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K-CONTINUITY IS EQUIVALENT TO K-EXACTNESS

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Abstract

Let A be a C^*-algebra. It is well known that the functor $B \mapsto A \otimes B$ of taking the minimal tensor product with A preserves inductive limits if and only if it is exact. C^*-algebras with this property play an important role in the structure and finite-dimensional approximation theory of C^*-algebras.

We consider a K-theoretic analogue of this result and show that the functor $B \mapsto K_0(A \otimes B)$ preserves inductive limits if and only if it is half-exact.

1. Introduction

We denote the *spatial* or *minimal* tensor product of C^*-algebras by the symbol $\otimes$ (cf. [22, Section IV.4], [3, Section 3.3]).

Let A be a C^*-algebra. We say that A is $\otimes$-*exact* if for every extension (i.e. short exact sequence)

$$0 \longrightarrow I \longrightarrow D \longrightarrow B \longrightarrow 0$$

of C^*-algebras, the natural sequence

$$0 \longrightarrow A \otimes I \longrightarrow A \otimes D \longrightarrow A \otimes B \longrightarrow 0$$

is exact (in the middle). We note that $\otimes$-exactness is generally referred to as just *exactness* in the literature. We chose our terminology for consistency within the paper.

Let M_n denote the C^*-algebra of $n \times n$ complex matrices. Letting

$$\prod_{n=0}^{\infty} M_n := \Big\{(a_n)_{n=0}^{\infty} \;\Big|\; a_n \in M_n \text{ for all } n \text{ and } \sup_n \|a_n\| < \infty\Big\}$$

and

$$\bigoplus_{n=0}^{\infty} M_n := \Big\{(a_n)_{n=0}^{\infty} \;\Big|\; a_n \in M_n \text{ for all } n \text{ and } \lim_{n\to\infty} \|a_n\| = 0\Big\},$$

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we get an extension

$$0 \longrightarrow \bigoplus_{n=0}^{\infty} M_n \longrightarrow \prod_{n=0}^{\infty} M_n \longrightarrow \prod_{n=0}^{\infty} M_n \Big/ \bigoplus_{n=0}^{\infty} M_n \longrightarrow 0.$$

E. Kirchberg proved the following fundamental result about $\otimes$-exactness. See [3] for more details.

THEOREM 1.1 (E. Kirchberg [12], [13]). *Let A be a C^*-algebra. The following statements are equivalent.*

(i) *The algebra A is $\otimes$-exact.*

(ii) *The sequence*

$$0 \longrightarrow A \otimes \bigoplus_{n=0}^{\infty} M_n \longrightarrow A \otimes \prod_{n=0}^{\infty} M_n \longrightarrow A \otimes \left(\prod_{n=0}^{\infty} M_n \Big/ \bigoplus_{n=0}^{\infty} M_n \right) \longrightarrow 0$$

is exact.

(iii) *The algebra A is nuclearly embeddable in the sense of [23].*

We remark that the implication (iii) $\Rightarrow$ (i) was proved by S. Wassermann in [24].

We say that A is $\otimes$-*continuous* if for every inductive sequence

$$B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \dots$$

of C^*-algebras, the natural (surjective) map

$$\varinjlim (A \otimes B_n) \longrightarrow A \otimes \varinjlim B_n$$

is an isomorphism, where $\varinjlim$ denotes the inductive limit functor.

The following result is well-known and follows from the equivalence (i) $\Leftrightarrow$ (ii) in the theorem above. N. Ozawa attributes it to E. Kirchberg.

THEOREM 1.2. *A C^*-algebra is $\otimes$-exact if and only if it is $\otimes$-continuous.*

In this paper, we consider a K-theoretic analogue of this result. See [25], [2], [16] for details about topological K-theory for C^*-algebras. We say that a C^*-algebra A is K-*exact* if for every extension

$$0 \longrightarrow I \longrightarrow D \longrightarrow B \longrightarrow 0$$

of C^*-algebras, the sequence

$$K_0(A \otimes I) \longrightarrow K_0(A \otimes D) \longrightarrow K_0(A \otimes B)$$

is exact in the middle. We say that a C^*-algebra A is K*-continuous* if for every inductive sequence

$$B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \dots$$

of C^*-algebras, the natural map

$$\varinjlim K_0(A \otimes B_i) \longrightarrow K_0(A \otimes \varinjlim B_i)$$

is an isomorphism.

The following is our main result.

THEOREM 1.3. *A C^*-algebra is K-exact if and only if it is K-continuous.*

All $\otimes$-exact C^*-algebras are K-exact. Examples of non-K-exact C^*-algebras were first constructed by G. Skandalis and play an important role in K-theory and KK-theory (cf. [19], [20], [9], [11]).

In section 2, we give a proof of Theorem 1.2, as we couldn't find a direct reference and also the proof of the implication Theorem 1.2($\Rightarrow$) is used in proof of Theorem 1.3($\Rightarrow$). In section 3, we study the notions of K-exactness and K-continuity and prove Theorem 1.3. We note that our proof of the implication Theorem 1.3($\Leftarrow$) uses [7, Theorem 3.11] in a crucial way.

2. Proof of Theorem 1.2

Let $\mathsf{N} := \{0, 1, \dots\}$ denote the set of non-negative integers. The following is obvious.

LEMMA 2.1. *Let A be a C^*-algebra and let*

$$B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \dots$$

be an inductive sequence of C^*-algebras. If the connecting maps are all injective, then the map

$$\varinjlim (A \otimes B_n) \longrightarrow A \otimes \varinjlim B_n$$

is an isomorphism.

LEMMA 2.2. *Consider an inductive sequence of extensions of C^*-algebras*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_n & \longrightarrow & D_n & \longrightarrow & B_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_n} & & \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \beta_n} & & \\ 0 & \longrightarrow & I_{n+1} & \longrightarrow & D_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & 0 \end{array}$$

(i) *The limit sequence*

$$0 \longrightarrow \varinjlim I_n \longrightarrow \varinjlim D_n \longrightarrow \varinjlim B_n \longrightarrow 0$$

is exact.

(ii) *Suppose that for every $n \in \mathsf{N}$, the extension*

$$0 \longrightarrow I_n \longrightarrow D_n \longrightarrow B_n \longrightarrow 0 \tag{2.1}$$

is split[1] *(i.e. the quotient map admits a $*$-homomorphic section) and the connecting maps ι_n and δ_n are injective. Then for any C^*-algebra A, the map*

$$\varinjlim(A \otimes B_n) \longrightarrow A \otimes \varinjlim B_n$$

is an isomorphism if and only if the sequence

$$0 \longrightarrow A \otimes \varinjlim I_n \longrightarrow A \otimes \varinjlim D_n \longrightarrow A \otimes \varinjlim B_n \longrightarrow 0$$

is exact.

PROOF. (i) is clear. For (ii), consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varinjlim(A \otimes I_n) & \longrightarrow & \varinjlim(A \otimes D_n) & \longrightarrow & \varinjlim(A \otimes B_n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & A \otimes \varinjlim I_n & \longrightarrow & A \otimes \varinjlim D_n & \longrightarrow & A \otimes \varinjlim B_n & \longrightarrow & 0 \end{array} \tag{2.2}$$

Since the connecting maps ι_n and δ_n are injective, the maps ι and δ are isomorphisms by Lemma 2.1. For any $n \in \mathsf{N}$, since (2.1) is split exact, the sequence

$$0 \longrightarrow A \otimes I_n \longrightarrow A \otimes D_n \longrightarrow A \otimes B_n \longrightarrow 0$$

is exact. Thus the top row is exact by (i). It follows that β is an isomorphism if and only if the bottom row is exact by five-lemma.

PROOF OF THEOREM 1.2. ($\Rightarrow$): Let A be a $\otimes$-exact C^*-algebra and let

$$B_0 \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \dots$$

be an inductive sequence.

Let $n \in \mathsf{N}$ and let $I_n := \bigoplus_{k=0}^{n-1} B_k$ and let $D_n := \bigoplus_{k=0}^{n} B_k$. Then the obvious inclusion and projection maps give a split extension

$$0 \longrightarrow I_n \longrightarrow D_n \longrightarrow B_n \longrightarrow 0.$$

[1] Locally split is enough for our purposes [3, Proposition 3.7.6]. See [8].

Let $\iota_n\colon I_n \to I_{n+1}$ denote the natural inclusion and let $\delta_n\colon D_n \to D_{n+1}$ denote the injective map given by

$$\begin{array}{ccc}
B_0 & \xrightarrow{\text{id}} & B_0 \\
\oplus & & \oplus \\
\vdots & \xrightarrow{\text{id}} & \vdots \\
\oplus & & \oplus \\
B_n & \xrightarrow{\text{id}} & B_n \\
& \searrow^{\beta_n} & \oplus \\
& & B_{n+1}
\end{array}$$

Then we get a map of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_n & \longrightarrow & D_n & \longrightarrow & B_n & \longrightarrow & 0 \\
& & \big\downarrow{\scriptstyle \iota_n} & & \big\downarrow{\scriptstyle \delta_n} & & \big\downarrow{\scriptstyle \beta_n} & & \\
0 & \longrightarrow & I_{n+1} & \longrightarrow & D_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & 0
\end{array}$$

Since A is $\otimes$-exact, the sequence

$$0 \longrightarrow A \otimes \varinjlim I_n \longrightarrow A \otimes \varinjlim D_n \longrightarrow A \otimes \varinjlim B_n \longrightarrow 0$$

is exact, hence the map

$$\varinjlim (A \otimes B_n) \longrightarrow A \otimes \varinjlim B_n$$

is an isomorphism by Lemma 2.2 (ii).

($\Leftarrow$): Conversely, let A be a $\otimes$-continuous C^*-algebra. Let M_n, $n \in \mathsf{N}$, denote the C^*-algebra of $n \times n$ complex matrices. Consider the following inductive system of split extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{k=0}^{n} M_k & \longrightarrow & \prod_{k=0}^{\infty} M_k & \longrightarrow & \prod_{k=n+1}^{\infty} M_k & \longrightarrow & 0 \\
& & \big\downarrow & & \big\| & & \big\downarrow & & \\
0 & \longrightarrow & \bigoplus_{k=0}^{n+1} M_k & \longrightarrow & \prod_{k=0}^{\infty} M_k & \longrightarrow & \prod_{k=n+2}^{\infty} M_k & \longrightarrow & 0
\end{array}$$

given by the obvious injection and projection maps. Since A is $\otimes$-continuous, the map

$$\varinjlim\Big(A \otimes \prod_{k=n+1}^{\infty} M_k\Big) \longrightarrow A \otimes \Big(\varinjlim \prod_{k=n+1}^{\infty} M_k\Big)$$

is an isomorphism, hence the sequence

$$0 \longrightarrow A \otimes \bigoplus_{n=0}^{\infty} M_n \longrightarrow A \otimes \prod_{n=0}^{\infty} M_n \longrightarrow A \otimes \Big(\prod_{n=0}^{\infty} M_n \Big/ \bigoplus_{n=0}^{\infty} M_n \Big) \longrightarrow 0$$

is exact by Lemma 2.2(ii). It follows that A is $\otimes$-exact (cf. [12]).

3. K-exactness and K-continuity

3.1. K-exactness

DEFINITION 3.1. We say that a C^*-algebra A is *K-exact* if for every extension

$$0 \longrightarrow I \longrightarrow D \longrightarrow B \longrightarrow 0$$

of C^*-algebras, the sequence

$$K_0(A \otimes I) \longrightarrow K_0(A \otimes D) \longrightarrow K_0(A \otimes B)$$

is exact in the middle.

REMARK 3.2. A C^*-algebra A is K-exact if and only if for every extension

$$0 \longrightarrow I \longrightarrow D \longrightarrow B \longrightarrow 0$$

of C^*-algebras, the natural six-term sequence

$$\begin{array}{ccccc} K_0(A \otimes I) & \longrightarrow & K_0(A \otimes D) & \longrightarrow & K_0(A \otimes B) \\ \uparrow & & & & \downarrow \\ K_1(A \otimes B) & \longleftarrow & K_1(A \otimes D) & \longleftarrow & K_1(A \otimes I) \end{array}$$

is exact (cf. [2, Theorem 21.4.4]). The latter condition is taken as the definition of K-exactness in [27].

EXAMPLE 3.3. $\otimes$-exact C^*-algebras are K-exact, by the half-exactness of K-theory (cf. [25, Theorem 6.3.2], [2, Theorem 5.6.1]).

EXAMPLE 3.4. C^*-algebras satisfying the Künneth formula of Schochet (cf. [18], [2, Theorem 23.1.3]) are K-exact by [4, Remark 4.3]. In particular, the full group C^*-algebra $C^*(F_2)$ of the free group F_2 on two generators is K-exact (but not $\otimes$-exact).

EXAMPLE 3.5. C^*-algebras that are K-nuclear in the sense of [19] are K-exact (cf. [19, Proposition 3.5]). See also [6].

Needless to say, not all C^*-algebras are K-exact.

EXAMPLE 3.6.

(1) Let Γ be an infinite countable discrete group with Khazdan property (T), Kirchberg property (F) and Akemann-Ostrand property (AO), such as a lattice in $\mathrm{Sp}(n, 1)$ (cf. [1]). The *full* group C^*-algebra $C^*(\Gamma)$ is not K-exact (G. Skandalis [20]).

(2) The reduced group C^*-algebra of a Gromov non-exact group is not K-exact (N. Ozawa, see [9, Remark 13]).

(3) The product $\prod_{n=0}^{\infty} M_n$ is not *K-exact* (N. Ozawa [15, Theorem A.1]).

(4) Let Γ be a finitely generated discrete group with property $\tau(\mathscr{L})$ (cf. [21, Definition 9] or [14]). If Γ has property $\tau(\mathscr{L}')$, where Let $\mathscr{L}' := \{N_1 \cap N_2 \mid N_1, N_2 \in \mathscr{L}\}$, then the uniform Roe algebra $C_u^*(X)$ of the expander $X := \sqcup_{N\in\mathscr{L}} \Gamma/N$ is not K-exact (J. Špakula [21, Theorem 2]).

DEFINITION 3.7. Let $C_0[0, 1)$ denote the commutative C^*-algebra of continuous functions on the interval $[0, 1]$ vanishing at $1 \in [0, 1]$,

$$\mathrm{ev}_0\colon \quad \begin{aligned} C_0[0,1) &\longrightarrow \mathbb{C} \\ f &\longmapsto f(0) \end{aligned}$$

denote the evaluation map at $0 \in [0, 1)$.

DEFINITION 3.8. Let $\phi\colon D \to B$ be a $*$-homomorphism of C^*-algebras. The *mapping cone* C_ϕ of ϕ is given by the pullback

$$\begin{array}{ccc} C_\phi & \longrightarrow & C_0[0,1) \otimes B \\ \downarrow & & \downarrow \scriptstyle{\mathrm{ev}_0 \otimes \mathrm{id}_B} \\ D & \xrightarrow{\ \phi\ } & B \end{array}$$

REMARK 3.9. Let $\phi\colon D \to B$ be a $*$-homomorphism of C^*-algebras. Then for any C^*-algebra A, there is a natural isomorphism $C_{\mathrm{id}_A \otimes \phi} \cong A \otimes C_\phi$. Indeed, since $\mathrm{ev}_0\colon C_0[0, 1) \to \mathbb{C}$ is admits a completely positive section, we have a map of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \otimes C_0(0,1) \otimes B & \longrightarrow & A \otimes C_\phi & \longrightarrow & A \otimes D & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \scriptstyle{\mathrm{id}_A \otimes \phi} & & \\ 0 & \longrightarrow & A \otimes C_0(0,1) \otimes B & \longrightarrow & A \otimes C_0[0,1) \otimes B & \longrightarrow & A \otimes B & \longrightarrow & 0, \end{array}$$

where $C_0(0, 1)$ denotes the kernel of ev_0. Now it is easy to see that the square on the right is a pullback square.

LEMMA 3.10 (cf. [9]). *A C^*-algebra A is K-exact if and only if for every extension*

$$0 \longrightarrow I \longrightarrow D \xrightarrow{q} B \longrightarrow 0$$

of separable C^*-algebras, the natural inclusion map $\iota\colon I \longrightarrow C_q$ induces an isomorphism

$$(\mathrm{id}_A \otimes \iota)_*\colon K_0(A \otimes I) \cong K_0(A \otimes C_q).$$

PROOF. Let A be a C^*-algebra and let

$$0 \longrightarrow I \longrightarrow D \xrightarrow{q} B \longrightarrow 0 \tag{3.1}$$

be an extension of (not necessarily separable) C^*-algebras.

(⇒): Suppose that A is K-exact. By the homotopy invariance of K-theory, we have $K_*(A \otimes C_0[0,1) \otimes B) = 0$. Hence, applying Remark 3.2 to the pullback extension

$$0 \longrightarrow I \xrightarrow{\iota} C_q \longrightarrow C_0[0,1) \otimes B \longrightarrow 0,$$

we see that $\mathrm{id}_A \otimes \iota$ induces an isomorphism $K_0(A \otimes I) \cong K_0(A \otimes C_q)$.

(⇐): Conversely, suppose that A satisfies the necessary condition in the lemma. We prove that the sequence

$$K_0(A \otimes I) \longrightarrow K_0(A \otimes D) \longrightarrow K_0(A \otimes B)$$

is exact in the middle.

If I, D and B are separable, then the exactness follows from the Puppe exact sequence (cf. [17] or [25, Lemma 6.4.8]) and the natural isomorphism $C_{\mathrm{id}_A \otimes q} \cong A \otimes C_q$ of Remark 3.9.

The general case is reduced to the separable case as follows. Let Λ denote the set of separable C^*-subalgebras of D, ordered by inclusion. Then Λ is a directed set. For each $E \in \Lambda$, we associate a subextension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \cap E & \longrightarrow & E & \longrightarrow & E/(I \cap E) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & D & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

It is clear that the inductive limit of the subextensions is the extension (3.1). Since all the connecting maps are injective, the proof is complete by the continuity of K-theory (cf. [25, Proposition 6.2.9]) and the exactness of the inductive limit functor for abelian groups (cf. [26, Theorem 2.6.15]).

The following result is inspired by the proof of [11, Theorem 5.4].

PROPOSITION 3.11. *A C^*-algebra A is K-exact if and only if the functor $B \mapsto K_0(A \otimes B)$, from the category of* separable *C^*-algebras to abelian groups, factors through the category E of Higson (cf. [10], [5]).*

PROOF. Let A be a C^* and let $F(B) := K_0(A \otimes B)$.

($\Rightarrow$): Suppose that A is K-exact. Then F is half-exact and since F is homotopy invariant and stable (under tensoring with the compacts), it factors through the category E by the universal property (cf. [5, Théorème 7]).

($\Leftarrow$): Suppose that F factors through E. For any extension

$$0 \longrightarrow I \longrightarrow D \xrightarrow{q} B \longrightarrow 0$$

of separable C^*-algebras, the inclusion $\iota\colon I \to C_q$ is an equivalence in E (cf. [5, Lemma 12]). Now Lemma 3.10 completes the proof.

3.2. K-continuity

DEFINITION 3.12. We say that a C^*-algebra A is *K-continuous* if for every inductive sequence

$$B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \dots$$

of C^*-algebras, the natural map

$$\varinjlim K_0(A \otimes B_i) \longrightarrow K_0(A \otimes \varinjlim B_i)$$

is an isomorphism.

REMARK 3.13. In Definition 3.12, we could use K_1 instead of K_0.

EXAMPLE 3.14. $\otimes$-continuous C^*-algebras are K-continuous, by the continuity of K-theory (cf. [25, Proposition 6.2.9], [2, 5.2.4, 8.1.5]).

EXAMPLE 3.15. C^*-algebras satisfying the Künneth formula of Schochet (cf. [18], [2, Theorem 23.1.3]) are K-continuous. Indeed, let A be a C^*-algebra satisfying the Künneth formula and let

$$B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \dots$$

be an inductive sequence of C^*-algebras. Then the top row in the diagram

$$\begin{array}{ccccccccc} 0 \to & \varinjlim K_*(A) \otimes K_*(B_i) & \to & \varinjlim K_*(A \otimes B_i) & \to & \varinjlim \operatorname{Tor}(K_*(A), K_*(B_i)) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & K_*(A) \otimes K_*(\varinjlim B_i) & \to & K_*(A \otimes \varinjlim B_i) & \to & \operatorname{Tor}(K_*(A), K_*(\varinjlim B_i)) & \to 0 \end{array}$$

is an extension of abelian groups by [26, Theorem 2.6.15] and the second row is an extension by the Künneth formula. The left and right vertical maps are isomorphisms by [26, Corollary 2.6.17] and thus the middle vertical map is also an isomorphism by five-lemma. Hence A is K-continuous.

3.3. *Proof of Main Theorem 1.3*

PROOF OF THEOREM 1.3. (⇒): Let A be a K-exact C^*-algebra and let

$$B_0 \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \dots$$

be an inductive sequence. We use the notations of the proof of Theorem 1.2 (⇒). Applying K-theory to the diagram (2.2), we get a map of *exact* sequences

$$\begin{array}{ccccccccc}
K_0(\varinjlim A\otimes I_n) & \to & K_0(\varinjlim A\otimes D_n) & \to & K_0(\varinjlim A\otimes B_n) & \to & K_1(\varinjlim A\otimes I_n) & \to & K_1(\varinjlim A\otimes D_n) \\
\downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\beta_*} & & \downarrow{\cong} & & \downarrow{\cong} \\
K_0(A\otimes \varinjlim I_n) & \to & K_0(A\otimes \varinjlim D_n) & \to & K_0(A\otimes \varinjlim B_n) & \to & K_1(A\otimes \varinjlim I_n) & \to & K_1(A\otimes \varinjlim D_n)
\end{array}$$

By five-lemma, the map β_* is an isomorphism.

(⇐): Conversely, let A be a K-continuous C^*-algebra. Then the functor $F(B) := K_0(A \otimes B)$, considered on the category of separable C^*-algebras, factors through the asymptotic homotopy category of Connes-Higson by [7, Theorem 3.11]. Since F is stable and satisfies Bott periodicity, it in fact factors through the category E. Hence by Proposition 3.11, A is K-exact.

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