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TRIBONACCI NUMBERS CLOSE TO THE SUM $2^a + 3^b + 5^c$

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Abstract

We show that there are exactly 22 solutions to the inequalities

$$0 \le T_n - 2^a - 3^b - 5^c \le 10,$$

where T_n denotes the n^{th} term ($n \ge 0$) of the Tribonacci sequence, and $0 \le a, b \le c$ are integers. All the solutions are explicitly determined.

1. Introduction

Let $k \ge 2$ be a positive integer. The so-called k-generalized Fibonacci sequence $\{F_n^{(k)}\}_{k=0}^{\infty}$ is defined by the initial values $F_0^{(k)} = \cdots = F_{k-2}^{(k)} = 0$, $F_{k-1}^{(k)} = 1$, and the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \qquad (n \ge k).$$

Naturally, $k = 2$ gives the Fibonacci numbers, $k = 3$ the Tribonacci sequence, etc. Since this paper analyzes a question linked to Tribonacci numbers therefore we recall their original notation: $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$, further

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \qquad (n \ge 3).$$

The problem of determining different type of numbers among the terms of a given linear recurrence has a long history and an extensive literature. It might have been started with the square Fibonacci numbers, for which one of the deepest results, due to [3], says that the only non-trivial full powers among them are $F_5 = 8$ and $F_{12} = 144$.

Exponential terms also occur in such diophantine equations. For instance, the authors in [6] showed that the equation $F_n = p^a \pm p^b + 1$ admits only finitely many, effectively computable, positive integer solutions (n, p, a, b), where p is a prime number, $n \ge 2$. Bravo and Luca [1] proved that the only non-trivial solution of the Diophantine equation $F_n^{(k)} = 2^m$ in positive integers

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n, k, m with $k \geq 2$ is $(n, k, m) = (6, 2, 3)$. Recently, the paper of Marques and Togbé [7] determines the Fibonacci and Lucas numbers of the form $2^a + 3^b + 5^c$.

The present paper targets a similar question for the Tribonacci sequence. More precisely, we solve the diophantine equation

$$T_n = 2^a + 3^b + 5^c + \delta$$

in the integers n, a, b, c and $0 \leq \delta \leq 10$ with $0 \leq a, b \leq c$. Basically, the method of Marques and Togbé is applicable, but two more difficulties appear. First, in the usage of Baker's method, instead of two terms, we work with three algebraic numbers. Therefore a theorem of Matveev [8] is recalled (Theorem 1.2). Secondly, the new variable δ extends the "radius" of the problem.

The main result of our article is the following.

THEOREM 1.1. *The Diophantine equation*

$$T_n = 2^a + 3^b + 5^c + \delta \tag{1}$$

has exactly 22 *solutions in non-negative integers* n, a, b, c *and* δ *with* $a, b \leq c$ *and* $\delta \leq 10$. *The solutions to* (1) *are shown in the following tables.*

n	5	6	6	7	7	7	7	7	9	9	9	11
a	0	0	0	1	0	1	0	0	2	1	0	3
b	0	0	0	1	1	0	0	0	2	2	2	2
c	0	1	0	1	1	1	1	0	2	2	2	3
δ	1	0	4	3	4	5	6	10	6	8	9	7

n	16	16	16	16	16	16	16	16	16	16
a	1	3	0	3	2	1	2	0	1	0
b	2	1	2	0	1	1	0	1	0	0
c	5	5	5	5	5	5	5	5	5	5
δ	0	0	1	2	4	6	6	7	8	9

In the forthcoming part we quote three important results which play a crucial role in the proof of Theorem 1.1. The first is the aforementioned theorem of Matveev [8].

THEOREM 1.2. *Denote by* $\eta_1, \ldots, \eta_k$ *algebraic numbers, neither* 0 *nor* 1, *by* $\log \eta_1, \ldots, \log \eta_k$ *determinations of their logarithms, by* D *the degree over* Q *of the number field* $\mathsf{K} = \mathsf{Q}(\eta_1, \ldots, \eta_k)$, *and by* $b_1, \ldots, b_k$ *rational integers. Furthermore let* $\kappa = 1$ *if* K *is real and* $\kappa = 2$ *otherwise. Choose*

$$A_i \geq \max\{Dh(\eta_i), |\log \eta_i|\} \qquad (1 \leq i \leq k),$$

where $h(\eta)$ denotes the absolute logarithmic Weil height of η and

$$B = \max\{1, \max\{|b_j| A_j / A_k : 1 \leq j \leq n\}\}.$$

Assume that $b_k \neq 0$ and $\log \eta_1, \ldots, \log \eta_k$ are linearly independent over $\mathbb{Z}$. *Then*

$$\log |b_1 \log \eta_1 + \cdots + b_k \log \eta_k| \geq -C(k, \kappa) C_0 W_0 D^2 \Omega,$$

with

$$\begin{aligned}
\Omega &= A_1 \cdots A_k, \\
C(k, \kappa) &= \frac{16}{k! \kappa} e^k (2k + 1 + 2\kappa)(k + 2)(4(k + 1))^{k+1} \left(\frac{1}{2} ek\right)^{\kappa}, \\
C_0 &= \log\big(e^{4.4k+7} k^{5.5} D^2 \log(eD)\big), \\
W_0 &= \log(1.5eBD \log(eD)).
\end{aligned}$$

The application of the work of Matveev provides a large ($\sim 10^{14}$) upper bound for the subscript n. Then we mean to reduce the upper bound by

THEOREM 1.3. *Suppose that M is a positive integer. Let γ be an irrational number and p/q a convergent of the continued fraction expansion of γ with $q > 6M$. Put $\varepsilon = \|\mu q\| - M \|\gamma q\|$, where μ is a real number and $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-m}$$

in positive integers m and n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leq m < M.$$

The preceding theorem is due to Dujella and Pethő [5]. Finally, we present a useful explicit form of k-Fibonacci numbers (see [4]). We will use only the case $k = 3$.

THEOREM 1.4. *For $F_n^{(k)}$ the n^{th} k-generalized Fibonacci number, one has*

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the roots of $x^k - x^{k-1} - \cdots - 1 = 0$.

One consequence (see [2], Lemma 1) of the above theorem is that

$$\alpha^{n-2} \le F_n^{(k)} \le \alpha^{n-1} \qquad (n \ge 1) \tag{2}$$

holds for the dominating zero α of the polynomial $x^k - x^{k-1} - \cdots - 1$.

2. Proof of the Theorem

Suppose that $n \ge 11$ satisfies (1) with some suitable integers a, b, c and δ. The case $k = 3$ of Theorem 1.4 gives

$$\frac{\alpha - 1}{4\alpha - 6}\alpha^{n-1} + \frac{\beta - 1}{4\beta - 6}\beta^{n-1} + \frac{\gamma - 1}{4\gamma - 6}\gamma^{n-1} = 2^a + 3^b + 5^c + \delta, \tag{3}$$

where $1 < \alpha \in \mathsf{R}$ is the dominant root of the characteristic polynomial $x^3 - x^2 - x - 1$ of the Tribonacci sequence, and the common absolute value of the complex conjugates β and γ is less than 3/4. Put $\alpha_1 = (\alpha - 1)/(4\alpha - 6)$, and define β_1 and γ_1 analogously. Equation (3) is equivalent to

$$\frac{\alpha_1\alpha^{n-1}}{5^c} - 1 = \frac{2^a + 3^b + \delta - \xi}{5^c},$$

where $\xi = \beta_1\beta^{n-1} + \gamma_1\gamma^{n-1}$ is obviously a real number, whose absolute value is less than 1 if $n \ge 0$. Therefore $\alpha_1\alpha^{n-1}/5^c > 1$ and consequently

$$\Lambda = \log\alpha_1 + (n - 1)\log\alpha - c\log 5 > 0.$$

By $n \ge 11$, the inequalities

$$\frac{2^a + 3^b + \delta - \xi}{5^c} < \frac{2^a + 3^b + 10 + 0.1}{5^c} < \frac{5/3 \cdot 5^{0.69c}}{5^c} = \frac{5}{3 \cdot 5^{0.31c}}$$

provide an upper bound for the difference $\alpha_1\alpha^{n-1}/5^c - 1$.

In the next part, we apply Theorem 1.2 with $\eta_1 = \alpha$, $\eta_2 = 5$, $\eta_3 = \alpha_1$ and $b_1 = n - 1$, $b_2 = -c$ and $b_3 = 1$. Note that as $\alpha \in \mathsf{R}$, $\kappa = 1$ holds; moreover $\alpha_1 \in \mathsf{Q}(\alpha)$ allows $\mathsf{K} = \mathsf{Q}(\alpha)$ and $D = 3$. Observe that the minimal polynomial of α_1 is $44x^3 - 44x^2 + 12x - 1$. We take $A_1 = 0.61$, $A_2 = 1.61$ and $A_3 = 3.79$. By [2], $T_n < \alpha^{n-1}$ holds, which together with $5^c < T_n$ shows

$$\frac{n-1}{c} > \frac{\log 5}{\log\alpha} > \frac{1.61}{0.61}.$$

Hence $B = 61(n - 1)/369$ is valid in Theorem 1.2. Clearly, α, 5 and α_1 are linearly independent over Z. By Theorem 1.2, with the values

$$\Omega = 3.73, \quad C(k,\kappa) = 6.5 \cdot 10^8, \quad C_0 = 29.2, \quad W_0 = 1.42,$$

we deduce

$$\Lambda > e^{-6.38\cdot 10^{11}(1.42+\log(n-1))}. \tag{4}$$

On the other hand,

$$\Lambda \leq \frac{\alpha_1\alpha^{n-1}}{5^c} - 1 < \frac{5}{3\cdot 5^{0.31c}} < e^{0.52-0.49c}. \tag{5}$$

Combining (4) and (5), we obtain

$$0.49c - 0.52 < 6.38\cdot 10^{11}(1.42+\log(n-1)). \tag{6}$$

Lemma 1 of [2] states $\alpha^{n-2} < T_n$, which together with the obvious inequalities $T_n < 2\cdot 5^c < 5^{c+1}$ $(n \geq 11)$ implies

$$(n-2)\frac{\log\alpha}{\log 5} - 1 < c.$$

Inserting this into (6), we get

$$0.18(n-2) - 1.01 < 6.38\cdot 10^{11}(1.42+\log(n-1)),$$

and then $n < 1.2\cdot 10^{14}$.

To reduce the upper bound on n we apply Theorem 1.3. First observe, that

$$\frac{0.52-0.49c}{\log 5} < 2.5\cdot 1.2^{-n}$$

follows via $c < (n-1)\log\alpha/\log 5 < 0.378n - 1.758$. Put $\gamma := \log\alpha/\log 5 \notin \mathbb{Q}$. Thus

$$0 < (n-1)\gamma - c + \frac{\log\alpha_1}{\log 5} < 2.5\cdot 1.2^{-n}.$$

Taking $M = 1.2\cdot 10^{14}$, we found that q_{33}, the denominator of the 33rd convergent of γ exceeds $6M$. Furthermore

$$\varepsilon = \left\| \frac{\log\alpha_1}{\log 5} q_{33} \right\| - 1.2\cdot 10^{14}\|\gamma q_{33}\| > 0.498.$$

Let $A = 2.5$ and $B = 1.2$ in Theorem 1.3. From the above inequalities we conclude

$$\frac{\log(Aq_{33}/\varepsilon)}{\log B} < 222.7.$$

That is $n \leq 223$.

To complete the proof of the Theorem 1.1, we verify the possible cases $0 \leq n \leq 223$ by computer separately for all the values of $0 \leq \delta \leq 10$.

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