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UNIMODULAR EQUIVALENCE OF ORDER AND CHAIN POLYTOPES

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Abstract

Order polytope and chain polytope are two polytopes that arise naturally from a finite partially ordered set. These polytopes have been deeply studied from viewpoints of both combinatorics and commutative algebra. Even though these polytopes possess remarkable combinatorial and algebraic resemblance, they seem to be rarely unimodularly equivalent. In the present paper, we prove the following simple and elegant result: the order polytope and chain polytope for a poset are unimodularly equivallent if and only if that poset avoid the 5-element "X" shape subposet. We also explore a few equivalent statements of the main result.

Introduction

A finite poset P (partially ordered set) yield naturally two poset polytopes. One is the order polytope $\mathscr{O}(P)$ and the other is the chain polytope $\mathscr{C}(P)$. From viewpoints of both combinatorics and commutative algebra, both polytopes $\mathscr{O}(P)$ and $\mathscr{C}(P)$ have been studied by many authors. For example, the combinatorial structure of $\mathscr{O}(P)$ and $\mathscr{C}(P)$ is explicitly discussed in Stanley [4]. Stanley also showed that $\mathscr{O}(P)$ and $\mathscr{C}(P)$ have the same volume and Ehrhart polynomial. On the other hand, in [1] and [3], it is shown that the toric ring of each of $\mathscr{O}(P)$ and $\mathscr{C}(P)$ is an algebra with straightening laws ([2, p. 124]) on the distributive lattice $L = \mathscr{J}(P)$, where $\mathscr{J}(P)$ is the set of all poset ideals of P, ordered by inclusion. It then turns out that the behavior of $\mathscr{O}(P)$ and $\mathscr{C}(P)$ is remarkably resemblant. However, $\mathscr{O}(P)$ and $\mathscr{C}(P)$ seem to be rarely unimodularly equivalent. In general, in the study on integral convex polytopes, the notion of unimodular equivalence is of importance. For example, if two integral polytopes $\mathscr{P}$ and $\mathscr{Q}$ are unimodularly equivalence, then their toric ideals ([6]) coincide. In the present paper, we provide a simple and elegant answer to this question: $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent if and only if the poset P avoid the 5-element "X" shape subposet (Theorem 2.1). We also proved that for $\mathscr{O}(P)$ and $\mathscr{C}(P)$, unimodularly equivalent, combinatorially equivalent, and affinely equivalent are the same (Corollary 2.3).

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The outline of this paper is as follows. In section 1, we first recall fundamental materials on order polytopes and edge polytopes from [4]. We refer the reader to [4] for detailed information on these polytopes. A crucial fact in our discussion is that the number of facets of $\mathscr{O}(P)$ is less than or equal to that of $\mathscr{C}(P)$ (Corollary 1.2). Then in Theorem 1.3 we give a characterization of a poset P for which the number of facets of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$. Clarly Theorem 1.3 give a necessary condition for $\mathscr{O}(P)$ and $\mathscr{C}(P)$ to be unimodularly equivalent. Then, in Theorem 2.1 of Section 2, we show that the necessary condition of Theorem 1.3 is, in fact, sufficient for the unimodular equivalence of $\mathscr{O}(P)$ and $\mathscr{C}(P)$.

1. The number of facets of chain and order polytopes

Let $P = \{x_1, \ldots, x_d\}$ be a finite poset. To each subset $W \subset P$, we associate $\rho(W) = \sum_{i \in W} \mathbf{e}_i \in \mathsf{R}^d$, where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the unit coordinate vectors of R^d. In particular $\rho(\emptyset)$ is the origin of R^d. A *poset ideal* of P is a subset I of P such that, for all x_i and x_j with $x_i \in I$ and $x_j \leq x_i$, one has $x_j \in I$. An *antichain* of P is a subset A of P such that x_i and x_j belonging to A with $i \neq j$ are incomparable. We say that x_j *covers* x_i if $x_i < x_j$ and $x_i < x_k < x_j$ for no $x_k \in P$. A chain $x_{j_1} < x_{j_2} < \cdots < x_{j_\ell}$ of P is called *saturated* if x_{j_q} covers $x_{j_{q-1}}$ for $1 < q \leq \ell$. A *maximal chain* is a saturated chain such that x_{j_1} is a minimal element and x_{j_ℓ} is a maximal element of the poset.

Recall that the *order polytope* of P is the convex polytope $\mathscr{O}(P) \subset \mathsf{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathsf{R}^d$ such that $0 \leq a_i \leq 1$ for every $1 \leq i \leq d$ together with

$$a_i \geq a_j$$

if $x_i \leq x_j$ in P. The *chain polytope* of P is the convex polytope $\mathscr{C}(P) \subset \mathsf{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathsf{R}^d$ such that $a_i \geq 0$ for every $1 \leq i \leq d$ together with

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k} \leq 1$$

for every maximal chain $x_{i_1} < x_{i_2} < \cdots < x_{i_k}$ of P.

One has $\dim \mathscr{O}(P) = \dim \mathscr{C}(P) = d$. Each vertex of $\mathscr{O}(P)$ is $\rho(I)$ such that I is a poset ideal of P ([4, Corollary 1.3]) and each vertex of $\mathscr{C}(P)$ is $\rho(A)$ such that A is an antichain of P ([4, Theorem 2.2]). In particular the number of vertices of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$. Moreover, the volume of $\mathscr{O}(P)$ and that of $\mathscr{C}(P)$ are equal to $e(P)/d!$, where $e(P)$ is the number of linear extensions of P ([4, Corollary 4.2]).

It follows from [4] that the facets of $\mathscr{O}(P)$ are the following:

- $x_i \geq 0$, where $x_i \in P$ is minimal;
- $x_j \leq 1$, where $x_j \in P$ is maximal;

- $x_i \geq x_j$, where x_j covers x_i,

and that the facets of $\mathscr{C}(P)$ are the following:

- $x_i \geq 0$ for all $x_i \in P$;
- $x_{i_1} + x_{i_2} + \cdots + x_{i_k} \leq 1$, where $x_{i_1} < x_{i_2} < \cdots < x_{i_k}$ is a maximal chain of P.

Notice that in order to make the expression clear, we use x_i instead of a_i to express the coordinates. Also for the facets even though they are hyperplanes, we still use inequalities instead of equalities since this will also provide the information on which side of the hyperplanes we are using.

Let $m_*(P)$ (resp. $m^*(P)$) denote the number of minimal (reps. maximal) elements of P and $h(P)$ the number of edges of the Hasse diagram ([5, p. 243]) of P. In other words, $h(P)$ is the number of pairs $(x_i, x_j) \in P \times P$ such that x_j covers x_i. Let $c(P)$ denote the number of maximal chains of P. It then follows immediately that

LEMMA 1.1. *The number of facets of* $\mathscr{O}(P)$ *is* $m_*(P) + m^*(P) + h(P)$ *and that of* $\mathscr{C}(P)$ *is* $d + c(P)$.

COROLLARY 1.2. *The number of facets of* $\mathscr{O}(P)$ *is less than or equal to that of* $\mathscr{C}(P)$.

PROOF. We work with induction on d, the number of elements of P. If P has one element then the statement is true, since both polytopes have 2 facets. Choose a minimal element α of P which is not maximal. By induction hypothesis, we have

$$m_*(P \setminus \{\alpha\}) + m^*(P \setminus \{\alpha\}) + h(P \setminus \{\alpha\}) \leq (d-1) + c(P \setminus \{\alpha\}). \tag{1}$$

Let $\beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t$ be the elements of P which cover α such that each β_i covers at least two elements of P and each γ_j covers no element of P except for α. Let N_i denote the number of saturated chains of the form $\beta_i < x_{j_1} < x_{j_2} < \cdots$. Then

$$\begin{aligned} m_*(P \setminus \{\alpha\}) &= m_*(P) - 1 + t; \\ m^*(P \setminus \{\alpha\}) &= m^*(P); \\ h(P \setminus \{\alpha\}) &= h(P) - (s+t); \\ c(P \setminus \{\alpha\}) &= c(P) - \sum_{i=1}^{s} N_i. \end{aligned}$$

Hence

$$c(P \setminus \{\alpha\}) \leq c(P) - s. \tag{2}$$

One has

$$m_*(P \setminus \{\alpha\}) + m^*(P \setminus \{\alpha\}) + h(P \setminus \{\alpha\}) = m_*(P) + m^*(P) + h(P) - (s+1)$$

and

$$d - 1 + c(P \setminus \{\alpha\}) \leq d - 1 + c(P) - s = d + c(P) - (s+1).$$

Thus, by virtue of the inequality (1), it follows that

$$m_*(P) + m^*(P) + h(P) \leq d + c(P),$$

as desired.

We now come to a combinatorial characterization of P for which the number of facets of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$.

THEOREM 1.3. *The number of facets of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$ if and only if the following poset*

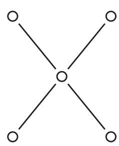

FIGURE 1

does not appear as a subposet ([5, p. 243]) *of P.*

PROOF. The number of facets of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$ if and only if, in the proof of Corollary 1.2, each of the inequalities (1) and (2) is an equality.

("If") Suppose that the poset of Figure 1 does not appear as a subposet of P. Then, in the proof of Corollary 1.2, one has $N_i = 1$ for $1 \leq i \leq s$. This is because we assume P avoids the 5-element "X" shape subposet, therefore, for each β as defined in Corollary 1.2, there exists unique saturated chain above β. Hence the inequality (2) is an equality. Moreover, the induction hypothesis guarantees that the inequalities (1) is an equality. Thus the number of facets of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$, as required.

("Only if") Suppose that the poset of Figure 1 appears as a subposet of P. It then follows from the 5-element "X" shape subposet that there exist $\delta, \xi, \mu, \phi, \psi$ of P such that (i) δ covers ξ and μ, (ii) $\delta < \phi$, $\delta < \psi$, and (iii) ϕ and ψ are incomparable. Now we want to use induction to show that in this case, the number of facets of $\mathscr{O}(P)$ is strictly smaller than $\mathscr{C}(P)$. First, if P is the poset shown in Figure 1, then the statement is true since $\mathscr{O}(P)$ has 8 facets and $\mathscr{C}(P)$ has 9. In the next step of our induction, notice that by the proof of

Corollary 1.2, in order to have the same number of facets for $\mathscr{O}(P)$ and $\mathscr{C}(P)$, we need both (1) and (2) to be equality. Therefore, to show $\mathscr{O}(P)$ has fewer facets than $\mathscr{C}(P)$, we only need to show one of (1) and (2) is a strict inequality.

- If neither ξ nor μ is a minimal element of P, then the poset of Figure 1 appears as a subposet of $P \setminus \{\alpha\}$, where α is any minimal element of P. Hence the induction hypothesis guarantees that the inequality (1) cannot be an equality.
- If either ξ or μ coincides with a minimal element α of P, then, in the proof of Corollary 1.2, one has $N_i > 1$ for some $1 \leq i \leq s$. Hence, the inequality (2) cannot be an equality.

Hence, at least one of the inequalities (1) and (2) cannot be an equality. Thus the number of facets of $\mathscr{O}(P)$ is less than that of $\mathscr{C}(P)$.

2. Unimodular equivalence

Let $\mathsf{Z}^{d\times d}$ denote the set of $d \times d$ integral matrices. Recall that a matrix $A \in \mathsf{Z}^{d\times d}$ is *unimodular* if $\det(A) = \pm 1$. Given integral polytopes $\mathscr{P} \subset \mathsf{R}^d$ of dimension d and $\mathscr{Q} \subset \mathsf{R}^d$ of dimension d, we say that $\mathscr{P}$ and $\mathscr{Q}$ are *unimodularly equivalent* if there exist a unimodular matrix $U \in \mathsf{Z}^{d\times d}$ and an integral vector $\mathbf{w} \in \mathsf{Z}^d$ such that $\mathscr{Q} = f_U(\mathscr{P}) + \mathbf{w}$, where f_U is the linear transformation of R^d defined by U, i.e., $f_U(\mathbf{v}) = \mathbf{v}\,U$ for all $\mathbf{v} \in \mathsf{R}^d$.

Now, we wish to solve our pending problem when $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent.

Theorem 2.1. *The order polytope $\mathscr{O}(P)$ and the chain polytope $\mathscr{C}(P)$ of a finite poset P are unimodularly equivalent if and only if the poset of Figure 1 of Theorem 1.3 does not appear as a subposet of P.*

Proof. ("Only if") If $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent, then in particular the number of facets of $\mathscr{O}(P)$ and that of $\mathscr{C}(P)$ coincides. Hence by virtue of Theorem 1.3 the poset of Figure 1 does not appear as a subposet of P.

("If") Let $P = \{x_1, \ldots, x_d\}$ and suppose that the poset of Figure 1 does not appear as a subposet of P. Fix $x \in P$ which is neither minimal nor maximal. Then at least one of the following conditions are satisfied:

- there is a unique saturated chain of the form $x = x_{i_0} > x_{i_1} > \cdots > x_{i_k}$, where x_{i_k} is a minimal element of P;
- there is a unique saturated chain of the form $x = x_{j_0} < x_{j_1} < \cdots < x_{j_\ell}$, where x_{i_ℓ} is a maximal element of P.

Now, identifying $x_1, \ldots, x_d$ with the coordinates of R^d, we introduce the affine map $\Psi : \mathsf{R}^d \to \mathsf{R}^d$ defined as follows:

- $\Psi(x_i) = 1 - x_i$ if $x_i \in P$ is minimal, but not maximal;
- $\Psi(x_i) = x_i$ if $x_i \in P$ is maximal;
- Let x_i be neither minimal nor maximal. If there is a unique saturated chain of the form $x = x_{i_0} > x_{i_1} > \cdots > x_{i_k}$, where x_{i_k} is a minimal element of P, then

$$\Psi(x_i) = 1 - x_{i_0} - x_{i_1} - \cdots - x_{i_k};$$

- Let x_i be neither minimal nor maximal. If there exist at least two saturated chains of the form $x_i = x_{i_0} > x_{i_1} > \cdots > x_{i_k}$, where x_{i_k} is a minimal element of P. Since the poset avoids "X" shape subposet, there is a unique saturated chain of the form $x_i = x_{j_0} < x_{j_1} < \cdots < x_{j_\ell}$, where x_{j_ℓ} is a maximal element of P, then

$$\Psi(x_i) = x_i + x_{j_1} + \cdots + x_{j_\ell}.$$

It is routine work to show that if $\mathscr{F}$ is a facet of $\mathscr{O}(P)$, then $\Psi(\mathscr{F})$ is a facet of $\mathscr{C}(P)$. We will prove this claim with the help of Example 2.2.

In fact, there are three types of facets for $\mathscr{O}(P)$:

(1) a minimal element $x \leq 1$;
(2) a maximal element $y \geq 0$;
(3) a cover relation $x \leq y$ if x covers y in P.

There are two types of facets for $\mathscr{C}(P)$:

(1') for each element in the poset $x \geq 0$;
(2') each maximal chain $\sum_{i \in C} x_i \leq 1$.

In Example 2.2, $x_1 \leq 1$ is mapped to $1 - x_1 \leq 1$, which is $x_1 \geq 0$. For type 3) facets $x \leq y$ of $\mathscr{O}(P)$, there are three cases. For any $x \in P$, if there is a unique saturated chain starting at x going down to a minimal element, we call x a *down element*, otherwise, if there exists at two such chains, we call x an *up element*. Then there are two cases for facets of the form $x \leq y$ of $\mathscr{O}(P)$.

(a) Both x and y are down elements, then this facet is sent to 1') facet of $\mathscr{C}(P)$: $x \geq 0$. In Example 2.2, $x_2 \leq x_1$ is mapped to $1 - x_1 - x_2 \leq 1 - x_1$, which is $x_2 \geq 0$.
(b) Both x and y are up elements, then this facet is sent to 1') facet of $\mathscr{C}(P)$: $y \geq 0$. In Example 2.2, $x_9 \leq x_7$ is mapped to $x_9 + x_{11} \leq x_7 + x_9 + x_{11}$, which is $x_7 \geq 0$.

(c) If x is up and y is down, then this facet is sent to a type 2') facet of $\mathscr{C}(P)$. In Example 2.2, $x_7 \leq x_2$ is mapped to $x_7 + x_9 + x_{11} \leq 1 - x_1 - x_2$, which is $x_1 + x_2 + x_7 + x_9 + x_{11} \leq 1$.

Hence $\Psi(\mathscr{O}(P)) = \mathscr{C}(P)$. Thus $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are affinely equivalent. Moreover, since $\Psi(\mathsf{Z}^n) = \mathsf{Z}^n$ and since the volume of $\mathscr{O}(P)$ coincides with that of $\mathscr{C}(P)$, it follows that $\mathscr{O}(P)$ and $\mathscr{C}(P)$ are unimodularly equivalent.

EXAMPLE 2.2. Consider the following poset.

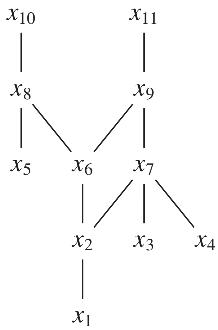

FIGURE 2

Both $\mathscr{O}(P)$ and $\mathscr{C}(P)$ have 17 facets. Facets of $\mathscr{O}(P)$ are

$$\begin{array}{llllll}
x_1 \leq 1, & x_3 \leq 1, & x_4 \leq 1, & x_5 \leq 1, & x_{10} \geq 0, & x_{11} \geq 0, \\
x_1 \geq x_2, & x_2 \geq x_7, & x_3 \geq x_7, & x_4 \geq x_7, & x_2 \geq x_6, & x_5 \geq x_8, \\
x_6 \geq x_8, & x_6 \geq x_9, & x_7 \geq x_9, & x_8 \geq x_{10}, & x_9 \geq x_{11}. &
\end{array}$$

Facets of $\mathscr{C}(P)$ are

$$\begin{array}{llllll}
x_1 \geq 0, & x_2 \geq 0, & x_3 \geq 0, & x_4 \geq 0, & x_5 \geq 0, & x_6 \geq 0, \\
x_7 \geq 0, & x_8 \geq 0, & x_9 \geq 0, & x_{10} \geq 0, & x_{11} \geq 0, &
\end{array}$$

$$\begin{array}{rr}
x_1 + x_2 + x_7 + x_9 + x_{11} \leq 1, & x_3 + x_7 + x_9 + x_{11} \leq 1, \\
x_4 + x_7 + x_9 + x_{11} \leq 1, & x_1 + x_2 + x_6 + x_8 + x_{10} \leq 1, \\
x_5 + x_8 + x_{10} \leq 1, & x_1 + x_2 + x_6 + x_9 + x_{11} \leq 1.
\end{array}$$

Here is the map Ψ defined in Theorem 2.1:

$$\begin{array}{lll}
x_1 \mapsto 1 - x_1, & x_2 \mapsto 1 - x_1 - x_2, & x_3 \mapsto 1 - x_3, \\
x_4 \mapsto 1 - x_4, & x_5 \mapsto 1 - x_5, & x_6 \mapsto 1 - x_6 - x_2 - x_1, \\
x_7 \mapsto x_7 + x_9 + x_{11}, & x_8 \mapsto x_8 + x_{10}, & x_9 \mapsto x_9 + x_{11}, \\
x_{10} \mapsto x_{10}, & x_{11} \mapsto x_{11}. &
\end{array}$$

COROLLARY 2.3. *Given a finite poset P, the following conditions are equivalent:*

(i) $\mathscr{O}(P)$ *and* $\mathscr{C}(P)$ *are unimodularly equivalent;*

(ii) $\mathscr{O}(P)$ *and* $\mathscr{C}(P)$ *are affinely equivalent;*

(iii) $\mathscr{O}(P)$ *and* $\mathscr{C}(P)$ *are combinatorially isomorphic;*

(iv) $\mathscr{O}(P)$ *and* $\mathscr{C}(P)$ *have the same* f*-vector* ([2, p. 12])*;*

(v) *The number of facets of* $\mathscr{O}(P)$ *is equal to that of* $\mathscr{C}(P)$*;*

(vi) *The poset of Figure 1 of Theorem 1.3 does not appear as a subposet of* P.

CONJECTURE 2.4. *Let P be a finite poset with $|P| = d > 1$. Let $f(\mathscr{O}(P)) = (f_0, f_1, \ldots, f_{d-1})$ denote the f-vector of $\mathscr{O}(P)$ and $f(\mathscr{C}(P)) = (f'_0, f'_1, \ldots, f'_{d-1})$ the f-vector of $\mathscr{C}(P)$. Then*

(a) $f_i \leq f'_i$ *for all* $1 \leq i \leq d-1$.

(b) *If* $f_i = f'_i$ *for some* $1 \leq i \leq d-1$*, then* $\mathscr{O}(P)$ *and* $\mathscr{C}(P)$ *are unimodularly equivalent.*

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