is a permanent cycle in the Adams SS and converges to a nontrivial element $\zeta_{n-1} \in \pi_{q(p^n+1)-3}S$.

Consider the composition of maps

$$f = (j_1j_2j_3\gamma^{s+3}i_3i_2i_1)(j_1j_2\beta^2i_2i_1)(\zeta_{n-1}).$$

Since ζ_{n-1} is represented by $h_0b_{n-1} \in \mathrm{Ext}_A^{3,q(p^n+1)}(\mathsf{Z}_p, \mathsf{Z}_p)$ in the Adams SS, then f is represented by

$$\varphi = (j_1j_2j_3\gamma^{s+3}i_3i_2i_1)_*(j_1j_2\beta^2i_2i_1)_*(h_0b_{n-1})$$

in the Adams SS. By using the Yoneda products, we know that the composition

$$\mathrm{Ext}_A^{0,0}(\mathsf{Z}_p, \mathsf{Z}_p) \xrightarrow{(i_2i_1)_*} \mathrm{Ext}_A^{0,0}(H^*V(1), \mathsf{Z}_p) \xrightarrow{(j_1j_2)_*(\beta^2)_*} \mathrm{Ext}_A^{2,2pq+q}(\mathsf{Z}_p, \mathsf{Z}_p)$$

is multiplication by $\widetilde{\beta}_2 \in \mathrm{Ext}_A^{2,2pq+q}(\mathsf{Z}_p, \mathsf{Z}_p)$, and the composition

$$\mathrm{Ext}_A^{0,0}(\mathsf{Z}_p, \mathsf{Z}_p) \xrightarrow{(i_3i_2i_1)_*} \mathrm{Ext}_A^{0,0}(H^*V(2), \mathsf{Z}_p) \xrightarrow{(j_1j_2j_3)_*(\gamma^{s+3})_*} \mathrm{Ext}_A^{s+3,q[(s+3)p^2+(s+2)p+(s+1)]+s}(\mathsf{Z}_p, \mathsf{Z}_p)$$

is also multiplication by $\widetilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,q[(s+3)p^2+(s+2)p+(s+1)]+s}(\mathsf{Z}_p, \mathsf{Z}_p)$.

Hence the composite map $\gamma_{s+3}\beta_2\zeta_{n-1} \in \pi_*S$ is represented by

$$\widetilde{\gamma}_{s+3}k_0h_0b_{n-1} \in \mathrm{Ext}_A^{s+8,t(s,n)}(\mathsf{Z}_p, \mathsf{Z}_p)$$

in the Adams SS.

From Theorem 3.1, we see that

$$\widetilde{\gamma}_{s+3}k_0h_0b_{n-1} \neq 0.$$

Moreover, from Theorem 3.2, it follows that $\widetilde{\gamma}_{s+3}k_0h_0b_{n-1}$ can not be hit by any differential in the Adams SS. Thus, the $\widetilde{\gamma}_{s+3}k_0h_0b_{n-1}$ survives nontrivially to a homotopy element of π_*S.

ACKNOWLEDGEMENT. This work was undertaken when the authors visited the institute of Mathematical Science at University of Copenhagen. We would like to express our deep thanks to the institute and Professor Jesper Møller for their hospitality. The first author was supported by the NSFC grant (Nos. 11301386, 11026197, 11226080), the Outstanding Youth Teacher Foundation of Tianjin(ZX110QN044) and the Doctor Foundation of Tianjin Normal University(52XB1011). The second author was supported by NSFC grant (No. 11001195) and Beiyang Elite Scholar Program of Tianjin University (No. 0903061016).

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