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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF VARIABLE EXPONENT ELLIPTIC SYSTEMS

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Abstract

We consider the system of differential equations

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)} f(u, v) & \text{in } \Omega, \\ -\Delta_{q(x)}v = \mu^{q(x)} g(u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathsf{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$, $1 < p(x), q(x) \in C^1(\bar{\Omega})$ are functions. $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian. We discuss the existence of a positive solution via sub-super solutions.

1. Introduction

The study of differential equatons and variational problems with variable exponent is a new and interesting topic.

It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [4], [14], [19]). Many results have been obtained on these kinds of problems, for example [1], [4], [5], [8], [9], [13]. The basic regularity results have been established in for the relevant model case, which already requires almost all the basic new ideas. Then in [9] Fan, by relying on the techniques of [1], [7], has extended these results, valid for the model case, to more general equations and up to the boundary. On the existence of solutions for elliptic systems with variable exponent, we refer to [13], [16]. In this paper, we mainly consider the existence of positive weak solutions for the system

$$\text{(P)} \begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)} f(u, v) & \text{in } \Omega, \\ -\Delta_{q(x)}v = \mu^{q(x)} g(u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathsf{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$, $1 < p(x), q(x) \in C^1(\bar{\Omega})$ are functions. The operator $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the

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$p(x)$-Laplacian and the corresponding equation is called a variable exponent equation. In particular, if $p(x) \equiv p$ (a constant), $\Delta_{p(x)}$ is the well-known p-Laplacian and the corresponding equation is called a constant exponent equation. There are many articles on the existence of solutions for constant exponent elliptic systems, for example [2], [5], [6]. Because of the nonhomogeneity of $p(x)$-Laplacian problems, the $p(x)$-Laplacian problems are more complicated than p-Laplacian problems, and many results and methods for p-Laplacian are invalid for $p(x)$-Laplacian; for example, if Ω is bounded, then the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx}$$

is zero in general, but under some special conditions it is not zero (see [12]). The first eigenvalue and the first eigenfunction of the $p(x)$-Laplacian do not exist in general. It is important in the study of p-Laplacian problems to have the existence of the first eigenfunction and the condition $\lambda_p > 0$. There are more difficulties in discussing the existence of solutions of variable exponent problems.

In [3], the authors discussed the existence of positive solutions of the system

$$\text{(I)} \begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)} F(x, u, v) & \text{in } \Omega, \\ -\Delta_{p(x)} v = \lambda^{p(x)} G(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p(x) \in C^1(\bar{\Omega})$ is a function, $F(x, u, v) = [g(x)a(u) + f(v)]$, $G(x, u, v) = [g(x)b(v) + h(u)]$, λ is a positive parameter and $\Omega \subset \mathsf{R}^N$ is a bounded domain.

In [18], the authors consider the existence and asymptotic behavior of positive weak solutions of the system

$$\text{(II)} \begin{cases} -\Delta_{p(x)} u = \lambda(u^{\alpha(x)} v^{\gamma(x)} + h_1(x)) & \text{in } \Omega, \\ -\Delta_{q(x)} v = \lambda(u^{\delta(x)} v^{\beta(x)} + h_2(x)) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

without any symmetry conditions.

The system (I) is called $(p(x), p(x))$-type and the systems (P) and (II) are called $(p(x), q(x))$-type, since there exist a $p(x)$-Laplacian and a $q(x)$-Laplacian in (P) and (II). There are some differences between the existence of positive solutions of $(p(x), q(x))$-type and $(p(x), p(x))$-type systems.

In this article, we consider the existence of positive solutions of the system

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)} f(u, v) & \text{in } \Omega, \\ -\Delta_{q(x)}v = \mu^{q(x)} g(u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p(x), q(x) \in C^1(\bar{\Omega})$ are functions, λ, μ are positive parameters and $\Omega \subset \mathsf{R}^N$ is a bounded domain.

To study $p(x)$-Laplacian problems, we need to mention some facts about the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and some properties of the $p(x)$-Laplacian (see [8], [15]). If $\Omega \subset \mathsf{R}^N$ is an open domain, we write

$$C_+(\Omega) = \{h : h \in C(\Omega), h(x) > 1 \text{ for } x \in \Omega\},$$
$$h^+ = \sup_{x\in\Omega} h(x), \quad h^- = \inf_{x\in\Omega} h(x), \quad \text{for any } \ h \in C(\Omega).$$

Throughout the article, we will assume that:

(H_1) $\Omega \subset \mathsf{R}^N$ is an open bounded domain with C^2 boundary $\partial\Omega$.

(H_2) $p, q \in C^1(\bar{\Omega})$ and $1 < p^- \le p^+$, $1 < q^- \le q^+$.

(H_3) $f, g \in C^1((0,\infty)\times(0,\infty)) \cap C([0,\infty)\times[0,\infty))$ are monotone functions such that $f_u, f_v, g_u, g_v \ge 0$ and $\lim_{u,v\to\infty} f(u, v) = \lim_{u,v\to\infty} g(u, v) = \infty$.

(H_4) For any positive constant M

$$\lim_{u\to+\infty} \frac{f\big[u, M(g(u,u))^{\frac{1}{(q^- - 1)}}\big]}{u^{p^- - 1}} = 0.$$

(H_5) $\lim_{u\to\infty} \frac{g(u,u)}{u^{q^- - 1}} = 0$.

Denote

$$L^{p(x)}(\Omega) = \left\{u \;\middle|\; u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

We recall that the norm on $L^{p(x)}(\Omega)$ is defined by

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

The Banach space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is called generalized Lebesgue space, and it is a separable, reflexive, and uniform convex Banach space (see [8, Theorems 1.10 and 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)} : |\nabla u| \in L^{p(x)}\}$, and it is equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall\, u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive, and uniformly convex Banach space (see [8, Theorem 2.1]). We define

$$(L(u), v) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx, \quad \forall\, v, u \in W_0^{1,p(x)}(\Omega),$$

then $L : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is a continuous, bounded, and strictly monotone operator, and it is a homeomorphism (see [11, Theorem 3.1]).

If $u, v \in (W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega))$, (u, v) is called a weak solution of (P) if it satisfies

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \lambda^{p(x)} f(u, v) \varphi \, dx, & \forall\, \varphi \in W_0^{1,p(x)}(\Omega), \\ \displaystyle\int_\Omega |\nabla v|^{q(x)-2} \nabla v \cdot \nabla \psi \, dx = \int_\Omega \mu^{q(x)} g(u, v) \psi \, dx, & \forall\, \psi \in W_0^{1,q(x)}(\Omega). \end{cases}$$

Define $A : W^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ as

$$\langle Au, \varphi \rangle = \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + l(x, u)\varphi) \, dx, \qquad \forall\, u \in W^{1,p(x)}(\Omega), \ \ \forall\, \varphi \in W_0^{1,p(x)}(\Omega),$$

where $l(x, u)$ is continuous on $\bar{\Omega} \times \mathsf{R}$, and $l(x, .)$ is increasing. It is easy to check that A is a continuous bounded mapping. Copying the proof of [17], we have the following lemma.

LEMMA 1.1 (Comparison Principle). *Let $u, v \in W^{1,p(x)}(\Omega)$ satisfying $Au - Av \geq 0$ in $(W_0^{1,p(x)}(\Omega))^*$, $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ (i.e., $u \geq v$ on $\partial\Omega$), then $u \geq v$ a.e. in Ω.*

Here and hereafter, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to the boundary of Ω.

Denote $d(x) = d(x, \partial\Omega)$ and $\partial\Omega_\epsilon = \{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$. Since $\partial\Omega$ is C^2 regularly, then there exists a constant $\delta \in (0, 1)$ such that $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, and $|\nabla d(x)| \equiv 1$.

Denote

$$v_1(x) = \begin{cases} \gamma d(x), \quad d(x) < \delta, \\ \gamma\delta + \displaystyle\int_\delta^{d(x)} \gamma\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} dt, \quad \delta \le d(x) < 2\delta, \\ \gamma\delta + \displaystyle\int_\delta^{2\delta} \gamma\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} dt, \quad 2\delta \le d(x). \end{cases}$$

Obviously, $0 \le v_1(x) \in C^1(\bar{\Omega})$. Considering

$$-\Delta_{p(x)} w(x) = \eta \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where η is a positive parameter.

LEMMA 1.2 (See [10]). *If positive parameter η is large enough and w is the unique solution of* (1.1)*, then we have*

(i) *For any $\theta \in (0, 1)$ there exists a positive constant C_1 such that*

$$C_1 \eta^{\frac{1}{p^+ - 1 + \theta}} \le \max_{x \in \bar{\Omega}} w(x);$$

(ii) *There exists a positive constant C_2 such that*

$$\max_{x \in \bar{\Omega}} w(x) \le C_2 \eta^{\frac{1}{p^- - 1}}.$$

2. Existence results

In the following, when there is no misunderstanding, we always use C_i to denote positive constants.

THEOREM 2.1. *On the conditions of* (H_1)–(H_5)*, then (P) has a positive solution when λ, μ are large enough.*

PROOF. We shall establish Theorem 2.1 by constructing a positive subsolution (Φ_1, Φ_2) and supersolution (z_1, z_2) of (P), such that $\Phi_1 \le z_1$ and $\Phi_2 \le z_2$. That is (Φ_1, Φ_2) and (z_1, z_2) satisfies

$$\begin{cases} \displaystyle\int_\Omega |\nabla\Phi_1|^{p(x)-2}\nabla\Phi_1 \cdot \nabla\varphi \, dx \le \int_\Omega \lambda^{p(x)} f(\Phi_1, \Phi_2)\varphi \, dx, \\ \displaystyle\int_\Omega |\nabla\Phi_2|^{q(x)-2}\nabla\Phi_2 \cdot \nabla\psi \, dx \le \int_\Omega \mu^{q(x)} g(\Phi_1, \Phi_2)\psi \, dx, \end{cases}$$

and

$$\begin{cases} \int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla\varphi\, dx \geq \int_\Omega \lambda^{p(x)} f(z_1, z_2)\varphi\, dx, \\ \\ \int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \cdot \nabla\psi\, dx \geq \int_\Omega \mu^{q(x)} g(z_1, z_2)\psi\, dx, \end{cases}$$

for all $(\varphi, \psi) \in (W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega))$ with $\varphi \geq 0$ and $\psi \geq 0$. According to the sub-supersolution method for $p(x)$-Laplacian equations (see [10]), then (P) has a positive solution.

Step 1. We construct a subsolution of (P).

Let $\sigma \in (0, \delta)$ is small enough. Denote

$$\phi_1(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt, \quad \sigma \leq d(x) < 2\delta, \\ \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt, \quad 2\delta \leq d(x). \end{cases}$$

$$\phi_2(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{q^- - 1}} dt, \quad \sigma \leq d(x) < 2\delta, \\ \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{q^- - 1}} dt, \quad 2\delta \leq d(x). \end{cases}$$

It is easy to see that $\phi_1, \phi_2 \in C^1(\bar{\Omega})$. Denote

$$\alpha = \min\left\{\frac{\inf p(x) - 1}{4(\sup|\nabla p(x)| + 1)}, \frac{\inf q(x) - 1}{4(\sup|\nabla q(x)| + 1)}, 1\right\},$$

$$\zeta = |f(0, 0)| + |g(0, 0)| + 1.$$

By computation

$$-\Delta_{p(x)}\phi_1 = \begin{cases} -k(ke^{kd(x)})^{p(x)-1}\Big[(p(x)-1) \\ \quad + \big(d(x)+\frac{\ln k}{k}\big)\nabla p \nabla d + \frac{\Delta d}{k}\Big], & d(x)<\sigma, \\ \Big\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^- -1} - \big(\frac{2\delta-d}{2\delta-\sigma}\big) \\ \quad \times \Big[\Big(\ln ke^{k\sigma}\big(\frac{2\delta-d}{2\delta-\sigma}\big)^{\frac{2}{p^- -1}}\Big)\nabla p \nabla d + \Delta d\Big]\Big\} \\ \quad \times (ke^{k\sigma})^{p(x)-1}\big(\frac{2\delta-d}{2\delta-\sigma}\big)^{\frac{2(p(x)-1)}{p^- -1}-1}, & \sigma<d(x)<2\delta, \\ 0, & 2\delta<d(x). \end{cases}$$

From (H_3), there exists a positive constant $M > 2$ such that

$$f(\phi_1,\phi_2) \geq 1 \text{ and } g(\phi_1,\phi_2)\geq 1, \quad \text{when } \phi_1,\phi_2 \geq M-1. \tag{2.1}$$

Let $\sigma = \frac{1}{k}\ln M$, then

$$\sigma k = \ln M. \tag{2.2}$$

If k is sufficiently large, from (2.2), we have

$$-\Delta_{p(x)}\phi_1 \leq -k^{p(x)}\alpha, \quad d(x) < \sigma. \tag{2.3}$$

Let $\lambda = \frac{\alpha}{\zeta+1}k$, then

$$k^{p(x)}\alpha \geq \lambda^{p(x)}\zeta,$$

from (2.1), (2.3), then we have

$$-\Delta_{p(x)}\phi_1 \leq -\lambda^{p(x)}\zeta \leq \lambda^{p(x)} f(\phi_1,\phi_2), \quad d(x) < \sigma. \tag{2.4}$$

Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, then there exists a positive constant C_3 such that

$$\begin{aligned} -\Delta_{p(x)}\phi_1 &\leq (ke^{k\sigma})^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^- -1}-1}\left|\left\{\frac{2(p(x)-1)}{(2\delta-\sigma)(p^- -1)}\right.\right. \\ &\qquad \left.\left. - \left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- -1}}\right)\nabla p\nabla d + \Delta d\right]\right\}\right| \\ &\leq C_3(ke^{k\sigma})^{p(x)-1}\ln k, \quad \sigma < d(x) < 2\delta. \end{aligned}$$

If k is sufficiently large, let $\lambda = \frac{\alpha}{\zeta+1}k$, we have

$$C_3(ke^{k\sigma})^{p(x)-1}\ln k = C_3(kM)^{p(x)-1}\ln k \leq \lambda^{p(x)},$$

then

$$-\Delta_{p(x)}\phi_1 \le \lambda^{p(x)}, \quad \sigma < d(x) < 2\delta. \tag{2.5}$$

Obviously

$$\lambda^{p(x)} \le \lambda^{p(x)} f(\phi_1, \phi_2), \quad \sigma < d(x) < 2\delta. \tag{2.6}$$

Combining (2.5), (2.5), we have

$$-\Delta_{p(x)}\phi_1 \le \lambda^{p(x)} \le \lambda^{p(x)} f(\phi_1, \phi_2), \quad \sigma < d(x) < 2\delta. \tag{2.7}$$

when λ is large enough.

Obviously

$$-\Delta_{p(x)}\phi_1 = 0 \le \lambda^{p(x)} f(\phi_1, \phi_2), \quad 2\delta < d(x). \tag{2.8}$$

Combining (2.4), (2.7), and (2.8), we can conclude that

$$-\Delta_{p(x)}\phi_1 \le \lambda^{p(x)} f(\phi_1, \phi_2), \quad \text{a.e. on } \Omega. \tag{2.9}$$

Similarly

$$-\Delta_{p(x)}\phi_2 \le \mu^{q(x)} g(\phi_1, \phi_2), \quad \text{a.e. on } \Omega. \tag{2.10}$$

From (2.9) and (2.10), we can see that (ϕ_1, ϕ_2) is a subsolution of (P).

Step 2. We construct a supersolution of (P).

We consider

$$\begin{cases} -\Delta_{p(x)} z_1 = \lambda^{p^+}\mu_1 & \text{in } \Omega, \\ -\Delta_{p(x)} z_2 = \mu^{q^+}\big[g(\beta(\lambda^{p^+}\mu_1), \beta(\lambda^{p^+}\mu_1))\mu_2\big] & \text{in } \Omega, \\ z_1 = z_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\beta = \beta(\lambda^{p^+}\mu_1) = \max_{x\in\bar\Omega} z_1(x)$. We shall prove that (z_1, z_2) is a supersolution for (P).

From Lemma 1.2, we have

$$\max_{x\in\bar\Omega} z_1(x) \le C_2\big[\lambda^{p^+}\mu_1\big]^{\frac{1}{p^- - 1}}$$

and

$$\max_{x\in\bar\Omega} z_2(x) \le C_2\big[\mu^{q^+} g\big((\beta(\lambda^{p^+}\mu_1), \beta(\lambda^{p^+}\mu_1)\big)\mu_2\big]^{\frac{1}{q^- - 1}}.$$

For $\psi \in W_0^{1,q(x)}(\Omega)$ with $\psi \geq 0$, it is easy to see that

$$\begin{aligned}(2.11)\quad \int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \cdot \nabla\psi\, dx \\ = \int_\Omega \mu^{q^+}\big[g\big(\beta(\lambda^{p^+}\mu_1), \beta(\lambda^{p^+}\mu_1)\big)\mu_2\big]\psi\, dx\end{aligned}$$

Since $\lim_{u\to\infty}\frac{g(u,u)}{u^{q^--1}} = 0$, when μ_1, μ_2 are sufficiently large, from Lemma 1.2 we have

$$\begin{aligned}(2.12)\quad &\int_\Omega \mu^{q^+}\big[g\big(\beta(\lambda^{p^+}\mu_1), \beta(\lambda^{p^+}\mu_1)\big)\mu_2\big]\psi\, dx \\ &\geq \mu^{q^+}\int_\Omega g\Big(\beta(\lambda^{p^+}\mu_1), \big[g\big(\beta(\lambda^{p^+}\mu_1), \beta(\lambda^{p^+}\mu_1)\big)\mu_2\big]^{\frac{1}{q^--1}}\Big)\psi\, dx \\ &\geq \mu^{q^+}\int_\Omega g(z_1, z_2)\psi\, dx.\end{aligned}$$

Hence

$$(2.13)\qquad \int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \cdot \nabla\psi\, dx \geq \mu^{q^+}\int_\Omega g(z_1, z_2)\psi\, dx.$$

Also

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla\varphi\, dx = \int_\Omega \lambda^{p^+}\mu_1\varphi\, dx$$

By (H_4), when μ_1, μ_2 are sufficiently large, combining Lemma 1.2 and (H_4), we have

$$\begin{aligned}\mu_1 &\geq \frac{1}{\lambda^{p^+}}\left[\frac{1}{C_2}\beta(\lambda^{p^+}\mu_1)\right]^{p^--1} \\ &\geq f\Big(\beta(\lambda^{p^+}\mu_1), C_2\big[\mu^{q^+}g\big(\beta(\lambda^{p^+}\mu_1), \beta(\lambda^{p^+}\mu_1)\big)\mu_2\big]^{\frac{1}{q^--1}}\Big).\end{aligned}$$

Then

$$\begin{aligned}&\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla\varphi\, dx \\ (2.14)\quad &\geq \int_\Omega \lambda^{p^+}f\Big(\beta(\lambda^{p^+}\mu_1), C_2\Big[\mu^{q^+}g(\beta(\lambda^{p^+}\mu_1), \beta(\lambda^{p^+}\mu_1))\mu_2\Big]^{\frac{1}{q^--1}}\Big)\varphi\, dx \\ &\geq \int_\Omega \lambda^{p^+}f(z_1, z_2)\varphi\, dx.\end{aligned}$$

According to (2.13) and (2.14), we can conclude that (z_1, z_2) is a supersolution for (P).

It only remains to prove that $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$.

In the definition of $v_1(x)$, let $\gamma = \frac{2}{\delta}\big(\max_{x\in\bar{\Omega}} \phi_1(x) + \max_{x\in\bar{\Omega}} |\nabla\phi_1(x)|\big)$. We claim that

$$\phi_1(x) \leq v_1(x), \quad \forall\, x \in \Omega. \tag{2.15}$$

From the definition of v_1, it is easy to see that

$$\phi_1(x) \leq 2 \max_{x\in\bar{\Omega}} \phi_1(x) \leq v_1(x), \quad \text{when } d(x) = \delta,$$

and

$$\phi_1(x) \leq 2 \max_{x\in\bar{\Omega}} \phi_1(x) \leq v_1(x), \quad \text{when } d(x) \geq \delta.$$

It only remains to prove that

$$\phi_1(x) \leq v_1(x), \quad \text{when } d(x) < \delta.$$

Since $v_1 - \phi_1 \in C^1(\overline{\partial\Omega_\delta})$, then there exists a point $x_0 \in \overline{\partial\Omega_\delta}$ such that

$$v_1(x_0) - \phi_1(x_0) = \min_{x\in\overline{\partial\Omega_\delta}} [v_1(x) - \phi(x)].$$

If $v_1(x_0) - \phi_1(x_0) < 0$, it is easy to see that $0 < d(x_0) < \delta$, and then

$$\nabla v_1(x_0) - \nabla\phi_1(x_0) = 0.$$

From the definition of v_1, we have

$$|\nabla v_1(x_0)| = \gamma = \frac{2}{\delta}\Big(\max_{x\in\bar{\Omega}} \phi_1(x) + \max_{x\in\bar{\Omega}} |\nabla\phi_1(x)|\Big) > |\nabla\phi_1(x_0)|.$$

It is a contradiction to $\nabla v_1(x_0) - \nabla\phi_1(x_0) = 0$. Thus (2.15) is valid.

Obviously, there exists a positive constant C_3 such that

$$\gamma \leq C_3\lambda.$$

Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, according to the proof of Lemma 1.2, then there exists a positive constant C_4 such that

$$-\Delta_{p(x)} v_1(x) \leq C_* \gamma^{p(x)-1+\theta} \leq C_4 \lambda^{p(x)-1+\theta}, \quad \text{a.e. in } \Omega, \text{ where } \theta \in (0, 1).$$

When $\eta \geq \lambda^{p^+}$ is large enough, we have

$$-\Delta_{p(x)} v_1(x) \leq \eta.$$

According to the comparison principle, we have

$$v_1(x) \le w(x), \quad \forall x \in \Omega. \tag{2.16}$$

From (2.15) and (2.16), when $\eta \ge \lambda^{p^+}$ and $\lambda \ge 1$ is sufficiently large, we have

$$\phi_1(x) \le v_1(x) \le w(x), \quad \forall x \in \Omega. \tag{2.17}$$

According to the comparison principle, when μ_1, μ_2 are large enough, we have

$$v_1(x) \le w(x) \le z_1(x), \quad \forall\, x \in \Omega.$$

Combining the definition of $v_1(x)$ and (2.17), it is easy to see that

$$\phi_1(x) \le v_1(x) \le w(x) \le z_1(x), \quad \forall\, x \in \Omega.$$

When $\mu_i \ge 1 (i = 1, 2)$ and λ, μ are large enough, from Lemma 1.2, we can see that $\beta(\lambda^{p^+}\mu_1)$ is large enough, then $\mu^{q^+}[g(\beta(\lambda^{p^+}\mu_1), \beta(\lambda^{p^+}\mu_1))]\mu_2$ is large enough. Similarly, we have $\phi_2 \le z_2$.

This completes the proof.

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