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THE SQUARE TERMS IN GENERALIZED LUCAS SEQUENCE WITH PARAMETERS P AND Q

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Abstract

Let P and Q be nonzero integers. Generalized Lucas sequence is defined as follows: $V_0 = 2$, $V_1 = P$, and $V_{n+1} = PV_n + QV_{n-1}$ for $n \geq 1$. We assume that P and Q are odd relatively prime integers. Firstly, we determine all indices n such that $V_n = kx^2$ and $V_n = 2kx^2$ when $k|P$. Then, as an application of our these results, we find all solutions of the equations $V_n = 3x^2$ and $V_n = 6x^2$. Moreover, we find integer solutions of some Diophantine equations.

1. Introduction

Let P and Q be nonzero integers. Generalized Fibonacci and Lucas sequences are defined as follows:

$$\begin{aligned} U_0(P, Q) &= 0,\\ U_1(P, Q) &= 1,\\ U_{n+1}(P, Q) &= PU_n(P, Q) + QU_{n-1}(P, Q), \end{aligned}$$

for $n \geq 1$, and

$$\begin{aligned} V_0(P, Q) &= 2,\\ V_1(P, Q) &= P,\\ V_{n+1}(P, Q) &= PV_n(P, Q) + QV_{n-1}(P, Q) \end{aligned}$$

for $n \geq 1$, respectively. $U_n(P, Q)$ and $V_n(P, Q)$ are called n'th generalized Fibonacci number and n'th generalized Lucas number, respectively. Since

$$U_n(-P, Q) = (-1)^{n-1}U_n(P, Q) \quad \text{and} \quad V_n(-P, Q) = (-1)^n V_n(P, Q),$$

it will be assumed that $P \geq 1$. Moreover, we will assume that $P^2 + 4Q > 0$. Instead of $U_n(P, Q)$ and $V_n(P, Q)$, we will use U_n and V_n, respectively.

The question of when, for which values of P and Q, U_n or V_n can be x^2 (or kx^2) has generated interest in the literature. Now we summarize briefly the relevant known facts. In [1], Cohn determined all indices n such that U_n or

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V_n is x^2 or $2x^2$ for $P = Q = 1$. The same author, in [2], [3], solved same problems when P is odd and $Q = \pm 1$. Moreover, in [6], Ribenboim and McDaniel showed that if P and Q are odd and relatively prime, and U_n or V_n is x^2 or $2x^2$, then $n \leq 12$. In [9], they solved the equation $V_n = kx^2$ for $P \equiv 1, 3 \pmod 8$, $Q \equiv 3 \pmod 4$, $(P, Q) = 1$ and all odd prime factors of k are congruent to 1 or 3 (mod 8) and under the assumption that the Jacobi symbol $\left(-\frac{V_{2^u}}{h}\right)$ is defined and equals 1 for each odd divisor h of k with $u \geq 1$.

More generally, we can recall the following theorem proved by Shorey and Stewart in [10]:

Let $k > 0$ be an integer, then there exists an effectively computable number $C > 0$, which depends on k, such that if $n > 0$ and $U_n = kx^2$ or $V_n = kx^2$, then $n < C$.

In this paper, we assume that P and Q are odd relatively prime integers. In this study, we determine all indices n such that $V_n = kx^2$ and $V_n = 2kx^2$ for all odd relatively prime integers P and Q under the assumption that $k|P$. After that, we solve the equations $V_n = 3x^2$ and $V_n = 6x^2$. Moreover, we find integer solutions of some Diophantine equations.

2. Preliminaries

We begin by listing the properties concerning generalized Fibonacci and Lucas numbers, which will be needed later.

$$V_{-n} = (-Q)^{-n} V_n, \tag{1}$$

$$V_{2n} = V_n^2 - 2(-Q)^n, \tag{2}$$

$$V_{3n} = V_n(V_n^2 - 3(-Q)^n), \tag{3}$$

$$\text{If } n \geq 0 \text{ is odd, then } (V_n, Q) = (V_{2n}, P) = 1, \tag{4}$$

$$2|V_n \iff 2|U_n \iff 3|n \tag{5}$$

for all natural number n.

$$\text{If } d = (m, n), \text{ then } (V_m, V_n) = \begin{cases} V_d & \text{if } m/d \text{ and } n/d \text{ are odd,} \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases} \tag{6}$$

$$\text{If } V_m \neq 1, \text{ then } V_m | V_n \iff m|n \text{ and } \frac{n}{m} \text{ is odd.} \tag{7}$$

$$\text{If } n \text{ is odd, then } V_n \equiv (-Q)^{\frac{n-1}{2}} P \pmod{P^2 + 4Q}. \tag{8}$$

All the above properties except for (8) are well known and can be found in [8]. The identity (8) is given in [4].

Now, we give some theorems and lemmas, which will be used in the proofs of the main theorems.

THEOREM 2.1 ([11], Corollaries 3.3 and 3.5). *Let* $n \in \mathbb{N} \cup \{0\}$ *and* $r \in \mathbb{Z}$. *Then*

$$V_{2mn+r} \equiv (-(-Q)^m)^n V_r \pmod{V_m} \tag{9}$$

for nonnegative integer m, and

$$V_{2mn+r} \equiv (-Q)^{mn} V_r \pmod{U_m} \tag{10}$$

for positive integer m such that $mn + r \geq 0$ *if* $Q \neq \pm 1$.

We can see that $8 | U_6$ and thus, using (10),

$$V_{12q+r} \equiv V_r \pmod{8} \tag{11}$$

for nonnegative integers q and r. It can be seen that if $Q \equiv 3, 7 \pmod{8}$, then

$$4 \nmid V_n \tag{12}$$

for every natural number n. When $Q \equiv 5 \pmod{8}$, it might be $8 | V_n$.

LEMMA 2.2 ([6], Lemma 3). *Let r be a positive integer. Then*

(i) $\left(\frac{2}{V_{2^r}}\right) = \begin{cases} -\left(\frac{-1}{Q}\right) & \text{if } r = 1, \\ 1 & \text{if } r \geq 2, \end{cases}$

(ii) $\left(\frac{-1}{V_{2^r}}\right) = -1,$

(iii) $\left(\frac{Q}{V_{2^r}}\right) = \left(\frac{-1}{Q}\right),$

(iv) If $r \geq 3$, then $\left(\frac{V_2}{V_{2^r}}\right) = \left(\frac{-1}{Q}\right),$

(v) $\left(\frac{P}{V_{2^r}}\right) = \begin{cases} \left(\frac{-2Q}{P}\right) & \text{if } r = 1, \\ \left(\frac{-2}{P}\right) & \text{if } r \geq 2, \end{cases}$

(vi) $\left(\frac{V_3}{V_{2^r}}\right) = \begin{cases} \left(\frac{-1}{Q}\right)\left(\frac{-2Q}{P}\right) & \text{if } r = 1, \\ \left(\frac{-2}{P}\right) & \text{if } r \geq 2, \end{cases}$

(vii) $\left(\frac{U_3}{V_{2^r}}\right) = \begin{cases} -\left(\frac{-1}{Q}\right) & \text{if } r = 1, \\ 1 & \text{if } r \geq 2, \end{cases}$

(viii) $\left(\frac{P^2 + 3Q}{V_{2^r}}\right) = \begin{cases} \left(\frac{-1}{Q}\right) & \text{if } r = 1, \\ 1 & \text{if } r \geq 2. \end{cases}$

If M is any divisor of P, then (v) implies that

$$\left(\frac{M}{V_{2^r}}\right) = \begin{cases} (-1)^{\left(\frac{M-1}{2}\right)}(-1)^{\left(\frac{M^2-1}{8}\right)}\left(\frac{Q}{M}\right) & \text{if } r = 1, \\ (-1)^{\left(\frac{M-1}{2}\right)}(-1)^{\left(\frac{M^2-1}{8}\right)} & \text{if } r \geq 2. \end{cases} \tag{13}$$

The following two lemmas can be proved by induction.

LEMMA 2.3. *If* $3 \nmid P$*, then*

$$V_{2^r} \equiv \begin{cases} 0 \pmod 3 & \text{if } r = 1 \text{ and } Q \equiv 1 \pmod 3, \\ 1 \pmod 3 & \text{if } r \geq 1,\ Q \equiv 0 \pmod 3 \text{ or } r = 2,\ Q \equiv 1 \pmod 3, \\ 2 \pmod 3 & \text{if } r = 2,\ Q \equiv 2 \pmod 3 \text{ or } r \geq 3,\ Q \equiv 1, 2 \pmod 3, \end{cases}$$

and if $3|P$*, then* $V_{2^r} \equiv 2 \pmod 3$ *for* $r \geq 2$*.*

LEMMA 2.4. *If* n *is an even positive integer, then* $V_n \equiv 2Q^{\frac{n}{2}} \pmod{P^2}$ *and if* n *is an odd positive integer, then* $V_n \equiv nPQ^{\frac{n-1}{2}} \pmod{P^2}$*.*

Lastly, we give the following two lemmas.

LEMMA 2.5. *Let* n *be a positive integer. If* $3|P$*, then* $3|V_n$ *iff* n *is odd. If* $3 \nmid P$*, then* $3|V_n$ *iff* $n \equiv 2 \pmod 4$ *and* $Q \equiv 1 \pmod 3$*.*

PROOF. If $3|P$, then, since $V_1 = P$, the properties (7) implies that $3|V_n$ iff n is odd. Assume that $3 \nmid P$. If $Q \equiv 0, 2 \pmod 3$, then it can be easily seen that $3 \nmid V_n$. If $Q \equiv 1 \pmod 3$, then, since $V_2 = P^2 + 2Q \equiv 0 \pmod 3$, the property (7) implies that $3|V_n$ iff $n \equiv 2 \pmod 4$. This completes the proof.

The following lemma can be proved by induction on r.

LEMMA 2.6. *Let* r *be a positive integer. Then*

$$V_{2^r} \equiv \begin{cases} Q^{2^{r-1}-1} V_2 \pmod A & \text{if } r \text{ is odd}, \\ -Q^{2^{r-1}-1}(P^2 + 3Q) \pmod A & \text{if } r \text{ is even}, \end{cases}$$

where $A = P^4 + 5P^2Q + 5Q^2$*.*

By Lemma 2.6, it can be shown that if $Q \equiv 3 \pmod 8$, then

$$\left(\frac{A}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{A}\right) = -1 \tag{14}$$

since $A = P^4 + 5P^2Q + 5Q^2 \equiv 5 \pmod 8$.

3. Main Theorems

In [12], Şiar and Keskin solved the equation $V_n = kx^2$ when $k|P$, P is odd, and $Q = 1$. Moreover, in [9], Ribenboim and McDaniel showed that for $n > 0$, the equation $V_n = kx^2$ has only the solutions $n = 1, 3$ under the assumptions mentioned in the introduction section. Now we improve to result of Ribenboim and McDaniel in [9].

From now on, we will assume that n and m are positive integers.

THEOREM 3.1. *Let $P = kM$ for some positive integers M and k with $k > 1$. If $V_n = kx^2$ for some integer x, then $n = 1$, $n = 3$ or $n = 5$.*

PROOF. Assume that $P = kM$ and $V_n = kx^2$. Then it is seen that n is odd by Lemma 2.4. Assume that $n > 3$. Then we can write $n = 4q + 1$ or $n = 4q + 3$ for some $q > 0$. From now on, we divide the proof into two cases.

Case 1: Let $\left(\frac{Q}{M}\right) = -1$. If $n = 4q + 1$, then

$$kx^2 = V_n = V_{4q+1} \equiv Q^{2q} P \pmod{P^2 + 4Q}$$

i.e.,

$$x^2 \equiv Q^{2q} M \pmod{P^2 + 4Q}$$

by (8) and this shows that $J = \left(\frac{M}{P^2+4Q}\right) = 1$. On the other hand, it is seen that $P^2 + 4Q \equiv 4Q \pmod{P}$ and therefore $P^2 + 4Q \equiv 4Q \pmod{M}$. Also it is clear that $P^2 + 4Q \equiv 5 \pmod{8}$. Hence since $\left(\frac{Q}{M}\right) = -1$, we get

$$1 = J = \left(\frac{M}{P^2 + 4Q}\right) = \left(\frac{P^2 + 4Q}{M}\right) = \left(\frac{4Q}{M}\right) = \left(\frac{Q}{M}\right) = -1,$$

which is impossible. If $n = 4q + 3$, then

$$kx^2 = V_n = V_{4q+3} \equiv -Q^{2q+1} P \pmod{P^2 + 4Q}$$

i.e.,

$$x^2 \equiv -Q^{2q+1} M \pmod{P^2 + 4Q}$$

by (8) and this shows that $J = \left(\frac{-QM}{P^2+4Q}\right) = 1$. Whereas, since $\left(\frac{Q}{P^2+4Q}\right) = \left(\frac{P^2+4Q}{Q}\right) = 1$, and $\left(\frac{M}{P^2+4Q}\right) = -1$, it follows that

$$1 = J = \left(\frac{-QM}{P^2 + 4Q}\right) = \left(\frac{-1}{P^2 + 4Q}\right)\left(\frac{Q}{P^2 + 4Q}\right)\left(\frac{M}{P^2 + 4Q}\right)$$
$$= (+1)(+1)(-1) = -1,$$

which is impossible.

Case 2: Let $\left(\frac{Q}{M}\right) = 1$. Firstly, assume that $Q \equiv 1, 5 \pmod{8}$. If we write $n = 4q + 1 = 2(2^r z) + 1$ for some odd integer z with $r \geq 1$, then

$$kx^2 = V_n \equiv -Q^{2^r z} P \pmod{V_{2^r}},$$

i.e.,

$$x^2 \equiv -Q^{2^r z} M \pmod{V_{2^r}}$$

by (9). This shows that $J=\left(\frac{-M}{V_{2^r}}\right)=1$. Assume that $M\equiv 1,3 \pmod 8$. Then

$$J=\left(\frac{-M}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)=(-1)(+1)=-1$$

by Lemma 2.2 and (13) since $\left(\frac{Q}{M}\right)=1$. This contradicts the fact that $J=1$. Assume that $M\equiv 5,7 \pmod 8$. If we write $n=4q+1=4(q+1)-3=2(2^r z)-3$ for some odd integer z with $r\geq 1$, then it can be similarly seen that

$$x^2\equiv Q^{2^r z-3}M(P^2+3Q) \pmod{V_{2^r}}$$

by (1) and (9). This shows that

$$J=\left(\frac{Q}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2+3Q}{V_{2^r}}\right)=1.$$

On the other hand, it is seen that

$$J=\left(\frac{Q}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2+3Q}{V_{2^r}}\right)=(+1)(-1)(+1)=-1$$

by Lemma 2.2 and (13) since $M\equiv 5,7 \pmod 8$ and $Q\equiv 1,5 \pmod 8$. This is a contradiction. If we write $n=4q+3=2(2^r z)+3$ for some odd integer z with $r\geq 1$, then

$$kx^2=V_n\equiv -Q^{2^r z}V_3 \pmod{V_{2^r}},$$

i.e.,

$$x^2\equiv -Q^{2^r z}M(P^2+3Q) \pmod{V_{2^r}}$$

by (9). This shows that

$$J=\left(\frac{-M(P^2+3Q)}{V_{2^r}}\right)=1.$$

Assume that $M\equiv 1,3 \pmod 8$. Then since $Q\equiv 1,5 \pmod 8$, it follows that

$$J=\left(\frac{-M(P^2+3)}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2+3Q}{V_{2^r}}\right)=-1$$

by Lemma 2.2. This contradicts the fact that $J=1$. Now assume that $M\equiv 5,7 \pmod 8$. If we write $n=4q+3=4(q+1)-1=2(2^r z)-1$ for some odd positive integer z with $r\geq 1$, then similar argument shows that

$$x^2\equiv Q^{2^r z-1}M \pmod{V_{2^r}}$$

by (1) and (9), and therefore $J = \left(\frac{Q}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = 1$. On the other hand, it is seen that $J = \left(\frac{Q}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -1$ by Lemma 2.2 and (13) since $M \equiv 5, 7 \pmod 8$ and $Q \equiv 1, 5 \pmod 8$. This contradicts the fact that $J = 1$.

Secondly, assume that $Q \equiv 3, 7 \pmod 8$. If $Q \equiv 7 \pmod 8$, then it can be seen that

$$kx^2 = V_n \equiv P, 6P \pmod 8,$$

i.e.,

$$x^2 \equiv M, 6M \pmod 8$$

by (11). This is impossible for $M \equiv 3, 5, 7 \pmod 8$. If $Q \equiv 3 \pmod 8$ and $n \not\equiv 5 \pmod 6$, then it can be seen that

$$kx^2 = V_n \equiv P, 2P \pmod 8,$$

i.e.,

$$x^2 \equiv M, 2M \pmod 8$$

by (11). This is also impossible for $M \equiv 3, 5, 7 \pmod 8$. Now assume that $M \equiv 1 \pmod 8$. If we write $n = 2(2^r z) \pm m$ for some odd positive integer z with $r \geq 2$ and $m = 1$ or 3, then

$$kx^2 = V_n \equiv \left(-Q^{2^r z} V_m\right) \text{ or } \left(Q^{2^r z - m} V_m\right) \pmod{V_{2^r}}$$

by (9) and (1). Writing the values of m in the last congruence, we get the Jacobi symbols

$$J_1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = 1,$$

$$J_2 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2 + 3Q}{V_{2^r}}\right) = 1,$$

$$J_3 = \left(\frac{Q}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = 1,$$

and

$$J_4 = \left(\frac{Q}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2 + 3Q}{V_{2^r}}\right) = 1.$$

Since $r \geq 2$ and $Q \equiv 3, 7 \pmod 8$, it follows that $J_1 = J_2 = J_3 = J_4 = -1$ for $M \equiv 1 \pmod 8$ by Lemma 2.2 and (13). This contradicts the fact that $J_1 = J_2 = J_3 = J_4 = 1$. Now let $Q \equiv 3 \pmod 8$ and $n = 6a + 5$ for some positive integer a. Then $n = 12t + 5$ or $n = 12t + 11$ for some positive integer t and thus

$$kx^2 = V_n \equiv 5P \pmod 8,$$

i.e.,

$$x^2 \equiv 5M \pmod 8$$

by (11). Moreover, it is obvious that x is odd by (12). Thus $M \equiv 5 \pmod 8$. Assume that $n = 12t + 5$. Then $n = 12t + 5 = 2(2^r z) + 5$ for some odd positive integer z with $r \geq 1$. Hence we get

$$kx^2 = V_n \equiv -Q^{2^r z} V_5 \pmod{V_{2^r}}$$

by (9) and from here, we get

$$x^2 \equiv -Q^{2^r z} MA \pmod{V_{2^r}},$$

where $A = P^4 + 5P^2Q + 5Q^2$. This shows that $J = \left(\frac{-MA}{V_{2^r}}\right) = 1$. On the other hand, by Lemma 2.2, (13) and (14), it follows that

$$1 = J = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{A}{V_{2^r}}\right) = \left(\frac{A}{V_{2^r}}\right) = -1,$$

which is impossible. Assume that $n = 12t + 11$. Thus we can write n as $n = 4c+3$ for some positive integer c. If c is odd, then $n = 4(c+1)-1 = 8b-1$ for some positive integer b. Hence

$$kx^2 = V_n \equiv -Q^{4b-1} P \pmod{V_2},$$

i.e.,

$$x^2 \equiv -Q^{4b-1} M \pmod{V_2}$$

by (9) and (1). By using Lemma 2.2 and (13), it can be seen that

$$1 = J = \left(\frac{-QM}{V_2}\right) = \left(\frac{-1}{V_2}\right)\left(\frac{Q}{V_2}\right)\left(\frac{M}{V_2}\right) = -1,$$

which is impossible. Assume that c is even. Then $c = 2^r z$ for some odd positive integer z with $r \geq 1$ and so $n = 4c + 3 = 2(2^{r+1} z) + 3$. If $r \geq 2$, then we get

$$kx^2 = V_n \equiv Q^{2^{r+1} z} V_3 \pmod{V_{2^r}},$$

i.e.,

$$x^2 \equiv Q^{2^{r+1} z} M(P^2 + 3Q) \pmod{V_{2^r}}$$

by (9). By using Lemma 2.2, it can be seen that

$$1 = J = \left(\frac{M(P^2 + 3Q)}{V_{2^r}}\right) = \left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2 + 3Q}{V_{2^r}}\right) = -1,$$

which is impossible. Now assume that $r = 1$. Then we can write $n = 8z+3 = 8(z+1) - 5 = 2(2^s t) - 5$ for some odd positive integer t with $s \geq 3$. Thus

$$kx^2 = V_n \equiv Q^{2^s t-5} V_5 \pmod{V_{2^s}},$$

which implies that

$$x^2 \equiv Q^{2^s t-5} MA \pmod{V_{2^s}}$$

by (9) and (1), where $A = P^4 + 5P^2Q + 5Q^2$. By using Lemma 2.2, (13) and (14), we get

$$1 = J = \left(\frac{Q^{2^s t-5} MA}{V_{2^s}}\right) = \left(\frac{Q}{V_{2^s}}\right)\left(\frac{M}{V_{2^s}}\right)\left(\frac{A}{V_{2^s}}\right) = -1$$

which is impossible. Therefore $a = 0$, i.e., $n = 5$. Then $kx^2 = V_5 = P(P^4 + 5P^2Q + 5Q^2)$ or $(P/k)(P^4 + 5P^2Q + 5Q^2) = x^2$. It can be seen that $\big((P/k),\, P^4 + 5P^2Q + 5Q^2\big) = 1$ or 5. This implies that either $P = ku^2$ and $P^4 + 5P^2Q + 5Q^2 = v^2$ or $P = 5ku^2$ and $P^4 + 5P^2Q + 5Q^2 = 5v^2$ for some integers u and v. Since $P^4 + 5P^2Q + 5Q^2 \equiv 6 + 5Q \pmod 8$, either $Q \equiv 7 \pmod 8$ or $Q \equiv 3 \pmod 8$. If $Q \equiv 7 \pmod 8$, then, by Lemma 2.2,

$$1 = \left(\frac{P^4 + 5P^2Q + 5Q^2}{V_2}\right) = \left(\frac{-Q^2}{V_2}\right) = -1,$$

which is impossible. If $P = 5ku^2$, $P^4 + 5P^2Q + 5Q^2 = 5v^2$ and $Q \equiv 3 \pmod 8$, it has solution for some values of P and Q. For example, $(P, Q) = (15, 2419)$ is a solution. This completes the proof.

In the above theorem, when $k = 1$, Ribenboim and McDaniel showed in [6] that the equation $V_n = x^2$ has solution only for $n = 1, 3, 5$.

THEOREM 3.2. *Let $k > 1$ and $k|P$. If $V_n = 2kx^2$ for some integer x, then $n = 3$.*

PROOF. Assume that $k|P$ and $V_n = 2kx^2$. Since $k|P$ and $2|V_n$, it is seen that n is odd by Lemma 2.4 and $3|n$ by (5), respectively. Thus $n = 3m$ for some odd positive integer m and therefore

$$V_n = V_{3m} = V_m(V_m^2 + 3Q^m) = 2kx^2$$

by (3). This shows that

$$(V_m/k)(V_m^2 + 3Q^m) = 2x^2.$$

It can be easily seen that $(V_m/k,\, V_m^2 + 3Q^m) = 1$ or 3 by (4). In both cases, we have $V_m^2 + 3Q^m = wu^2$ for some integer u with $w \in \{1, 2, 3, 6\}$. Thus, since

$V_{2m} = V_m^2 + 2Q^m$ by (2), we obtain $V_{2m} + Q^m = wu^2$ with $w \in \{1, 2, 3, 6\}$. Now assume that $m > 1$. Then we can write $2m = 2(2^r z \pm 1) = 2(2^r z) \pm 2$ for some odd positive integer z with $r \geq 2$. Hence,

$$\begin{aligned} wu^2 &= V_{2m} + Q^m \\ &\equiv \left(-Q^{2^r z} V_2 + Q^{2^r z+1}\right) \text{ or } \left(-Q^{2^r z-2} V_2 + Q^{2^r z-1}\right) \pmod{V_{2^r}} \end{aligned}$$

by (9). This shows that

$$wu^2 \equiv \left(-Q^{2^r z} U_3\right) \text{ or } \left(-Q^{2^r z-2} U_3\right) \pmod{V_{2^r}}.$$

Consequently, we have the Jacobi symbol $J = \left(\frac{-wU_3}{V_{2^r}}\right) = 1$. On the other hand, we know that $\left(\frac{-1}{V_{2^r}}\right) = -1$, $\left(\frac{2}{V_{2^r}}\right) = 1$, and $\left(\frac{U_3}{V_{2^r}}\right) = 1$ by Lemma 2.2 since $r \geq 2$. Besides, when $w = 3$ or 6, since $V_m^2 + 3Q^m = wu^2$ and m is odd, it follows that $3|V_m$ and therefore $3|P$ by Lemma 2.5. Thus

$$\left(\frac{3}{V_{2^r}}\right) = -\left(\frac{V_{2^r}}{3}\right) = -\left(\frac{2}{3}\right) = 1$$

by Lemma 2.3 and so,

$$\left(\frac{6}{V_{2^r}}\right) = \left(\frac{2}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right) = 1.$$

These show that

$$J = \left(\frac{-wU_3}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{w}{V_{2^r}}\right)\left(\frac{U_3}{V_{2^r}}\right) = -1$$

for $w \in \{1, 2, 3, 6\}$. This contradicts the fact that $J = 1$. Then $m = 1$, and therefore $n = 3$. Thus, from the equation $V_n = 2kx^2$, we obtain $(P/k)(P^2 + 3Q)/2 = x^2$, and this equation has solution for some values of P and Q. This completes the proof.

Now, we can give the following two corollaries.

Corollary 3.3. *If $V_n = 3x^2$ for some integer x, then $n = 1$, $n = 2$, $n = 3$ or $n = 5$. $V_1 = 3x^2$ iff $P = 3a^2$; $V_2 = 3x^2$ iff $P^2 + 2Q = 3a^2$; $V_3 = 3x^2$ iff $P = a^2$ and $P^2 + 3Q = 3b^2$; $V_5 = 3x^2$ iff $P = 15a^2$ and $P^4 + 5P^2Q + 5Q^2 = 5b^2$ for some integers a and b.*

Proof. Assume that $3 \nmid P$. Since $3|V_n$, it follows that $n \equiv 2 \pmod 4$ and also $Q \equiv 1 \pmod 3$ by Lemma 2.5. Firstly, let $Q \equiv 1, 5 \pmod 8$. If $n = 2$, then $V_n = V_2 = P^2 + 2Q = 3x^2$. This equation has solution for some values of P and Q. If $n = 6$, then $3x^2 = V_6 = V_3^2 + 2Q^3$ by (2). This implies that since V_3 is even and $Q \equiv 1, 5 \pmod 8$,

$$3x^2 = V_3^2 + 2Q^3 \equiv 2 \pmod 4,$$

which is impossible. Then we can write $n = 16c \pm 2$ or $n = 16c \pm 6$ for some positive integer c. Assume that $n = 16c \pm 6$. Then

$$3x^2 = V_n = V_{16c\pm 6} \equiv \left(Q^{8c} V_6\right) \text{ or } \left(Q^{8c-6} V_6\right) \pmod{V_4}.$$

by (9). Moreover, it can be easily shown that $V_6 \equiv -Q^2 V_2 \pmod{V_4}$. Hence we get

$$3x^2 \equiv \left(-Q^{8c+2} V_2\right) \text{ or } \left(-Q^{8c-4} V_2\right) \pmod{V_4}.$$

In both cases, it follows that $J = \left(\frac{-3V_2}{V_4}\right) = 1$. On the other hand, since $Q \equiv 1 \pmod 3$, it is seen that $V_4 \equiv 1 \pmod 3$ by Lemma 2.3. Then

$$\left(\frac{3}{V_4}\right) = \left(\frac{V_4}{3}\right)(-1)^{\frac{V_4-1}{2}} = -1$$

since $\left(\frac{-1}{V_4}\right) = -1$ by Lemma 2.2. Also $V_4 \equiv -2Q^2 \pmod{V_2}$ by (2) and thus since $Q \equiv 1, 5 \pmod 8$, we get

$$\begin{aligned}\left(\frac{V_2}{V_4}\right) &= \left(\frac{V_4}{V_2}\right)(-1)^{\left(\frac{V_4-1}{2}\right)\left(\frac{V_2-1}{2}\right)} = \left(\frac{-2Q^2}{V_2}\right)(-1)\\ &= \left(\frac{-1}{V_2}\right)\left(\frac{2}{V_2}\right)(-1) = -1\end{aligned}$$

by Lemma 2.2. These imply that

$$J = \left(\frac{-3V_2}{V_4}\right) = \left(\frac{-1}{V_4}\right)\left(\frac{3}{V_4}\right)\left(\frac{V_2}{V_4}\right) = (-1)(-1)(-1) = -1.$$

This contradicts the fact that $J = 1$. Assume that $n = 16c \pm 2$. If we write n as $n = 2(2^r z) \pm 2$ for some odd z with $r \geq 3$, then it is seen that

$$3x^2 = V_n \equiv \left(-Q^{2^r z} V_2\right) \text{ or } \left(-Q^{2^r z-2} V_2\right) \pmod{V_{2^r}}$$

by (9) and (1). In both cases, it follows that $J = \left(\frac{-3V_2}{V_{2^r}}\right) = 1$. On the other hand,

$$\left(\frac{V_2}{V_{2^r}}\right) = \left(\frac{-1}{Q}\right) = 1$$

by Lemma 2.2 since $Q \equiv 1, 5 \pmod 8$. Moreover, $V_{2^r} \equiv 2 \pmod 3$ by Lemma 2.3 since $Q \equiv 1 \pmod 3$. Then

$$\left(\frac{3}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{3}\right)(-1)^{\left(\frac{V_{2^r}-1}{2}\right)\left(\frac{3-1}{2}\right)} = \left(\frac{2}{3}\right)(-1) = 1.$$

Hence we get

$$J = \left(\frac{-3V_2}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{V_2}{V_{2^r}}\right) = -1,$$

which is a contradiction. Now let $Q \equiv 3, 7 \pmod 8$. Then it is seen that

$$3x^2 = V_n \equiv 2, 7 \pmod 8$$

by (11) since $n \equiv 2 \pmod 4$. This shows that

$$x^2 \equiv 5, 6 \pmod 8,$$

which is impossible.

Now assume that $3|P$. Then $n = 1$, $n = 3$ or $n = 5$ by Theorem 3.1. If $n = 1$, then $V_1 = P = 3x^2$. It is obvious that this is a solution. If $n = 3$, then it follows that $V_3 = P(P^2 + 3Q) = 3x^2$. This equation has solution for some values of P and Q. If $n = 5$, then it follows that $V_5 = P(P^4+5P^2Q+5Q^2) = 3x^2$. This equation has solution for some values of P and Q. For example, $(P, Q) = (15, 2419)$ is a solution. This completes the proof.

COROLLARY 3.4. *If $V_n = 6x^2$ for some integer x, then $n = 3$. $V_3 = 6x^2$ iff $P = a^2$ and $P^2 + 3Q = 6b^2$ for some integers a and b.*

PROOF. Assume that $V_n = 6x^2$. If $3|P$, then, since $V_n = 2(3x^2)$, it follows that $n = 3$ by Theorem 3.2 and therefore $V_3 = P(P^2 + 3Q) = 6x^2$. This shows that $P(P^2 + 3Q)/6 = x^2$ since $3|P$ and $P^2 + 3Q$ is even. It is obvious that $(P, (P^2 + 3Q)/6) = 1$. Thus, we obtain $P = a^2$ and $P^2 + 3Q = 6b^2$ for some integers a and b. Now let $3 \nmid P$. Then, since $3|V_n$ and $2|V_n$, it follows that $n \equiv 2 \pmod 4$ and also $Q \equiv 1 \pmod 3$ by Lemma 2.5 and $3|n$ by (5), respectively. This implies that $n = 12q + 6$ for some integer $q \geq 0$. Thus

$$6x^2 = V_n = V_{12q+6} \equiv 2 \pmod 8$$

by (11) and from here, it follows that

$$3x^2 \equiv 1 \pmod 4,$$

which is impossible. This completes the proof.

The following corollary can be seen from Corollary 3.3 and also can be found in [12].

COROLLARY 3.5. *Let $Q = 1$. If $V_n = 3x^2$ for some integer x, then $n = 1$ or $n = 2$.*

COROLLARY 3.6. *Let $Q = -1$. If $V_n = 3x^2$ for some integer x, then $n = 1$.*

PROOF. Assume that $V_n = 3x^2$. Then it is seen that $n = 1$ or $n = 2$ by Corollary 3.3 since $Q = -1$. If $n = 2$, then $V_2 = P^2 - 2 = 3x^2$. This implies that $P^2 \equiv 2 \pmod 3$, which is impossible. This completes the proof.

COROLLARY 3.7. *Let $Q = \pm 1$. Then there is no integer x such that $V_n = 6x^2$.*

PROOF. Assume that $V_n = 6x^2$. If $Q = 1$, then the proof can be found in [12]. If $Q = -1$, then $6x^2 = V_6 = V_3^2 - 2$ by (2). This shows that $V_3^2 \equiv 2 \pmod 3$, which is impossible.

Now we give solutions of some Diophantine equations using the above corollaries.

COROLLARY 3.8. *Let P be odd integer. Then the equation $9x^4 - (P^2+4)y^2 = \pm 4$ has positive integer solutions only when $P = 3a^2$ or $P = U_{m+1}(4, -1) + U_m(4, -1)$ with $m \geq 0$.*

PROOF. Assume that $9x^4 - (P^2 + 4)y^2 = \pm 4$ for some positive integers x and y. Then by Corollary 1 in [5], we get $(3x^2, y) = (V_n(P, 1), U_n(P, 1))$ for some $n \geq 1$. Thus $V_n = 3x^2$ and therefore $n = 1$ or $n = 2$ by Corollary 3.5. If $n = 1$, then $V_1 = P = 3x^2$ and $y = U_1 = 1$. If $n = 2$, then $V_2 = P^2 + 2 = 3x^2$. That is, $P^2 - 3x^2 = -2$. It can be shown that all positive integer solutions of the equation $u^2 - 3v^2 = -2$ are given by

$$(u, v) = \big(U_{m+1}(4, -1) + U_m(4, -1), U_{m+1}(4, -1) - U_m(4, -1)\big)$$

with $m \geq 0$. Therefore we get $P = U_{m+1}(4, -1) + U_m(4, -1)$ for some $m \geq 0$. This completes the proof.

Using Corollaries 1, 2, and 3 in [5], it is easy to get the following corollaries.

COROLLARY 3.9. *Let $P \geq 3$ be odd. Then the equation $9x^4 - (P^2-4)y^2 = 4$ has integer solution only when $P = 3a^2$.*

COROLLARY 3.10. *Let P be odd. The equation $36x^4 - (P^2 + 4)y^2 = \pm 4$ or $36x^4 - (P^2 - 4)y^2 = 4$ has no integer solutions.*

COROLLARY 3.11. *Let P be odd and $P^2 + 4$ a square free integer. Then the equation $9x^4 - 3Px^2y - y^2 = \pm(P^2 + 4)$ has integer solution only when $P = 3a^2$ or $P = U_{m+1}(4, -1) + U_m(4, -1)$ with $m \geq 0$.*

COROLLARY 3.12. *Let* $P \geq 3$ *be odd and* $P^2 - 4$ *a square free integer. Then the equation* $9x^4 - 3Px^2y + y^2 = -(P^2 - 4)$ *has integer solution only when* $P = 3a^2$.

COROLLARY 3.13. *Let* P *be odd and* $P^2 + 4$ *square free. Then the equation* $36x^4 - 6Px^2y - y^2 = \pm(P^2 + 4)$ *has no solutions. If* $P \geq 3$ *and* $P^2 - 4$ *is square free, then the equation* $36x^4 - 6Px^2y + y^2 = -(P^2 - 4)$ *has no integer solutions.*

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