MATH. SCAND. 118 (2016), 119–151

AFFINE MODULES AND THE DRINFELD CENTER

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Abstract

Given a finite index subfactor, we show that the *affine morphisms at zero level* in the affine category over the planar algebra associated to the subfactor is isomorphic to the fusion algebra of the subfactor as a $*$-algebra. This identification paves the way to analyze the structure of affine P-modules with weight zero for any subfactor planar algebra P (possibly having infinite depth). Further, for irreducible depth two subfactor planar algebras, we establish an additive equivalence between the category of affine P-modules and the center of the category of N-N-bimodules generated by $L^2(M)$; this partially verifies a conjecture of Jones and Walker.

1. Introduction

The standard invariant of a subfactor, which – in certain situations turns out to be a complete invariant – has been described in many seemingly different ways, for instance as a certain category of bimodules (see [1]), as lattices of finite dimensional C^* algebras satisfying certain properties (see [18]), as an algebraic system comprising of graphs, fusion rules and quantum $6j$ symbols (see [16]) or as a planar algebra (see [9]). In fact, the theory of planar algebras was initiated by Jones as a tool to study subfactors. The graphical calculus of pictures on a plane turned out to be extremely handy in analyzing the combinatorial data present in a subfactor. Although intimately connected to the theory of subfactors from the outset, planar algebra soon became a subject in its own merit. Moreover, quite recently it has found connections with the theories of random matrices and free probability as well, see [8].

Further, in [10], Jones introduced the notion of 'modules over a planar algebra' or 'annular representations', wherein he explicitly obtained all the irreducible modules over the Temperley-Lieb planar algebras for index greater than 4. Modules over planar algebras have been used in constructing subfactors of index less than 4, namely the subfactors with principal graphs, E_6 and E_8, see [10]. More recently, they have also found application in constructing the Haagerup subfactor, see [17]. Such modules for the group planar algebras were

* Das and Ghosh were supported by K. U. Leuven BOF Research Grant OT/08/032, and Gupta was supported by ERC Starting Grant VNALG-200749 and the Institute of Mathematical Sciences (Chennai).

Received 14 August 2013.

studied by the second-named author in [6] where an equivalence (as additive categories) was established between the category of annular representations over a group planar algebra (that is, planar algebra associated to the fixed point subfactor arising from an outer action of a finite group) and the representation category of a non-trivial quotient of the quantum double of the group, over a certain ideal. The appearance of a non-trivial quotient was due to the fact that the isotopy on annular tangles need not preserve the boundaries of the external and the distinguished internal discs. On the other hand, *affine isotopy* (introduced in [12]) does preserve the boundaries of the annulus; in fact, the category of affine modules of a group planar algebra becomes equivalent to the representation category of the quantum double of the group. Affine modules for the Temperley-Lieb planar algebras were studied in [12]. Certain finiteness results for affine modules of finite depth planar algebras were also established in [7].

The work in this paper was motivated by an attempt to understand the subfactor analogue of a conjecture made by Kevin Walker in the world of TQFTs, see [5]. Its analogue in the theory of subfactors was suggested by Vaughan Jones as follows:

> *The category of affine representations of a finite depth subfactor planar algebra is equivalent to the Drinfeld center of the bimodule category associated to the subfactor.*

This is evident in the case of group planar algebra where the center is equivalent to the representation category of the quantum double of the group. This conjecture can be important from various angles. If it is true, one can hope to use these tools (namely, affine representations) to obtain the quantum invariant of the fusion category associated to a finite depth subfactor. It would also be interesting to investigate the case of infinite depth subfactors and verify a generalized version of the conjecture.

As a step towards this conjecture, we first established an isomorphism between the affine morphisms *at zero level* (defined at the beginning of Section 3) and the fusion algebra of the bimodule category; as suggested to the second named author by Vaughan Jones and Dietmar Bisch. Later, this helped us in constructing affine modules with weight zero. Moreover, we verify the above conjecture in the case of irreducible depth two subfactors.

We now briefly describe the organization of this paper.

Section 2 begins with a brief recollection (mainly from [9] and [3]) of certain basic aspects of planar algebras and their relationship with subfactors and setting up some notation. For the sake of completeness, in the second part of Section 2, we present a detailed description of the affine category over a planar algebra.

Section 3 is devoted to proving one of the main theorems in this article, namely Theorem 3.1, the proof of which is divided in three subsections. In the first part, we find a nice spanning set (indexed by the isomorphism classes of irreducible bimodules appearing in the standard invariant of the subfactor) for the space of affine morphisms at zero level. Here, we crucially use a family of affine tangles, namely, the $\Psi^m_{\varepsilon k,\eta l}$'s and the fact that any affine morphism comes from the action of one of these affine tangles on regular tangles. In the second part, we obtain an equivalence relation on planar tangles induced by the effect of affine isotopy. We use this equivalence relation to show the linear independence of the spanning set, in the last part.

The canonical trace in the fusion algebra induces, via Theorem 3.1, a faithful tracial state on the space of affine morphisms at zero level. In Section 4, we first give a pictorial formulation of this trace. With this faithful trace at our disposal, we consider the left regular representation of the affine morphisms at zero level, which immediately produces a canonical pair of Hilbert affine P-modules (which we call *regular*); here P is not assumed to be of finite depth. Interestingly, in the case of finite depth subfactor planar algebras, it turns out that any weight zero irreducible Hilbert affine P-module is isomorphic to a submodule of one of the above regular Hilbert affine P-modules. We next analyze the finite von Neumann algebras generated by the affine morphisms at zero level in their GNS representations with respect to the faithful traces considered above. Moreover, to every left module over these von Neumann algebras, we uniquely associate a Hilbert affine P-module with weight zero; we use Connes fusion techniques and the above regular Hilbert affine P-modules at zero level for these constructions.

Section 5 deals with the study of Hilbert affine modules over irreducible depth two subfactor planar algebras P which (by the Ocneanu-Szymanski theorem [19]) basically arise from actions of finite dimensional Kac algebras, the skein theory of which has been described in [15], [4]. We recall the structure maps of the Kac algebras coming from P and the definition of the quantum double DH of a finite dimensional Hopf $*$-algebra H (from [14]). We then construct an explicit isomorphism between the quantum double of P_{+2} and the affine morphism space at level one, $AP_{+1,+1}$. Using this isomorphism and the normalized Haar functional on DP_{+2}, we build a Hilbert affine P-module V which is generated by its 1 space $V_1 = AP_{+1,+1}$ and contains all irreducible Hilbert affine P-modules. This gives a one-to-one correspondence between the isomorphism classes of irreducible Hilbert affine P-modules and that of $V_1 \cong DP_{+2}$. Thus, we prove the Jones-Walker conjecture in the case of irreducible depth two subfactors. We end this article with some questions related to the monoidal structure on affine modules and the Jones-Walker conjecture in a more general case.

ACKNOWLEDGEMENTS. The second named author would like to thank Vaughan Jones for suggesting the problem of Section 3 and also for patiently listening to a preliminary version of a proof of Theorem 3.1 during the von Neumann Algebras meeting at Oberwolfach in 2008.

2. Preliminaries

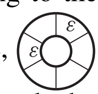
2.1. Some useful notation

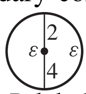
Like any other article on planar algebras, this paper will also be full of pictorial calculations, it thus makes sense to have a convenient notation that would simplify diagrams. Keeping this in mind, we will freely borrow some notation from [3] which we briefly recall below.

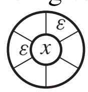
We will not give the definition of planar algebra which can be found in [9]; however, we will be consistent with the notation described in [7], [3].

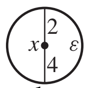
(1) We will consider the natural binary operation on $\{-,+\}$ given by $++ := +$, $+- := -$, $-+ := -$ and $-- := +$. Notation such as $(-)^l$ has to be understood in this context.

(2) We will denote the set of all possible colors of discs in tangles by $\mathrm{Col} := \{\varepsilon k : \varepsilon \in \{+,-\}, k \in \mathsf{N}_0\}$ where $\mathsf{N}_0 := \mathsf{N} \cup \{0\}$.

(3) In a tangle, we will replace (isotopically) parallel strings by a single strand labelled by the number of strings, and an internal disc with color εk will be replaced by a bold dot with the sign ε placed at the angle corresponding to the distinguished boundary components of the disc. For example, 

will be replaced by 

. In a similar token, if P is a planar algebra, we will replace a P-labelled internal disc by a bold dot with the label being placed at the angle corresponding to the distinguished boundary component of the disc; for instance, 

will be replaced by 

where $x \in P_{\varepsilon 3}$. We will reserve alphabets like x, y, z to denote elements of P, ε, η, ν to denote a sign, and k, l, m to denote a natural number to avoid confusion. It should be clear from the context what a bold dot or a string in a picture is labelled by.

(4) We set some notation for a set of 'generating tangles' (that is, tangles which generate all tangles by composition) in Figure 1.

(5) $\mathcal{T}_{\varepsilon k}$ (resp., $\mathcal{T}_{\varepsilon k}(P)$) will denote the set of tangles (resp., P-labelled tangles) which has εk as the color of the external disc; $\mathcal{P}_{\varepsilon k}(P)$ will

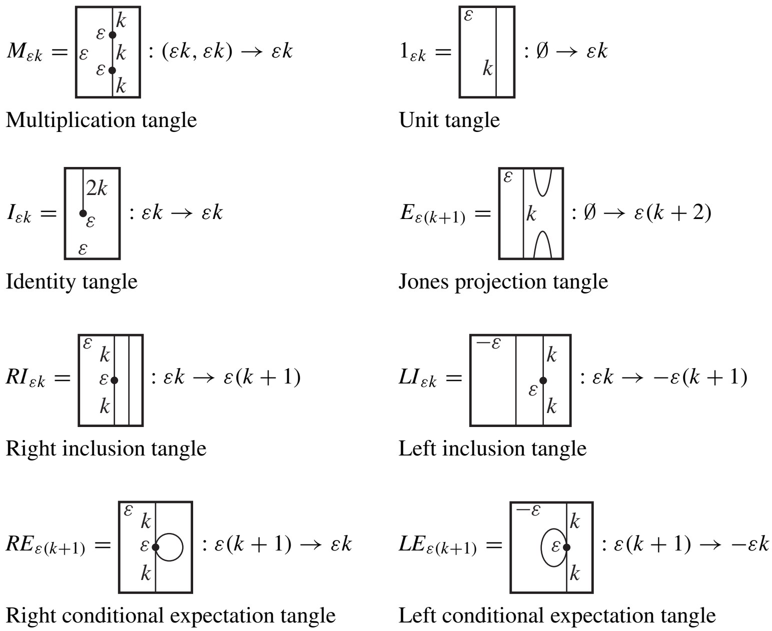


FIGURE 1. Generating tangles

be the vector space with $\mathcal{T}_{\varepsilon k}(P)$ as a basis. The action of P induces a linear map $\mathcal{P}_{\varepsilon k}(P) \ni T \overset{P}{\longmapsto} P_T \in P_{\varepsilon k}$.

We now recall (from [9]) the notion of the *n-th cabling* of a planar algebra P with modulus (δ_-, δ_+), denoted by $c_n(P)$. For a tangle T, let $c_n(T)$ be the tangle obtained by (a) replacing every string by n many strings parallel to it, and (b) putting n consecutive caps on the distinguished boundary component of every negatively signed (internal or external) disc and around the minus sign which is then replaced by a plus sign.

Vector spaces: For all colors εk, $c_n(P)_{\varepsilon k} := \operatorname{Range}(P_{c_n(I_{\varepsilon k})})$.

Action of tangles: For all tangles T, $c_n(P)_T := \left[\prod_{l=1}^{n} \delta_{(-)^l}\right]^{-w} P_{c_n(T)}$ where w is the number of negatively signed internal disc(s) of T.

Note that $c_1(P)$ is isomorphic to P, $c_m(c_n(P)) = c_{mn}(P)$ and $c_n(P)$ has modulus $\left(\prod_{l=1}^{n} \delta_{(-)^l}, \prod_{l=1}^{n} \delta_{(-)^{l+1}}\right)$.

2.2. *Planar algebras and subfactors*

In this section, we will recall certain basic facts about subfactors and its interplay with planar algebras. For the rest of this section, let $M_{-1} := N \subset M =: M_0$ be a subfactor with $\delta^2 := [M : N] < \infty$ $(\delta > 0)$ and $\{M_k\}_{k \geq 1}$ be a tower of basic constructions with $\{e_k \in \mathscr{P}(M_k)\}_{k\geq 1}$ being a set of Jones projections. Borrowing notation from [13], for each $k \geq 1$, set $e_{[-1,k]} := \delta^{k(k+1)}(e_{k+1}e_k \cdots e_1)(e_{k+2}e_{k+1} \cdots e_2) \cdots (e_{2k+1}e_{2k} \cdots e_{k+1}) \in N' \cap M_{2k+1}$, $e_{[0,k]} := \delta^{k(k-1)}(e_{k+1}e_k \cdots e_2)(e_{k+2}e_{k+1} \cdots e_3) \cdots (e_{2k}e_{2k-1} \cdots e_{k+1}) \in M' \cap M_{2k}$ and $v_k := \delta^k e_k e_{k-1} \cdots e_1 \in N' \cap M_k$. Then, the tower of II_1 factors $N \subset M_k \subset M_{2k+1}$ (resp., $M \subset M_k \subset M_{2k}$) is an instance of basic construction with $e_{[-1,k]}$ (resp., $e_{[0,k]}$) as Jones projection, that is, there exists an isomorphism $\varphi_{-1,k} : M_{2k+1} \longrightarrow \mathcal{L}_N(L^2(M_k))$ (resp., $\varphi_{0,k} : M_{2k} \longrightarrow \mathcal{L}_M(L^2(M_k))$) given by

$$\varphi_{-1,k}(x_{2k+1})\hat{x}_k = \delta^{2(k+1)} \widehat{E_{M_k}(x_{2k+1}x_k e_{[-1,k]})}$$
$$\text{(resp.,} \quad \varphi_{0,k}(x_{2k})\hat{x}_k = \delta^{2k} \widehat{E_{M_k}(x_{2k}x_k e_{[0,k]})}\,)$$

for all $x_i \in M_i$, $i = k, 2k, 2k+1$, which is identity restricted to M_k and sends $e_{[-1,k]}$ (resp., $e_{[0,k]}$) to the projection with range $L^2(N)$ (resp., $L^2(M)$). Also, $\varphi_{-1,k}(M_i' \cap M_{2k+1}) = {}_{M_i}\mathcal{L}_N(L^2(M_k))$ (resp., $\varphi_{0,k}(M_i' \cap M_{2k}) = {}_{M_i}\mathcal{L}_M(L^2(M_k))$) and $\varphi_{0,k} = \varphi_{-1,k}\big|_{M_{2k}}$ for all $k \geq 0$, $-1 \leq i \leq k$.

We now state the 'extended Jones' theorem' which provides an important link between finite index subfactors and planar algebras. This was first established for extremal finite index subfactors in [9]. Later, it was extended to arbitrary finite index subfactors in [2], [11], [3]. As mentioned above, we will follow the set up of [3].

THEOREM 2.1. *P defined by $P_{\varepsilon k} = N' \cap M_{k-1}$ or $M' \cap M_k$ according as $\varepsilon = +$ or $-$, has a unique unimodular bimodule planar algebra structure with the $*$-structure given by the usual $*$ of the relative commutants such that for each $k \in \mathsf{N}_0$,*

(1) *the action of multiplication tangles is given by the usual multiplication in the relative commutants,*

(2) *the action of the left inclusion tangle LI_{-k} is given by the usual inclusion $M' \cap M_k \subset N' \cap M_k$,*

(3) *the action of the right inclusion tangle RI_{+k} is given by the usual inclusion $M_{k-1} \subset M_k$,*

(4) $P_{E_{+(k+1)}} = \delta e_{k+1}$,

(5) $P_{LE_{+(k+1)}} = \delta^{-1} \sum_i b_i^* x b_i$ *for all* $x \in P_{+(k+1)}$,

where $\{b_i\}_i$ is a left Pimsner-Popa basis for the subfactor $N \subset M$. (P will be referred as the planar algebra associated to the tower $\{M_k\}_{k\geq -1}$ with Jones projections $\{e_k\}_{k\geq 1}$.)

REMARK 2.2. Apart from the action of the tangles given in conditions (1)–(5), it is also worth mentioning the actions of a few other useful tangles, namely,

(a) $P_{RE_{+k}} = \delta E^{M_{k-1}}_{M_{k-2}}\big|_{P_{+k}}$,

(b) $P_{TR^r_{+k}} = \delta^k \operatorname{tr}_{M_{k-1}}\big|_{P_{+k}}$,

(c) $\delta^{-k} P_{TR^l_{+2l}}$ (resp., $\delta^{-k} P_{TR^l_{+(2l-1)}}$) is given by the trace on $P_{+2l} = N' \cap M_{2l-1}$ (resp., $P_{+(2l-1)} = N' \cap M_{2l-2}$) induced by the canonical trace on ${}_N\mathcal{L}(L^2(M_{l-1}))$ via the map $\varphi_{-1,l-1}$ (resp., $\varphi_{0,l-1}$) where $TR^l_{\varepsilon k}$ (resp., $TR^r_{\varepsilon k}$) denotes the left (resp., right) trace tangle as described in Figure 2.

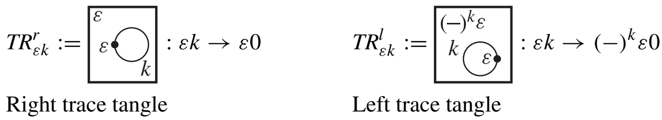


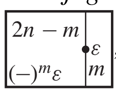
FIGURE 2. Trace tangles

COROLLARY 2.3. (a) $P_{E'_{-k}}(y) = \delta \sum_i b_i^* e_1 y e_1 b_i$ *for all* $y \in P_{-k} = M' \cap M_k$, *where* $E'_{-k} = LI_{+(k-1)} \circ LE_{-k}$ *and* $\{b_i\}_i$ *is a left Pimsner-Popa basis for* $N \subset M$,

(b) *the n-th dual of P,* $\lambda_n(P) =$ *the planar algebra associated to the tower* $\{M_{k+n}\}_{k\geq -1}$ *with Jones projections* $\{e_{k+n}\}_{k\geq 1}$.

If $e_{[l,k+l]}$ denotes the projection obtained by replacing each $e_\bullet$ in the defining equation of $e_{[0,k]}$ (as above), by $e_{l+\bullet}$, then $M_l \subset M_{k+l} \subset M_{2k+l}$ is an instance of basic construction with $e_{[l,k+l]}$ as Jones projection.

REMARK 2.4. An easy consequence of Corollary 2.3 (b) and Theorem 2.1 is $c_n(P) =$ the planar algebra associated to the tower $\{M_{n(k+1)-1}\}_{k\geq -1}$ with Jones projections $\{e_{[n(k-1)-1,nk-1]}\}_{k\geq 1}$.

PROPOSITION 2.5. *If* J_k *denotes the canonical conjugate-linear unitary operator on* $L^2(M_k)$ *and* $R^m_{\varepsilon n}$ *denotes the tangle* , *then for all* $k \geq 0$, *we have:*

(a) $\varphi_{-1,k}\big(P_{R^{2k+2}_{+(2k+2)}}(x)\big) = J_k \varphi_{-1,k}(x^*) J_k$ *and*

$$\operatorname{Range} \varphi_{-1,k}\big(P_{R^{2k+2}_{+(2k+2)}}(p)\big) \overset{N\text{-}N}{\cong} \overline{\operatorname{Range} \varphi_{-1,k}(p)}$$

for all $x \in P_{+(2k+2)}$, $p \in \mathscr{P}(P_{+(2k+2)})$,

(b) $\varphi_{-1,k}\big(P_{R^{2k+1}_{+(2k+1)}}(x)\big) = J_k \varphi_{0,k}(x^*) J_k$ *and*

$$\operatorname{Range} \varphi_{-1,k}\big(P_{R^{2k+1}_{+(2k+1)}}(p)\big) \overset{M\text{-}N}{\cong} \overline{\operatorname{Range} \varphi_{-1,k}(p)}$$

for all $x \in P_{+(2k+1)}$, $p \in \mathscr{P}(P_{+(2k+1)})$,

(c) $\varphi_{0,k}\big(P_{R^{2k+1}_{-(2k+1)}}(x)\big) = J_k\varphi_{-1,k}(x^*)J_k$ *and*

$$\text{Range}\,\varphi_{0,k}\big(P_{R^{2k+1}_{-(2k+1)}}(p)\big) \overset{N\text{-}M}{\cong} \overline{\text{Range}\,\varphi_{-1,k}(p)}$$

for all $x \in P_{-(2k+1)}$, $p \in \mathscr{P}(P_{-(2k+1)})$,

(d) $\varphi_{0,k}\big(P_{R^{2k}_{-2k}}(x)\big) = J_k\varphi_{0,k}(x^*)J_k$ *and*

$$\text{Range}\,\varphi_{0,k}\big(P_{R^{2k}_{-2k)}}(p)\big) \overset{M\text{-}M}{\cong} \overline{\text{Range}\,\varphi_{0,k}(p)}$$

for all $x \in P_{-2k}$, $p \in \mathscr{P}(P_{-2k})$.

PROOF. The isomorphism in the second part in each of (a), (b), (c) and (d), follows from the first part using [1, Proposition 3.11]. For the first parts, it is enough to establish only for (a) because all others can be deduced using conditions (2) and (3) in Theorem 2.1, and the relation $\varphi_{0,k} = \varphi_{-1,k}\big|_{M_{2k}}$.

First, we will prove part (a) for $k = 0$. Note that if $\{b_i\}_i$ is a left Pimsner-Popa basis for $N \subset M$, then

$$\begin{aligned} P_{R^2_{+2}}(x) = P_{R^1_{-2}}\big(P_{R^1_{+2}}(x)\big) &= P_{R^1_{-2}}\left(\delta \sum_i b_i^* x e_2 e_1 b_i\right) \\ &= \delta^4 \sum_i E_{M_1}(e_2e_1b_i^*xe_2e_1b_i) \end{aligned}$$

where we use the conditions of Theorem 2.1 in a decomposition of the rotation tangle into the generating ones $R^1_{+2} = LE_{+3} \circ M_{+3}(RI_{+2}, M_{+3}(E_{+2}, RI_{+2} \circ E_{+1}))$ (resp., $RE_{+3} \circ M_{+3}(M_{+3}(E_{+2}, RI_{+2} \circ E_{+1}), LI_{-2})$) for establishing the second (resp., third) equality. For $y \in M$, note that

$$\begin{aligned} \varphi_{-1,0}\big(P_{R^2_{+2}}(x)\big)\hat{y} &= \delta^6 \sum_i E_M(e_2e_1b_i^*xe_2e_1b_iye_1)\hat{} \\ &= \delta^6 E_M(e_2e_1yxe_2e_1)\hat{} = \delta^2 E_M(e_1yx)\hat{} \\ &= J_0\varphi_{-1,0}(x^*)J_0\hat{y}. \end{aligned}$$

Now, let $k > 0$. Using the above and Remark 2.4, we obtain

$$\varphi_{-1,k}\big(P_{R^{2k+2}_{+(2k+2)}}(x)\big) = \varphi_{-1,k}\big(c_{k+1}(P)_{R^2_{+2}}(x)\big) = J_k\varphi_{-1,k}(x^*)J_k.$$

We will make repeated use of the following standard facts, whose proof can be found in [1].

LEMMA 2.6 ([1]). *For each* $k \geq 0$*, and* $X \in \{N, M\}$*, we have:*

(1) $\text{Range}\,\varphi_{-1,k}(p) \overset{X\text{-}N}{\cong} \text{Range}\,\varphi_{-1,k+1}(pe_{2k+3})$ *for all* $p \in \mathscr{P}(X' \cap M_{2k+1})$,

(2) $\text{Range}\,\varphi_{0,k}(p) \overset{X\text{-}M}{\cong} \text{Range}\,\varphi_{0,k+1}(pe_{2k+2})$ *for all* $p \in \mathscr{P}(X' \cap M_{2k})$.

From this, one can easily deduce the following.

COROLLARY 2.7. *For* $k > l \geq 0$ *and* $X \in \{N, M\}$, *the following holds:*

(1) *For all* $p \in \mathscr{P}(X' \cap M_{2k+1})$ *and* $q \in \mathscr{P}(X' \cap M_{2l+1})$ *satisfying*

$$\text{Range}\,\varphi_{-1,k}(p) \overset{X\text{-}N}{\cong} \text{Range}\,\varphi_{-1,l}(q),$$

p is MvN-equivalent to $qe_{2l+3}\cdots e_{2k+1}$ *in* $X' \cap M_{2k+1}$.

(2) *For all* $p \in \mathscr{P}(X' \cap M_{2k})$ *and* $q \in \mathscr{P}(X' \cap M_{2l})$ *satisfying*

$$\text{Range}\,\varphi_{0,k}(p) \overset{X\text{-}M}{\cong} \text{Range}\,\varphi_{0,l}(q),$$

p is MvN-equivalent to $qe_{2l+2}\cdots e_{2k}$ *in* $X' \cap M_{2k}$.

2.3. Affine Category over a Planar Algebra

In this subsection, for the sake of self containment, we recall (from [7]) in some detail what we mean by the *affine category over a planar algebra* and the corresponding *affine morphisms* (with slight modifications).

DEFINITION 2.8. For each $\varepsilon, \eta \in \{+, -\}$ and $k, l \geq 0$, an $(\varepsilon k, \eta l)$-*affine tangular picture* consists of the following:

- finitely many (possibly none) non-intersecting subsets $D_1, \ldots, D_b$ (referred as *discs*) of the interior of the rectangular annular region $RA := [-2, 2] \times [-2, 2] \setminus (-1, 1) \times (-1, 1)$, each of which is homeomorphic to the unit disc and has even number of marked points on its boundary, numbered clockwise,
- non-interescting paths (called *strings*) in $RA \setminus \left[\bigsqcup_{i=1}^{b} \text{Int}(D_i)\right]$, which are either loops or meet the boundaries of the discs or RA exactly at two distinct points in $\left\{\left(\frac{i}{2k}, 1\right) : 0 \leq i \leq 2k-1\right\} \sqcup \left\{\left(\frac{j}{2l}, 2\right) : 0 \leq j \leq 2l-1\right\} \sqcup \{\text{marked points on the discs}\}$ in such a way that every point in this set must be an endpoint of a string,
- a checker-board shading on the connected components of $\text{Int}(RA) \setminus \left[\left(\bigsqcup_{i=1}^{1} D_i\right) \cup \{\text{strings}\}\right]$ such that the component near the point $(0, -1)$ (resp., $(0, -2)$) is unshaded or shaded according as ε (resp., η) is $+$ or $-$.

DEFINITION 2.9. An *affine isotopy* of an affine tangular picture is a map $\varphi\colon [0, 1] \times RA \to RA$ such that

(1) $\varphi(t, \cdot)$ is a homeomorphism of RA, for all $t \in [0, 1]$;

(2) $\varphi(0, \cdot) = \mathrm{id}_{RA}$; and

(3) $\varphi(t, \cdot)\big|_{\partial(RA)} = \mathrm{id}_{\partial(RA)}$ for all $t \in [0, 1]$.

Two affine tangular pictures are said to be *affine isotopic* if one can be obtained from the other using an affine isotopy preserving checker-board shading and the distinguished boundary components of the discs. It may be noted here that condition (3) in Definition 2.9 distinguishes affine isotopy from *annular isotopy* (see [12], [7]).

DEFINITION 2.10. An $(\varepsilon k, \eta l)$-*affine tangle* is the affine isotopy class of an $(\varepsilon k, \eta l)$-affine tangular picture.

Time and again, for the sake of convenience, we will abuse terminology by referring to an affine tangular picture as an affine tangle (corresponding to its affine isotopy class) and the figures might not be sketched to the scale but are clear enough to avoid any ambiguity. In Figure 3, for each $\varepsilon k, \eta l \in \mathrm{Col}$ and $m \in \mathsf{N}_{\varepsilon,\eta} := 2\mathsf{N}_0 + \delta_{\varepsilon,-\eta}$ (where δ represents the Kronecker delta), we draw a specific affine tangle called $\Psi^m_{\varepsilon k,\eta l}$, where the labels next to strings have the same significance as that explained in (3) of Section 2.1. This affine tangle will play an important role in the following discussions.

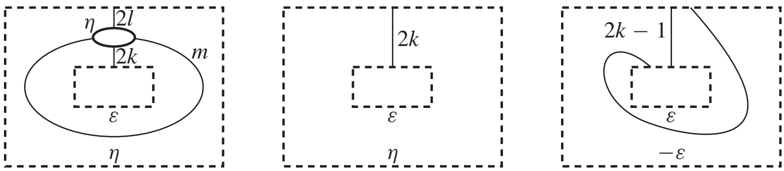


FIGURE 3. Some useful affine tangles. $(\varepsilon k, \eta l \in \mathrm{Col}, k, l \in \mathsf{N}_0, m \in \mathsf{N}_{\varepsilon,\eta})$

NOTATION. For each $\varepsilon, \eta \in \{+, -\}$ and $k, l \geq 0$, let

- $\mathcal{AT}_{\varepsilon k,\eta l}$ denote the set of all $(\varepsilon k, \eta l)$-affine tangles, and
- $\mathcal{A}_{\varepsilon k,\eta l}$ denote the complex vector space with $\mathcal{AT}_{\varepsilon k,\eta l}$ as a basis.

The composition of affine tangles $T \in \mathcal{AT}_{\varepsilon k,\eta l}$ and $S \in \mathcal{AT}_{\xi m,\varepsilon k}$ is given by $T \circ S := \frac{1}{2}(2T \cup S) \in \mathcal{AT}_{\xi m,\eta l}$ (diagrammatically which just amounts to plugging in S in the distinguished internal rectangle of T and erasing the boundary); this composition is linearly extended to the level of the vector spaces $\mathcal{A}_{\varepsilon k,\eta l}$'s.

The following pictorial observation (see [6]) comes in extremely handy, while working with affine morphisms.

REMARK 2.11. For each $A \in \mathcal{AT}_{\varepsilon k,\eta l}$, there exists $m \in \mathsf{N}_{\varepsilon,\eta}$ and $T \in \mathcal{T}_{\eta(k+l+m)}$ such that $A = \Psi^m_{\varepsilon k,\eta l}(T)$ where $\Psi^m_{\varepsilon k,\eta l}(T)$ is the isotopy class of the affine tangular picture obtained by inserting T in the disc of $\Psi^m_{\varepsilon k,\eta l}$.

In the above remark, the m can be chosen as large as one wants and the insertion method extends linearly to a linear map $\Psi^m_{\varepsilon k,\eta l} : \mathcal{P}_{\eta(k+l+m)} \to \mathcal{A}_{\varepsilon k,\eta l}$, and for each $A \in \mathcal{A}_{\varepsilon k,\eta l}$, there is an $m \in \mathsf{N}_0$ and an $X \in \mathcal{P}_{\eta(k+l+m)}$ such that $A = \Psi^m_{\varepsilon k,\eta l}(X)$. Let P be a planar algebra. An $(\varepsilon k, \eta l)$-affine tangle is said to be *P-labelled* if each disc is labelled by an element of $P_{\nu m}$ where νm is the color of the corresponding disc. Let $\mathcal{AT}_{\varepsilon k,\eta l}(P)$ denote the collection of all *P-labelled* $(\varepsilon k, \eta l)$-affine tangles, and let $\mathcal{A}_{\varepsilon k,\eta l}(P)$ be the vector space with $\mathcal{AT}_{\varepsilon k,\eta l}(P)$ as a basis. Composition of *P-labelled* affine tangles also makes sense as above and extends to their complex span. Note that $\Psi^m_{\varepsilon k,\eta l}$ also induces a linear map from $\mathcal{P}_{\eta(k+l+m)}(P)$ into $\mathcal{A}_{\varepsilon k,\eta l}(P)$. Moreover, from 2.11, we may conclude that, for each $A \in \mathcal{A}_{\varepsilon k,\eta l}(P)$, there is an $m \in \mathsf{N}_{\varepsilon,\eta}$ and an $X \in \mathcal{P}_{\eta(k+l+m)}(P)$ such that $A = \Psi^m_{\varepsilon k,\eta l}(X)$.

Now, consider the set

$$\mathcal{W}_{\varepsilon k,\eta l} := \bigcup_{m \in \mathsf{N}_0} \left\{ \Psi^m_{\varepsilon k,\eta l}(X) : X \in \mathcal{P}_{\eta(k+l+m)}(P) \text{ s.t. } P_X = 0 \right\}.$$

It is straight forward, see [7], to observe that $\mathcal{W}_{\varepsilon k,\eta l}$ is a vector subspace of $\mathcal{A}_{\varepsilon k,\eta l}(P)$.

Define the category AP by:

- $\mathrm{ob}(AP) := \{\varepsilon k : \varepsilon \in \{+,-\}, k \in \mathsf{N}_0\}$,
- $\mathrm{Mor}_{AP}(\varepsilon k, \eta l) := \frac{\mathcal{A}_{\varepsilon k,\eta l}(P)}{\mathcal{W}_{\varepsilon k,\eta l}}$ (also denoted by $AP_{\varepsilon k,\eta l}$),
- composition of morphisms is induced by the composition of *P-labelled* affine tangles (see [7]),
- the identity morphism of εk is given by the class $[A1_{\varepsilon k}]$, Figure 3.

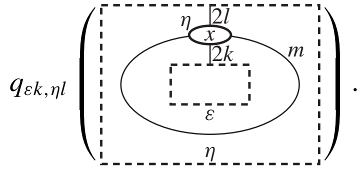
AP is a C-linear category and is called the *affine category over P* and the morphisms in this category are called *affine morphisms*.

For $\varepsilon k, \eta l \in \mathrm{Col}$ and $m \in \mathsf{N}_{\varepsilon,\eta}$, consider the composition map

$$\psi^m_{\varepsilon k,\eta l} : P_{\eta(k+l+m)} \xrightarrow{I_{\eta(k+l+m)}} \mathcal{P}_{\eta(k+l+m)}(P) \xrightarrow{\Psi^m_{\varepsilon k,\eta l}} \mathcal{A}_{\varepsilon k,\eta l}(P) \xrightarrow{q_{\varepsilon k,\eta l}} AP_{\varepsilon k,\eta l}$$

where the map $I_{\eta(k+l+m)} : P_{\eta(k+l+m)} \to \mathcal{P}_{\eta(k+l+m)}(P)$ is obtained by labelling the internal disc of the identity tangle $I_{\eta(k+l+m)}$ (defined in Figure 1) by a vector in $P_{\eta(k+l+m)}$, and $q_{\varepsilon k,\eta l} : \mathcal{A}_{\varepsilon k,\eta l}(P) \to AP_{\varepsilon k,\eta l}$ is the quotient map. Note that

$\psi^m_{\varepsilon k,\eta l}$ is indeed linear, although $I_{\eta(k+l+m)}$ is not. Pictorially, $\psi^m_{\varepsilon k,\eta l}(x)$ looks like



REMARK 2.12. For each $a \in AP_{\varepsilon k,\eta l}$, there exists $m \in \mathsf{N}_{\varepsilon,\eta}$ and $x \in P_{\eta(k+l+m)}$ such that $a = \psi^m_{\varepsilon k,\eta l}(x)$.

$*$-STRUCTURE. If P is a $*$-planar algebra, then each $\mathcal{P}_{\varepsilon k}(P)$ becomes a $*$-algebra where $*$ of a labelled tangle is given by $*$ of the unlabelled tangle whose internal discs are labelled with $*$ of the labels. Further, one can define $*$ of an affine tangular picture by reflecting it inside out such that the reflection of the distinguished boundary segment of any disc becomes the same for the disc in the reflected picture; this also extends to the *P-labelled* affine tangles as in the case of *P-labelled* tangles. Clearly, $*$ is an involution on the space of P-labelled affine tangles, which can be extended to a conjugate linear isomorphism $* : \mathcal{A}_{\varepsilon k,\eta l}(P) \to \mathcal{A}_{\eta l,\varepsilon k}(P)$ for all colours εk, ηl. Moreover, it is readily seen that $*(\mathcal{W}_{\varepsilon k,\eta l}) = \mathcal{W}_{\eta l,\varepsilon k}$; so the category AP inherits a $*$-category structure.

DEFINITION 2.13. Let P be a planar algebra.

(1) A C-linear functor V from AP to $\mathcal{V}ec$ (the category of complex vector spaces) is said to be an *affine P-module*, that is, there exists a vector space $V_{\varepsilon k}$ for each $\varepsilon k \in$ Col and a linear map $AP_{\varepsilon k,\eta l} \ni a \overset{V}{\longmapsto} V_a \in \mathrm{Mor}_{\mathcal{V}ec}(V_{\varepsilon k}, V_{\eta l})$ for every $\varepsilon k, \eta l \in$ Col such that compositions and identities are preserved. (V_a will be referred as the action of the affine morphism a.)

(2) For a $*$-planar algebra P, a *$*$-affine P-module* is a C-linear functor V from AP to the category of inner product spaces such that $\langle \xi, a\eta\rangle = \langle a^*\xi, \eta\rangle$ for all affine morphisms a, and ξ and η in appropriate $V_{\varepsilon k}$'s.

(3) A $*$-affine P-module V will be called *Hilbert affine P-module* if $V_{\varepsilon k}$'s are Hilbert spaces.

An affine module is said to be *bounded* (resp., *locally finite*) if the actions of the affine morphisms are bounded operators with respect to the norm coming from the inner product (resp., $V_{\varepsilon k}$'s are finite dimensional). By closed graph theorem, a Hilbert affine P-module is automatically bounded; conversely, every bounded $*$-affine P-module can be completed to a Hilbert affine module.

Below, we give a list of some standard facts on Hilbert affine P-modules for a $*$-planar algebra P with modulus. The proofs of the facts, as stated here, are straight-forward exercises (for analogous statements on annular tangles see

[10]). If V is a Hilbert affine P-module and $S_{\varepsilon k} \subset V_{\varepsilon k}$ for $\varepsilon k \in \mathrm{Col}$, then one can consider the 'submodule of V generated by $S = \coprod_{\varepsilon k \in \mathrm{Col}} S_{\varepsilon k}$' (denoted by $[S]$) given by $\left\{[S]_{\eta l} := \overline{\operatorname{span}\left\{\bigcup_{\varepsilon k \in \mathrm{Col}} V_{AP_{\varepsilon k,\eta l}}(S_{\varepsilon k})\right\}}^{\|\cdot\|}\right\}_{\eta l}$ which is also the smallest submodule of V containing S.

REMARK 2.14. Let V be a Hilbert affine P-module and W be an $AP_{\varepsilon k,\varepsilon k}$-submodule of $V_{\varepsilon k}$ for some $\varepsilon k \in \mathrm{Col}$. Then,

(1) V is irreducible if and only if $V_{\varepsilon k}$ is irreducible $AP_{\varepsilon k,\varepsilon k}$-module for all $\varepsilon k \in \mathrm{Col}$ if and only if $[v] = V$ for all $0 \neq v \in V$.

(2) W is irreducible $\Leftrightarrow$ $[W]$ is an irreducible submodule of V.

REMARK 2.15. If V and W are Hilbert affine P-modules for which there exists an $\varepsilon k \in \mathrm{Col}$ such that $V = [V_{\varepsilon k}]$ and there exists an $AP_{\varepsilon k,\varepsilon k}$-linear isometry $\theta : V_{\varepsilon k} \to W_{\varepsilon k}$, then θ extends uniquely to an isometry (of Hilbert affine P-modules) $\tilde{\theta} : V \to W$.

For $\varepsilon = \{+, -\}$, we will also consider *Hilbert ε-affine P-module V* consisting of the Hilbert spaces $V_{\pm 0}, V_1, V_2, \ldots$ equipped with a $*$-preserving action of affine morphisms as follows:

$$\left.\begin{aligned} AP_{\varepsilon k,\varepsilon l} \times V_k &\to V_l \\ AP_{\varepsilon k,\eta 0} \times V_k &\to V_{\eta 0} \\ AP_{\eta 0,\varepsilon l} \times V_{\eta 0} &\to V_l \\ AP_{\eta 0,\nu 0} \times V_{\eta 0} &\to V_{\nu 0} \end{aligned}\right\} \qquad \text{for all} \quad k, l \in \mathsf{N}, \eta, \nu \in \{\pm\}.$$

REMARK 2.16. The restriction map from the set of isomorphism classes of Hilbert affine P-modules to that of the Hilbert ε-affine P-modules, is a bijection.

To see this, consider an irreducible Hilbert $+$-affine P-module V. Define $(\operatorname{Ind} V)_{\varepsilon k} := V_k$ and $(\operatorname{Ind} V)_{\varepsilon 0} := V_{\varepsilon 0}$ (as Hilbert spaces) and the action of affine morphisms by $AP_{\varepsilon k,\eta l} \times (\operatorname{Ind} V)_{\varepsilon k} \ni (a, v) \longmapsto (r_{\eta l} \circ a \circ r_{\varepsilon k}^{-1}) \cdot v \in (\operatorname{Ind} V)_{\eta l}$ where

$$r_{\nu s} = \begin{cases} A1_{\nu s}, & \text{if } s = 0 \text{ or } \nu = + \\ AR_{\nu s}, & \text{otherwise.} \end{cases}$$

$A1_{\nu s}$ and $AR_{\nu s}$ being the affine tangles mentioned in Figure 3.

For every affine P-module V, $\dim(V_{+k}) = \dim(V_{-k})$ for all $k \geq 1$ and it increases as k increases. This motivates the following definition:

DEFINITION 2.17. The *weight of V* is defined to be the smallest non-negative integer k such that V_{+k} or V_{-k} is nonzero.

3. Affine morphisms at zero level

In this section, we will be interested in understanding the *affine morphisms at zero level* of a $*$-planar algebra P, that is, in the space

$$AP_{0,0} := \begin{bmatrix} AP_{+0,+0} & AP_{-0,+0} \\ AP_{+0,-0} & AP_{-0,-0} \end{bmatrix},$$

which has a natural $*$-algebra structure induced by matrix multiplication with respect to composition of affine morphisms and the $*$ as discussed before. On the other hand, given a finite index subfactor $N \subset M$, for each $\varepsilon, \eta \in \{+,-\}$, we set $V_{\varepsilon,\eta} :=$ {isomorphism classes of irreducible X_η-X_ε bimodules appearing in the standard invariant} $=$ {isomorphism classes of irreducible subbimodules of ${}_{X_\eta}L^2(M_k)_{X_\varepsilon}$ for some $k \in \mathsf{N}_0$} where X_+ (resp., X_-) denotes N (resp., M). Then, the usual matrix multiplication with respect to appropriate relative tensor products and the matrix adjoint with respect to the contragradients of bimodules induce a natural $*$-algebra structure on the space

$$\mathcal{F}_{N\subset M} := \begin{bmatrix} \mathsf{C}V_{+,+} & \mathsf{C}V_{-,+} \\ \mathsf{C}V_{+,-} & \mathsf{C}V_{-,-} \end{bmatrix}.$$

We will aim to prove the following:

THEOREM 3.1. *Let $N \subset M$ be a finite index subfactor and P be its associated planar algebra. Then,*

$$AP_{0,0} \cong \mathcal{F}_{N\subset M}$$

as $*$-algebras.

3.1. A spanning set for $AP_{0,0}$

In this subsection, P will always denote the planar algebra associated to the tower of basic construction $\{M_k\}_{k\in\mathsf{N}}$ of a finite index subfactor $N \subset M$ with Jones projections $\{e_k\}_{k\in\mathsf{N}}$, and $\psi^m_{\varepsilon,\eta}$ will denote the linear map $\psi^m_{\varepsilon 0,\eta 0}$ introduced right before Remark 2.12. We first list some elementary yet useful properties of the ψ-maps.

LEMMA 3.2. *For $\varepsilon, \eta \in \{+,-\}$ and $k \in \mathsf{N}_{\varepsilon,\eta}$, $\psi^k_{\varepsilon 0,\eta 0}(p) \neq 0$ for all nonzero $p \in \mathcal{P}(P_{\eta k})$.*

PROOF. Let $\omega_{\varepsilon,\eta} : \mathcal{A}_{\varepsilon 0,\eta 0}(P) \to \mathcal{P}_{\eta 0}(P)$ be the map defined by sending an affine tangle $[A] \in \mathcal{AT}_{\varepsilon 0,\eta 0}$ to the tangle obtained by ignoring the internal rectangle in A. Note that $\mathcal{W}_{\varepsilon 0,\eta 0} \subset \ker(P \circ \omega_{\varepsilon,\eta})$; thus, $P \circ \omega_{\varepsilon,\eta}$ induces a linear map $\omega'_{\varepsilon,\eta} : AP_{\varepsilon 0,\eta 0} \to P_{\eta 0} \cong \mathsf{C}$. Clearly, $\omega'_{\varepsilon,\eta} \circ \psi^k_{\varepsilon 0,\eta 0} = P_{TR^r_{\eta k}}$. This proves the lemma.

LEMMA 3.3. *Let* $\varepsilon, \eta \in \{+, -\}$ *and* $k \in \mathsf{N}_{\varepsilon,\eta}$.

(1) *The map* $\psi^k_{\varepsilon,\eta}$ *is tracial, (that is,* $\psi^k_{\varepsilon,\eta}(xy) = \psi^k_{\varepsilon,\eta}(yx)$ *for all* $x, y \in P_{\eta k}$*) and hence, factors through the center of* $P_{\eta k}$.

(2) $\psi^k_{\varepsilon,\eta}(x) = \psi^{k+2}_{\varepsilon,\eta}(x e_{(k+1+\delta_{\eta=-})})$ *for all* $x \in P_{\eta k}$.

PROOF. Both follow from simple application of affine isotopy and also using the relation between the Jones projections and the Jones projection tangles, in (2).

From Corollary 2.7 and Lemma 3.3, we deduce the following where, for convenience, we use $\varphi_{\varepsilon k}$ to denote $\varphi_{-1,\frac{k}{2}-1}$ or $\varphi_{0,\frac{k-1}{2}}$ (resp., $\varphi_{0,\frac{k}{2}}$ or $\varphi_{-1,\frac{k-1}{2}}$) according as k is even or odd if $\varepsilon = +$ (resp., $\varepsilon = -$).

COROLLARY 3.4. *Let* $\varepsilon, \eta \in \{+, -\}$ *and* $k, l \in \mathsf{N}_{\varepsilon,\eta}$. *Then, for all* $p \in \mathscr{P}(P_{\eta k})$ *and* $q \in \mathscr{P}(P_{\eta l})$ *satisfying* $\operatorname{Range}\varphi_{\eta k}(p) \overset{X_\eta\text{-}X_\varepsilon}{\cong} \operatorname{Range}\varphi_{\eta l}(q)$, *we have* $\psi^k_{\varepsilon,\eta}(p) = \psi^l_{\varepsilon,\eta}(q)$.

DEFINITION 3.5. The *weight* of a projection $p \in P_{\varepsilon k}$ for even (resp., odd) k, denoted by $wt(p)$, is defined to be the smallest even (resp., odd) integer l such that there exists a projection $q \in P_{\varepsilon l}$ satisfying $\operatorname{Range}(\varphi_{\varepsilon k}(p)) \cong \operatorname{Range}(\varphi_{\varepsilon l}(q))$ as X_ε-$X_{(-)^k\varepsilon}$-bimodules.

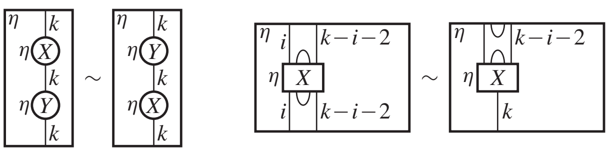
Let $\mathcal{S}_{\varepsilon k}$ be a maximal set of non-equivalent minimal projections in $P_{\varepsilon k}$ with weight k for all colors εk.

REMARK 3.6. In view of Remark 2.12, Lemma 3.3 and Corollary 3.4, we observe that $AP_{\varepsilon 0,\eta 0}$ is spanned by the set $\bigcup_{k\in\mathsf{N}_{\varepsilon,\eta}} \{\psi^k_{\varepsilon,\eta}(p) : p \in \mathcal{S}_{\eta k}\}$ for $\varepsilon, \eta \in \{+, -\}$.

We shall, in fact, see that these sets are linearly independent and hence form bases.

3.2. Equivalence on tangles induced by affine isotopy

For $\varepsilon, \eta \in \{+, -\}$, set $\mathcal{T}_{\varepsilon,\eta} := \bigsqcup_{l\in\mathsf{N}_{\varepsilon,\eta}} \mathcal{T}_{\eta l}(P)$. Define the equivalence relation $\sim$ on $\mathcal{T}_{\varepsilon,\eta}$ generated by the relations given by the pictures in Figure 4.



(i) (ii)

FIGURE 4. Equivalence relation $\sim$. ($\eta k \in \mathrm{Col}$, $0 \le i \le k-2$, $X, Y \in \mathcal{T}_{\eta k}(P)$)

The following topological lemma involving this equivalent relation will be crucial in the forthcoming section.

LEMMA 3.7. *If* $\varepsilon, \eta \in \{+, -\}$, $k, l \in \mathsf{N}_{\varepsilon,\eta}$ *and* $S \in \mathcal{T}_{\eta k}(P)$, $T \in \mathcal{T}_{\eta l}(P)$, *then* $\Psi^k_{\varepsilon 0,\eta 0}(S) = \Psi^l_{\varepsilon 0,\eta 0}(T)$ *if and only if* $X \sim Y$.

PROOF. If $S \sim T$ either by relation (i) or (ii) in Figure 4, then using affine isotopy, we easily see that $\Psi^k_{\varepsilon 0,\eta 0}(U) = \Psi^l_{\varepsilon 0,\eta 0}(V)$. For the 'only if' part, consider pictures $\hat{S}$ and $\hat{T}$ in the isotopy class of S and T respectively, and $\hat{\Psi}^m_{\varepsilon 0,\eta 0}$ as in Figure 3 to represent $\Psi^m_{\varepsilon 0,\eta 0}$ for $m = k, l$. Since $\Psi^k_{\varepsilon 0,\eta 0}(S) = \Psi^l_{\varepsilon 0,\eta 0}(T)$, we have an affine isotopy $\varphi : [0, 1] \times RA \to RA$ (as in Definition 2.9) such that $\varphi\big(1, \hat{\Psi}^k_{\varepsilon 0,\eta 0}(\hat{X})\big) = \hat{\Psi}^l_{\varepsilon 0,\eta 0}(\hat{Y})$. Let p be the straight path in RA joining the points $(0, -1)$ and $(0, -2)$ and suppose $\tilde{p} := \varphi(1, p)$ which is also a simple path in RA joining the same two points. Note that cutting $\hat{\Psi}^k_{\varepsilon 0,\eta 0}(\hat{X})$ (resp., $\hat{\Psi}^l_{\varepsilon 0,\eta 0}(\hat{Y})$) along the path p and straightening gives $\hat{X}$ (resp., $\hat{Y}$), as shown in Figure 5.

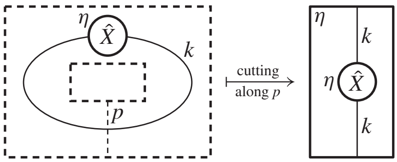


FIGURE 5. Cutting along a simple path

Further, since φ is an affine isotopy, even if we cut $\hat{\Psi}^l_{\varepsilon 0,\eta 0}(\hat{Y})$ along $\tilde{p}$, we still obtain $\hat{X}$ (upto planar isotopy). Let A_0 denote the affine tangular picture $\hat{\Psi}^l_{\varepsilon 0,\eta 0}(\hat{Y})$ and $\mathcal{SP}(A_0)$ denote the set of those simple paths in RA with end points $(0, -1)$ and $(0, -2)$ such that they (a) do not meet any disc in A_0, (b) intersect the set of strings discretely and non-tangentially, and (c) are equivalent to the straight path p via some affine isotopy. Clearly, $p, \tilde{p} \in \mathcal{SP}(A_0)$.

Analogous to the equivalence relation $\sim$ on $\mathcal{T}_{\varepsilon,\eta}$, we consider a equivalence relation $\sim$ on $\mathcal{SP}(A_0)$ generated by the local moves as shown in Figure 6. Note that cuts along two paths related by move (i) give same labelled tangles (upto tangle isotopy); and cuts along paths related by moves (ii) and $\{$(iii), (iii)$'\}$ correspond to equivalence relations (i) and (ii) of Figure 4, respectively. Thus, it is enough to show that the paths p and $\tilde{p}$ are equivalent under this relation which will imply $X \sim Y$. It is not hard to prove that p can obtained from $\tilde{p}$ by applying finitely many moves of the above types. We will not give a complete proof of this fact here; however, one can extract a detailed proof from the strategy used in the proof of [6, Proposition 2.8] which proves the same type of statement but for 'annular tangles' where the isotopy has no restriction on the internal and external boundaries as in affine isotopy. So, one has to make

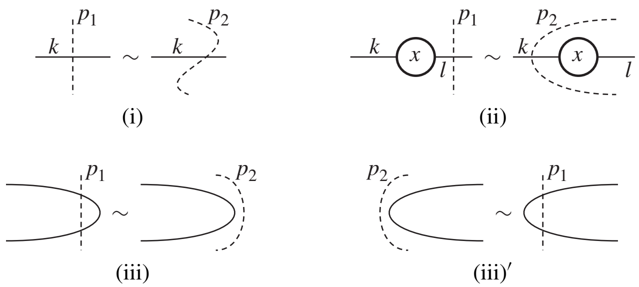


FIGURE 6. Equivalence relation on $\mathcal{SP}(A_0)$. ($k, l \in \mathsf{N}_0$ such that $(k+l) \in 2\mathsf{N}_0$, $x \in P_{\pm(\frac{k+l}{2})}$)

necessary modifications, namely, ignoring the rotation move in [6] but even this is not an issue for us because we are working with affine morphism from $\varepsilon 0$ to $\eta 0$ and no strings are attached to the boundary of RA. This completes the proof of the lemma.

3.3. *Proof of Theorem 3.1*

We first set up the following notation:

NOTATION. For $p \in \mathscr{P}_{\min}(\mathcal{Z}(P_{\varepsilon l}))$ and $\eta = (-)^l \varepsilon$, we write $v^p_{\eta,\varepsilon} \in V_{\eta,\varepsilon}$ for the isomorphism class of the X_ε-X_η bimodule Range $\varphi_{\varepsilon l}(p_0)$ for any $p_0 \in \mathscr{P}_{\min}(P_{\varepsilon l})$ with $p_0 \leq p$.

Now, for $\varepsilon, \eta \in \{+, -\}$ and $k \in \mathsf{N}_{\varepsilon,\eta}$, consider the map

$$P_{\eta k} \ni x \xmapsto{\gamma^k_{\varepsilon,\eta}} \sum_{p \in \mathscr{P}_{\min}(\mathcal{Z}(P_{\eta k}))} \sqrt{\dim(pP_{\eta k})} \left[\frac{\mathrm{tr}_{M_{(k-\delta_{\eta=+})}}(xp)}{\mathrm{tr}_{M_{(k-\delta_{\eta=+})}}(p)} \right] v^p_{\varepsilon,\eta} \in \mathsf{C}V_{\varepsilon,\eta}.$$

REMARK 3.8. The above definition directly implies $\gamma^k_{\varepsilon,\eta}(p_0) = v^p_{\varepsilon,\eta}$ for all $p \in \mathscr{P}_{\min}(\mathcal{Z}(P_{\eta k}))$ and $p_0 \in \mathscr{P}_{\min}(P_{\eta k})$ satisfying $p_0 \leq p$.

LEMMA 3.9. *If* $\varepsilon, \eta \in \{+, -\}$ *and* $k \in \mathsf{N}_{\varepsilon,\eta}$, *then*

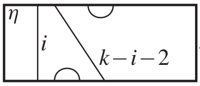
(1) $\gamma^k_{\varepsilon,\eta}$ *is tracial,*

(2) $\gamma^k_{\varepsilon,\eta}(x) = \gamma^{k+2}_{\varepsilon,\eta}(x e_{(k+1+\delta_{\eta=-})})$ *for all* $x \in P_{\eta k}$.

PROOF. Note that any partial isometry in $P_{\eta k}$ with orthogonal initial and final projections, is in the kernel of $\gamma^k_{\varepsilon,\eta}$; this along with Remark 3.8 imply (1).

For (2), let $\left\{e^p_{i,j} : p \in \mathscr{P}_{\min}(\mathcal{Z}(P_{\eta k})), 1 \leq i, j \leq \sqrt{\dim(pP_{\eta k})}\right\}$ be a system of matrix units for $P_{\eta k}$. Fix a $p \in \mathscr{P}_{\min}(\mathcal{Z}(P_{\eta k}))$. Then, by (1), $\gamma^k_{\varepsilon,\eta}(e^p_{i,j}) = 0 = \gamma^{k+2}_{\varepsilon,\eta}(e^p_{i,j}\, e_{k+1})$ for all $1 \leq i \neq j \leq \sqrt{\dim(pP_{\eta k})}$. It is easy to check that $e^p_{i,i} e_{(k+1+\delta_{\eta=-})}$ is a minimal projection; let $\tilde{p}$ be its central support in $P_{\eta(k+2)}$. By Remark 3.8 and Lemma 2.6, we have $\gamma^k_{\varepsilon,\eta}(e^p_{i,i}) = v^p_{\varepsilon,\eta} = v^{\tilde{p}}_{\varepsilon,\eta} = \gamma^{k+2}_{\varepsilon,\eta}(e^p_{i,i} e_{(k+1+\delta_{\eta=-})})$.

Corollary 3.10. *If $\varepsilon, \eta \in \{+,-\}$, $k, l \in \mathsf{N}_{\varepsilon,\eta}$, $S \in \mathcal{T}_{\eta k}(P)$ and $T \in \mathcal{T}_{\eta l}(P)$ such that $S \sim T$, then $\gamma^k_{\varepsilon,\eta}(P_S) = \gamma^l_{\varepsilon,\eta}(P_T)$.*

Proof. If $S \sim T$ by relation (i), as shown in Figure 4, then part (1) of Lemma 3.9 does the job. Suppose S and T denote the tangles on the left and the right sides of relation (ii) in Figure 4 respectively, and let $Z =$


. Then, we have

$$\begin{aligned}\gamma^{k-2}_{\varepsilon,\eta}(P_S) &= \gamma^k_{\varepsilon,\eta}(P_S e_{(k-1+\delta_{\eta=-})}) = \delta^{-1}\gamma^k_{\varepsilon,\eta}(P_Z P_X P_{Z^*}) \\ &= \delta^{-1}\gamma^k_{\varepsilon,\eta}(P_{Z^*} P_Z P_X) = \gamma^k_{\varepsilon,\eta}(P_T)\end{aligned}$$

where we use parts (2) and (1) of Lemma 3.9 to obtain the first and third equalities.

We are now just one step away from establishing the required isomorphism. For $\varepsilon, \eta \in \{+,-\}$, consider the map

$$\mathcal{AT}_{\varepsilon 0, \eta 0}(P) \ni A \overset{\Lambda_{\varepsilon,\eta}}{\longmapsto} \gamma^k_{\varepsilon,\eta}(P_T) \in \mathsf{C}V_{\varepsilon,\eta}$$

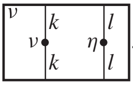
where (by Remark 2.11) $A = \Psi^k_{\varepsilon 0, \eta 0}(T)$ for some $k \in \mathsf{N}_{\varepsilon,\eta}$ and $T \in \mathcal{T}_{\eta k}(P)$. $\Lambda_{\varepsilon,\eta}$ is indeed a well-defined map due to Corollary 3.10. Extend this map linearly to $\Lambda_{\varepsilon,\eta} : \mathcal{A}_{\varepsilon 0,\eta 0}(P) \to \mathsf{C}V_{\varepsilon,\eta}$. Note that $\Lambda_{\varepsilon,\eta}(A) = \gamma^k_{\varepsilon,\eta}(P_X)$ whenever $A = \Psi^k_{\varepsilon 0,\eta 0}(X)$ and $X \in \mathcal{P}_{\eta k}(P)$; this implies $\mathcal{W}_{\varepsilon 0, \eta 0} \subset \ker \Lambda_{\varepsilon,\eta}$. Thus, each $\Lambda_{\varepsilon,\eta}$ induces a linear map $\lambda_{\varepsilon,\eta} : AP_{\varepsilon 0,\eta 0} \longrightarrow \mathsf{C}V_{\varepsilon,\eta}$, that is, $\Lambda_{\varepsilon,\eta} = \lambda_{\varepsilon,\eta} \circ q_{\varepsilon,\eta}$.

Proof of Theorem 3.1. Define

$$\lambda := \begin{bmatrix} \lambda_{+,+} & \lambda_{-,+} \\ \lambda_{+,-} & \lambda_{-,-} \end{bmatrix}.$$

We will show that $\lambda : AP_{0,0} \longrightarrow \mathcal{F}_{N \subset M}$ is a $*$-algebra isomorphism. Clearly, λ is linear. Now, for $\varepsilon, \eta \in \{+,-\}$, $k \in \mathsf{N}_{\varepsilon,\eta}$ and $p \in \mathcal{S}_{\eta k}$ (defined before Remark 3.6), let $\tilde{p}$ denote the central support of p in $P_{\eta k}$. Note that $\lambda_{\varepsilon,\eta}(\psi_{\varepsilon,\eta}(p)) = \Lambda_{\varepsilon,\eta}(\Psi_{\varepsilon,\eta}(I_{\eta k}(p))) = \gamma_{\varepsilon,\eta}(p) = v^{\tilde{p}}_{\varepsilon,\eta} \in V_{\varepsilon,\eta}$ where the first two equalities follow easily unravelling the definitions and the last one comes from Remark 3.8. On the other hand, from Corollary 2.7 and definition of $V_{\varepsilon,\eta}$, we get $\left\{v^{\tilde{p}}_{\varepsilon,\eta} : k \in \mathsf{N}_{\varepsilon,\eta}, p \in \mathcal{S}_{\eta k}\right\} = V_{\varepsilon,\eta}$. This and Remark 3.6, imply that $\lambda_{\varepsilon,\eta}$ is injective as well as surjective.

A closer look at the $*$-structures of $\mathcal{F}_{N\subset M}$ (resp., $AP_{0,0}$) reveals $\left[v^{\widetilde{p}}_{\varepsilon,\eta}\right]^* = v^{\widetilde{q}}_{\eta,\varepsilon}$ using Proposition 2.5 (resp., $[\psi_{\varepsilon,\eta}(p)]^* = \psi_{\eta,\varepsilon}(q)$) where $q = P_{R^k_{\eta k}}(p)$ for all $\varepsilon, \eta \in \{+,-\}$, $k \in \mathsf{N}_{\varepsilon,\eta}$ and $p \in \mathcal{S}_{\eta k}$. Hence, λ is $*$-preserving.

It remains to show that λ is an algebra homomorphism. Note that for $\varepsilon, \eta, \nu \in \{+,-\}$, $k \in \mathsf{N}_{\nu,\eta}$, $l \in \mathsf{N}_{\eta,\varepsilon}$, $x \in P_{\nu k}$ and $y \in P_{\eta l}$, we have $\psi_{\varepsilon,\nu}(P_{H_{\nu k,\eta l}}(x, y)) = \psi_{\eta,\nu}(x) \circ \psi_{\varepsilon,\eta}(y)$ where the tangle $H_{\nu k,\eta l}$ is given by . So, one needs to check $\operatorname{Range} \varphi_{\nu(k+l)}(P_{H_{\nu k,\eta l}}(p,q)) \cong \operatorname{Range} \varphi_{\nu k}(p) \otimes_{X_\eta} \operatorname{Range} \varphi_{\eta l}(q)$ as X_ν-X_ε-bimodules where $p \in \mathscr{P}(P_{\nu k})$ and $q \in \mathscr{P}(P_{\eta l})$. One way of seeing this is by translating some results in [1, Theorem 4.6] in the language of planar algebras. However, this isomorphism comes for free from the isomorphism between P and the normalized bimodule planar algebra associated to ${}_N L^2(M)_M$, established in the proof of [3, Theorem 5.4].

Hence, λ is a $*$-algebra isomorphism.

4. Affine modules with zero weight

In this section, we will analyze the affine P-modules with weight zero for any subfactor planar algebra P (possibly having infinite depth).

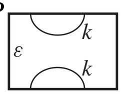
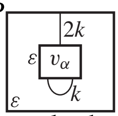
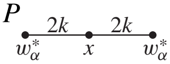
Throughout this section, ε will denote an element of $\{\pm\}$ and P will continue to be the planar algebra associated to a finite index subfactor $N \subset M$. Let us consider the trace on the algebra $\mathsf{C}V_{\varepsilon,\varepsilon}$ (introduced in Section 3) given by $V_{\varepsilon,\varepsilon} \ni v \xrightarrow{\omega_\varepsilon} \delta_{v=1_\varepsilon} \in \mathsf{C}$ where 1_ε is the isomorphism class of the trivial bimodule in $V_{\varepsilon,\varepsilon}$. Clearly, ω_ε is positive definite. By the isomorphism in Theorem 3.1, ω_ε induces a positive definite trace on $AP_{\varepsilon 0,\varepsilon 0}$. In the following lemma, we present a pictorial interpretation of ω_ε.

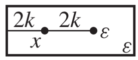
LEMMA 4.1. *For all $k \in \mathsf{N}_0$ and $x \in P_{\varepsilon 2k}$, we have*

$$\omega_\varepsilon\big(\psi^{2k}_{\varepsilon 0,\varepsilon 0}(x)\big) = \delta^{-k} \sum_\alpha P_{\underset{w^*_\alpha}{\bullet} \overset{2k}{\text{---}} \underset{x}{\bullet} \overset{2k}{\text{---}} \underset{w^*_\alpha}{\bullet}} \in P_{\varepsilon 0} \cong \mathsf{C}$$

where $\{w_\alpha\}$ is any orthonormal basis of $P_{\varepsilon k}$ with repect to the canonical trace (that is, the normalized picture trace).

PROOF. Let $\{E^i_{\alpha,\beta} : 0 \le i \le n, 1 \le \alpha, \beta \le d_i\}$ be a system of matrix units of the finite dimensional C^*-algebra $P_{\varepsilon 2k}$ where i gives the indexing of the matrix summands and d_i is the order of i-th summand; further, let us assume the 0-th summand is the one whose minimal projections correspond to $1_\varepsilon \in V_{\varepsilon,\varepsilon}$. Now, there exist scalars $x^i_{\alpha,\beta}$ such that $x = \sum_i \sum_{\alpha,\beta} x^i_{\alpha,\beta} E^i_{\alpha,\beta}$. So, by the isomorphism in Theorem 3.1 and definition of ω_ε, we get $\omega_\varepsilon\big(\psi^{2k}_{\varepsilon 0,\varepsilon 0}(x)\big) = \sum_\alpha x^0_{\alpha,\alpha}$.

Consider the minimal projection $p := \delta^{-k} P$  in $P_{\varepsilon 2k}$, which also corresponds to $1_\varepsilon \in V_{\varepsilon,\varepsilon}$. Let $v_\alpha \in P_{\varepsilon 2k}$ such that $v_\alpha v_\alpha^* = E^0_{\alpha,\alpha}$ and $v_\alpha^* v_\alpha = p$. Set $w_\alpha := P$  $\in P_{\varepsilon k}$. It easily follows from $v_\alpha^* v_\beta = \delta_{\alpha=\beta} p$ that $\{w_\alpha\}_\alpha$ is an orthonormal subset of $P_{\varepsilon k}$ with respect to the canonical trace. On the other hand, $v_\alpha^* x v_\alpha = x^0_{\alpha,\alpha} p$ implies P  $= \delta^k x^0_{\alpha,\alpha} 1_{P_{\varepsilon 0}}$. It only remains to show that $\{w_\alpha\}_{1 \leq \alpha \leq d_0}$ spans $P_{\varepsilon k}$. For this, we use Frobenius reciprocity for bimodules and get $d_0 = \dim(P_{\varepsilon k})$.

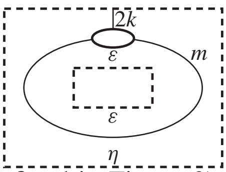
Independence from the choice of an orthonormal basis of $P_{\varepsilon k}$, follows from the equation $\omega_\varepsilon(\psi^{2k}_{\varepsilon 0,\varepsilon 0}(x)) = \sum_\alpha \langle w_\alpha, f_x(w_\alpha)\rangle_{P_{\varepsilon k}}$ where $f_x : P_{\varepsilon k} \to P_{\varepsilon k}$ is the linear operator given by the action of the semi-labelled tangle .

We now define $H^\varepsilon_{\eta k} := AP_{\varepsilon 0,\eta k}$ for all $\eta k \in \mathrm{Col}$. $H^\varepsilon = \left\{H^\varepsilon_{\eta k}\right\}_{\eta k \in \mathrm{Col}}$ forms an affine P-module with action of affine morphisms given by composition. Define a sesquilinear form on the affine module H^ε in the following way:

$$\langle h_1, h_2\rangle := \omega_\varepsilon(h_1^* \circ h_2) \qquad \text{where} \quad h_1, h_2 \in H^\varepsilon_{\eta k} \text{ and } \eta k \in \mathrm{Col}\,.$$

THEOREM 4.2. *H^ε is a bounded $*$-affine P-module with inner product given by the above form. Hence, its completion will be a Hilbert affine P-module.*

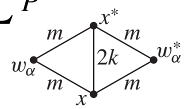
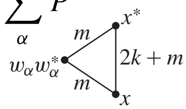
PROOF. We first need to check whether the form is positive definite, that is, $\omega_\varepsilon(h^* \circ h) > 0$ for $0 \neq h \in H^\varepsilon_{\eta k} = AP_{\varepsilon 0,\eta k}$. For each $m \in \mathsf{N}_{\varepsilon,\eta}$, set

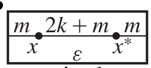
$\Phi^m_{\varepsilon,\eta k} :=$  which is the same as the unlabelled affine tangle $\Psi^m_{\varepsilon 0,\eta k}$ (defined in Figure 3) except there is a certain rotation on the internal disc. Let $\varphi^m_{\varepsilon,\eta k} : P_{\varepsilon(m+k)} \to AP_{\varepsilon 0,\eta k}$ be the linear map induced by the affine tangle $\Phi^m_{\varepsilon,\eta k}$. Note that using affine isotopy, we can obtain

$$(1) \qquad \varphi^m_{\varepsilon,\eta k}\big(P_{RI^k_{\varepsilon m}}(y)x\big) = \varphi^m_{\varepsilon,\eta k}\big(x P_{RI^k_{\varepsilon m}}(y)\big) \qquad \text{for all} \quad x \in P_{\varepsilon(m+k)}$$

and $y \in P_{\varepsilon m}$ where $RI^k_{\varepsilon m} : \varepsilon m \to \varepsilon(k+m)$ is the tangle obtained from $RI_{\varepsilon m}$ (in Figure 1) by replacing the straight vertical string on the right by k parallel strings. Considering a path algebra model of $\mathsf{C} \cong P_{\varepsilon 0} \hookrightarrow P_{\varepsilon m} \xhookrightarrow{P_{RI^k_{\varepsilon m}}} P_{\varepsilon(m+k)}$ and using Equation 1, we may deduce $\mathrm{Range}\,\varphi^m_{\varepsilon,\eta k} = \varphi^m_{\varepsilon,\eta k}(P'_{\varepsilon m} \cap P_{\varepsilon(m+k)})$.

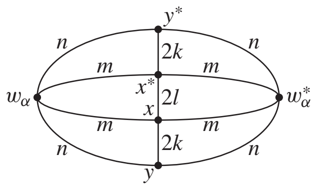
This along with Remark 2.12 implies that there exist $m \in \mathbb{N}_0$ and $0 \neq x \in P'_{\varepsilon m} \cap P_{\varepsilon(m+k)}$ such that $h = \varphi^m_{\varepsilon,\eta k}(x)$. By Lemma 4.1, we get

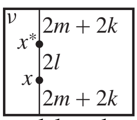
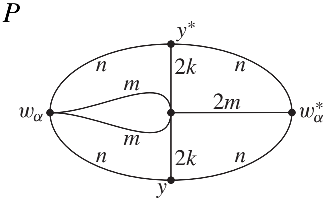
$$\omega_\varepsilon(h^* \circ h) = \delta^{-m} \sum_\alpha P$$

$$= \delta^{-m} \sum_\alpha P$$

$$= \delta^{-m} \sum_\alpha P_{TR^r_{\varepsilon m}}(w_\alpha w_\alpha^* y) = \sum_\alpha \langle w_\alpha, y w_\alpha\rangle_{P_{\varepsilon m}}$$

where $y = P$  and $\{w_\alpha\}_\alpha$ is an orthonormal basis of $P_{\varepsilon m}$ with respect to the canonical trace. The second equality follows from $[x, P_{\varepsilon m}] = 0$. Note that y is a positive element of $P_{\varepsilon m}$ and nonzero too since $P_{TR^r_{\varepsilon m}}(y) = P_{TR^r_{\varepsilon(m+k)}}(xx^*) \neq 0$. Thus, $\omega_\varepsilon(h^* \circ h) = \sum_\alpha \|y^{1/2} w_\alpha\|^2 > 0$.

The $*$-preserving condition $\langle a \circ h_1, h_2\rangle = \langle h_1, a^* \circ h_2\rangle$ holds trivially. Hence, H^ε is a $*$-affine module.

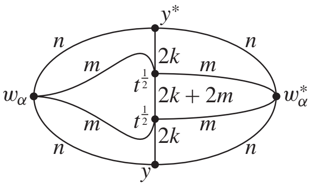
Boundedness of the action of affine morphisms: This part is relevant only if depth of P is infinite since for finite depth planar algebras, H^ε will be locally finite (see [7, Proof of Theorem 6.11]). Let $a = \psi^m_{\eta k, \nu l}(x) \in AP_{\eta k, \nu l}$ and $h = \varphi^n_{\varepsilon, \eta k}(y) \in H^\varepsilon_{\eta k} = AP_{\varepsilon 0, \eta k}$ where $x \in P_{\nu(k+l+m)}$ and $y \in P_{\varepsilon(k+n)}$. Now, $\|a \circ h\|^2 = \omega_\varepsilon(h^* \circ a^* \circ a \circ h)$ which, using Lemma 4.1, can be expressed as

$$\delta^{-(m+n)} \sum_\alpha P$$

$$= \gamma(s)$$

where $\{w_\alpha\}_\alpha$ is an orthonormal basis of $P_{\varepsilon(m+n)}$ with respect to the canonical trace, s is the element P  $\in P_{\nu 2(m+k)}$ and $\gamma : P_{\nu 2(m+k)} \to \mathbb{C}$ is the linear functional induced by the P-action of (the linear combination of semi-labelled tangles) $\delta^{-(m+n)} \sum_\alpha P$  . Note that s

is a positive element of $P_{\nu 2(m+k)}$. Also, γ is positive semi-definite because for a positive $t \in P_{\nu 2(m+k)}$, we have

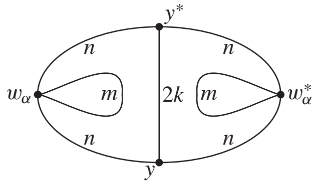
$$\gamma(t) = \delta^{-(m+n)} \sum_{\alpha} P$$



$$= \left\| \psi^m_{\eta k, \nu(m+k)}(t^{1/2}) \circ h \right\|^2 \geq 0$$

where the norm comes from the inner product in the first part. Thus,

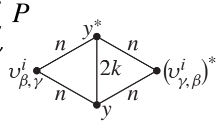
$$\|a \circ h\|^2 \leq \|s\| \gamma(1) = \|s\| \delta^{-(m+n)} \sum_{\alpha} P$$



.

We will now choose a special orthonormal basis of $P_{\varepsilon(m+n)}$. Let $\left\{E^i_{\beta,\gamma} : 0 \leq i \leq n, 1 \leq \beta, \gamma \leq d_i\right\}$ be a system of matrix units of the finite dimensional C^*-algebra $P_{\varepsilon n}$. Note that $\left\{v^i_{\beta,\gamma} := c_i E^i_{\beta,\gamma}\right\}_{i,\beta,\gamma}$ is an orthonormal basis in $P_{\varepsilon n}$ where c_i's are normalizing scalars, and thereby, $\left\{P_{RI^m_{\varepsilon n}}(v^i_{\beta,\gamma})\right\}_{i,\beta,\gamma}$ forms an orthonormal set in $P_{\varepsilon(m+n)}$. On the other hand, any $w \in P_{\varepsilon(m+n)}$ which is orthogonal to this set, must satisfy $\left\langle v^i_{\beta,\gamma}, P_{RE^m_{\varepsilon(m+n)}}(w) v^{i'}_{\beta',\gamma'}\right\rangle = 0$ where $RE^m_{\varepsilon(m+n)} : \varepsilon(m+n) \to \varepsilon n$ is the tangle obtained from the 'right conditional expectation tangle' $RE_{\varepsilon(m+n)}$ (described in Figure 1) replacing the single string with both endpoints attached to the internal disc, by m many parallel strings; thus, $P_{RE^m_{\varepsilon(m+n)}}(w) = 0$. This implies

$$\|a \circ h\|^2 \leq \|s\| \delta^m \delta^{-n} \sum_{i,\beta,\gamma} P$$



$$= \delta^m \|s\| \|h\|^2.$$

Clearly, $\|s\|$ is independent of h. Hence, the action of a is bounded.

COROLLARY 4.3. *If P has finite depth, then for every irreducible $AP_{\varepsilon 0, \varepsilon 0}$-module G, there exists a unique (upto affine module isomorphism) irreducible Hilbert affine submodule of H^ε, with the $\varepsilon 0$ space being isomorphic to G as an $AP_{\varepsilon 0,\varepsilon 0}$-module. Moreover, any irreducible Hilbert affine P-module with weight zero is isomorphic to a submodule of H^+ or H^-.*

PROOF. Finiteness of the depth of P and positive definiteness of ω_ε provide $AP_{\varepsilon 0,\varepsilon 0}$ with a finite dimensional C^*-algebra structure (using [7, Proof of Theorem 6.11]). Now, $H^\varepsilon_{\varepsilon 0} = AP_{\varepsilon 0,\varepsilon 0}$ is the regular $AP_{\varepsilon 0,\varepsilon 0}$-module, and by Wedderburn-Artin, $H^\varepsilon_{\varepsilon 0}$ contains all irreducible $AP_{\varepsilon 0,\varepsilon 0}$-modules (and hence, G too) as submodules. By Remark 2.14 (2), the submodule $[G]$ of H^ε, generated by G, is irreducible. Uniqueness follows from Remark 2.15.

For the second statement, consider an irreducible Hilbert affine P-module V with weight zero. Without loss of generality, let $V_{+0} \neq \{0\}$ which is also irreducible $AP_{+1,+1}$-module. By Remarks 2.14 and 2.15 and the first part, $V = [V_{+0}]$ sits inside H^+ as a submodule.

Next, we will investigate Hilbert affine P-modules which are generated by their $(+0)$- or (-0)- spaces where depth of P is not necessarily finite. The finite depth case is completely determinded by Corollary 4.3 which will not work in infinite depth because any irreducible $AP_{\varepsilon 0,\varepsilon 0}$-module might not be isomorphic to a submodule of $H^\varepsilon_{\varepsilon 0}$. However, the easiest example of an irreducible Hilbert affine P-module, namely, the planar algebra P itself, does sit inside both H^+ and H^- as a submodule. It is the submodule of H^ε generated by the one-dimensional orthogonal complement of the kernel of the linear homomorphism (which actually gives the dimension function via the isomorphism in Theorem 3.1)

$$AP_{\varepsilon 0,\varepsilon 0} \ni \psi^{2k}_{\varepsilon 0,\varepsilon 0}(x) \mapsto P_{TR^r_{\varepsilon 2k}}(x) \in P_{\varepsilon 0} \cong \mathbb{C}$$

for $k \in \mathbb{N}_0$, $x \in P_{\varepsilon 2k}$.

Let us denote the completion of H^ε by $\mathcal{H}^\varepsilon$ with $\mathcal{H}^\varepsilon_a$ being the unique extension of H^ε_a for all affine morphisms a (see Theorem 4.2). Then, $L^\varepsilon := (\mathcal{H}^\varepsilon_{AP_{\varepsilon 0,\varepsilon 0}})'' \subset \mathcal{B}(\mathcal{H}^\varepsilon_{\varepsilon 0})$ becomes a finite von Neumann algebra on which ω_ε extends to a faithful normal tracial state given by $\tilde{\omega}_\varepsilon := \langle \hat{1}, \cdot(\hat{1})\rangle : L^\varepsilon \to \mathbb{C}$. Note that $H^\varepsilon_{\eta k}$ has a right $AP_{\varepsilon 0,\varepsilon 0}$-module structure. Now, for all $a \in AP_{\varepsilon 0,\varepsilon 0}$, $b \in AP_{\varepsilon 0,\eta k}$, we have

$$\begin{aligned}\|b \circ a\|^2 &= \omega_\varepsilon(a^* \circ b^* \circ b \circ a) = \tilde{\omega}_\varepsilon(y\mathcal{H}^\varepsilon_{a\circ a^*} y)\\ &\leq \|\mathcal{H}^\varepsilon_a\|^2 \tilde{\omega}_\varepsilon(y^2) = \|\mathcal{H}^\varepsilon_a\|^2 \|b\|^2\end{aligned}$$

where $y \in L^\varepsilon$ is the positive square root of $\mathcal{H}^\varepsilon_{b^*\circ b}$. So, for all $\eta k \in \mathrm{Col}$, the right action of any element $a \in AP_{\varepsilon 0,\varepsilon 0}$ on $H^\varepsilon_{\eta k}$ is bounded as well; let $\rho^\varepsilon_{\eta k}(a) \in \mathcal{B}(\mathcal{H}^\varepsilon_{\eta k})$ denote its unique extension.

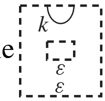
LEMMA 4.4. *For all $\eta k \in \mathrm{Col}$, the anti-algebra $*$-homomorphism $\rho^\varepsilon_{\eta k}$: $AP_{\varepsilon 0,\varepsilon 0} \to \mathcal{B}(\mathcal{H}^\varepsilon_{\eta k})$ extends to a normal anti-algebra $*$-homomorphism from L^ε to $\mathcal{B}(\mathcal{H}^\varepsilon_{\eta k})$. Moreover, it is faithful for all $\eta k \neq -\varepsilon 0$.*

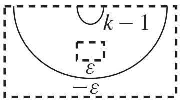
Proof. Note that $\mathcal{H}^{\varepsilon}_{a} \circ \rho^{\varepsilon}_{\eta k}(b) = \rho^{\varepsilon}_{\nu l}(b) \circ \mathcal{H}^{\varepsilon}_{a}$ for all $a \in AP_{\eta k,\nu l}$, $b \in AP_{\varepsilon 0,\varepsilon 0}$. Set $W_{\eta k} := (\rho^{\varepsilon}_{\eta k}(AP_{\varepsilon 0,\varepsilon 0}))'' \subset \mathcal{B}(\mathcal{H}^{\varepsilon}_{\eta k})$. Since $L^{\varepsilon} \supset \mathcal{H}^{\varepsilon}_{AP_{\varepsilon 0,\varepsilon 0}} \ni \mathcal{H}^{\varepsilon}_{a} \mapsto \rho^{\varepsilon}_{\varepsilon 0}(a) = J\mathcal{H}^{\varepsilon}_{a^*}J \in \rho^{\varepsilon}_{\varepsilon 0}(AP_{\varepsilon 0,\varepsilon 0}) \subset W_{\varepsilon 0}$ is an anti-algebra $*$-isomorphism (where J is the canonical conjugate linear unitary on $\mathcal{H}^{\varepsilon}_{\varepsilon 0}$), it is enough to show that

$$W_{\varepsilon 0} \supset \rho^{\varepsilon}_{\varepsilon 0}(AP_{\varepsilon 0,\varepsilon 0}) \ni \rho^{\varepsilon}_{\varepsilon 0}(a) \overset{\alpha_{\eta k}}{\longmapsto} \rho^{\varepsilon}_{\eta k}(a) \in \rho^{\varepsilon}_{\eta k}(AP_{\varepsilon 0,\varepsilon 0}) \subset W_{\eta k}$$

extends to a surjective normal $*$-homomorphism which is also injective for all $\eta k \neq -\varepsilon 0$.

Case 1: Suppose $\eta k = \varepsilon 0$. This case is trivial.

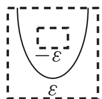
Case 2: Suppose $k > 0$. Let $c_{\varepsilon 0,\eta k} \in AP_{\varepsilon 0,\eta k}$ denote the affine tangle 

or 

according as $\eta = \varepsilon$ or $\eta = -\varepsilon$. Note that $U := \delta^{-k/2}\mathcal{H}^{\varepsilon}_{c_{\varepsilon 0,\eta k}} : \mathcal{H}^{\varepsilon}_{\varepsilon 0} \to \mathcal{H}^{\varepsilon}_{\eta k}$ is an isometry. Let $p := UU^* = \delta^{-k}\mathcal{H}^{\varepsilon}_{c_{\varepsilon 0,\eta k} \circ c^*_{\varepsilon 0,\eta k}} \in \mathcal{P}(\mathcal{H}^{\varepsilon}_{\eta k})$; clearly, $p \in W'_{\eta k}$. It is easy to check that the central support of p in $W_{\eta k}$ is 1 (using the fact $\mathcal{H}^{\varepsilon}_{AP_{\eta k,\eta k}}\rho^{\varepsilon}_{\eta k}(AP_{\varepsilon 0,\varepsilon 0})p(H^{\varepsilon}_{\eta k}) = H^{\varepsilon}_{\eta k}$). Thus, $W_{\eta k} \ni x \mapsto xp \in pW_{\eta k}$ is an isomorphism. This gives us an injective $*$-algebra homomorphism $W_{\eta k} \ni x \overset{\alpha}{\longmapsto} U^*xU \in \mathcal{B}(\mathcal{H}^{\varepsilon}_{\varepsilon 0})$. Range α is a von Neumann algebra since α is normal. On the other hand, $\alpha(\rho^{\varepsilon}_{\eta k}(a)) = \rho^{\varepsilon}_{\varepsilon 0}(a)$ for all $a \in AP_{\varepsilon 0,\varepsilon 0}$. So, Range $\alpha = W_{\varepsilon 0}$. Hence, $\alpha_{\eta k}$ is given by α^{-1}.

Case 3: Suppose $\eta k = -\varepsilon 0$. It is enough to show that

$$W_{\varepsilon 1} \supset \rho^{\varepsilon}_{\varepsilon 1}(AP_{\varepsilon 0,\varepsilon 0}) \ni \rho^{\varepsilon}_{\varepsilon 1}(a) \mapsto \rho^{\varepsilon}_{-\varepsilon 0}(a) \in W_{-\varepsilon 0}$$

extends to a normal $*$-homomorphism. For this, set $c_{-\varepsilon 0,\varepsilon 1} :=$ 

$\in AP_{-\varepsilon 0,\varepsilon 1}$ and $U := \delta^{-1/2}\mathcal{H}^{\varepsilon}_{c_{-\varepsilon 0,\varepsilon 1}} : \mathcal{H}^{\varepsilon}_{-\varepsilon 0} \to \mathcal{H}^{\varepsilon}_{\varepsilon 1}$. Note that $U^*U = 1$. Let $p := UU^* = \delta^{-1}\mathcal{H}_{c_{-\varepsilon 0,\varepsilon 1} \circ c^*_{-\varepsilon 0,\varepsilon 1}} \in \mathcal{P}(\mathcal{H}_{\varepsilon 1})$; clearly, $p \in W'_{\varepsilon 1}$. So, there exists a normal $*$-homomorphism $W_{\varepsilon 1} \ni x \overset{\alpha}{\longmapsto} U^*xU \in \mathcal{B}(\mathcal{H}_{-\varepsilon 0})$. Note that $\alpha(\rho^{\varepsilon}_{\varepsilon 1}(a)) = \rho^{\varepsilon}_{-\varepsilon 0}(a)$ for all $a \in AP_{\varepsilon 0,\varepsilon 0}$; this implies Range $\alpha = W_{-\varepsilon 0}$.

We now proceed towards finding the kernel of the extension of $\rho^{\varepsilon}_{-\varepsilon 0}$ to L^{ε}, which we denote with the same symbol. For this, consider the $*$-closed two sided ideal $I^{\varepsilon} := AP_{-\varepsilon 0,\varepsilon 0} \circ AP_{\varepsilon 0,-\varepsilon 0}$ in $AP_{\varepsilon 0,\varepsilon 0}$. Thus, $\overline{\mathcal{H}^{\varepsilon}_{I^{\varepsilon}}}$ (with respect to weak operator topology (WOT) in $\mathcal{B}(\mathcal{H}^{\varepsilon}_{\varepsilon 0})$) becomes a $*$-closed, WOT-closed two-sided ideal in L^{ε}; let z_{ε} be the central projection of L^{ε} such that $\overline{\mathcal{H}^{\varepsilon}_{I^{\varepsilon}}} = z_{\varepsilon}L^{\varepsilon}$.

LEMMA 4.5. $\ker \rho^{\varepsilon}_{-\varepsilon 0} = (1 - z_{\varepsilon})L^{\varepsilon}$.

PROOF. If $x \in L^{\varepsilon}$, then $\rho^{\varepsilon}_{-\varepsilon 0}(x) = 0$ if and only if

$$\begin{aligned} 0 = \langle \hat{c}, \rho^{\varepsilon}_{-\varepsilon 0}(x)\hat{d}\rangle &= \langle \hat{1}, \mathcal{H}^{\varepsilon}_{c^*}\rho^{\varepsilon}_{-\varepsilon 0}(x)\mathcal{H}^{\varepsilon}_{d}\hat{1}\rangle = \langle \hat{1}, \mathcal{H}^{\varepsilon}_{c^*\circ d}\rho^{\varepsilon}_{\varepsilon 0}(x)\hat{1}\rangle \\ &= \langle \hat{1}, \mathcal{H}^{\varepsilon}_{c^*\circ d}\rho^{\varepsilon}_{\varepsilon 0}(x)\hat{1}\rangle = \langle \hat{1}, \mathcal{H}^{\varepsilon}_{c^*\circ d} x\hat{1}\rangle \\ &= \tilde{\omega}_{\varepsilon}(\mathcal{H}^{\varepsilon}_{c^*\circ d}\, x) \end{aligned}$$

for all $c, d \in AP_{\varepsilon 0, -\varepsilon 0}$. Thus, by WOT-continuity of $\tilde{\omega}_{\varepsilon}$, we get $x \in \ker \rho^{\varepsilon}_{-\varepsilon 0}$ if and only if $\tilde{\omega}_{\varepsilon}(z_{\varepsilon} x x^*) = 0$ which is equivalent to $x z_{\varepsilon} = 0$ (using faithfulness of $\tilde{\omega}_{\varepsilon}$). This give the required equation.

THEOREM 4.6. *For every left L^{ε}-module $\mathcal{K}$, there exists a unique Hilbert affine P-module, say $[\mathcal{K}]$, such that $\mathcal{K}$ and $[\mathcal{K}]_{\varepsilon 0}$ are isometrically isomorphic as $AP_{\varepsilon 0,\varepsilon 0}$-modules and $[[\mathcal{K}]_{\varepsilon 0}] = [\mathcal{K}]$. Further, $[\mathcal{K}]_{-\varepsilon 0} = \{0\}$ if and only if the action of z_{ε} on $\mathcal{K}$, is zero.*

PROOF. Uniqueness easily follows from Remark 2.15. Next, we consider the space of bounded vectors, $\mathcal{K}^0$ which will be dense in $\mathcal{K}$ and have a left L^{ε}-valued inner product ${}_{L^{\varepsilon}}\langle \cdot, \cdot\rangle$ satisfying $\tilde{\omega}_{\varepsilon} \circ {}_{L^{\varepsilon}}\langle \cdot, \cdot\rangle = \langle \cdot, \cdot\rangle$. On the other hand, $\mathcal{H}^{\varepsilon}_{\eta k}$ gets a right L^{ε}-module structure from Lemma 4.4; so, $(\mathcal{H}^{\varepsilon}_{\eta k})^0$ (the space of bounded vectors of $\mathcal{H}^{\varepsilon}_{\eta k}$) will have a right L^{ε}-valued inner product compatible with $\tilde{\omega}_{\varepsilon}$. We now use Connes-fusion techniques to build $[\mathcal{K}]_{\eta k} := \mathcal{H}^{\varepsilon}_{\eta k} \otimes_{L^{\varepsilon}} \mathcal{K}$. The action of $a \in AP_{\eta k, \nu l}$ is given by $[\mathcal{K}]_a := \mathcal{H}^{\varepsilon}_a \otimes_{L^{\varepsilon}} \mathrm{id}_{\mathcal{K}} : [\mathcal{K}]_{\eta k} \to [\mathcal{K}]_{\nu l}$. This makes $[\mathcal{K}]$ a Hilbert affine P-module.

For the remaining part, first note that $(AP_{\varepsilon 0, \eta k})\hat{}$ sits inside $(\mathcal{H}^{\varepsilon}_{\eta k})^0$ and is dense in $\mathcal{H}^{\varepsilon}_{\eta k}$. Thereby, $\mathrm{span}\{\hat{a} \otimes_{L^{\varepsilon}} \zeta = [\mathcal{K}]_a(\hat{1} \otimes_{L^{\varepsilon}} \zeta) : a \in AP_{\varepsilon 0, \eta k}, \zeta \in \mathcal{K}^0\}$ becomes a dense subset in $[\mathcal{K}]_{\eta k}$. Thus, $[[\mathcal{K}]_{\varepsilon 0}] = [\mathcal{K}]$. The map $\mathcal{K}^0 \ni \zeta \mapsto \hat{1} \otimes_{L^{\varepsilon}} \zeta \in [\mathcal{K}]_{\varepsilon 0}$ extends to a surjective $AP_{\varepsilon 0,\varepsilon 0}$-linear isometry from $\mathcal{K}$ to $[\mathcal{K}]_{\varepsilon 0}$.

Observe that $[\mathcal{K}]_{-\varepsilon 0} = 0$ if and only if $0 = \langle \hat{a} \otimes L^{\varepsilon}\zeta, \hat{b} \otimes_{L^{\varepsilon}} \zeta\rangle = \langle \zeta, (a^* \circ b)\zeta\rangle$ for all $a, b \in AP_{\varepsilon 0, -\varepsilon 0}$, $\zeta \in \mathcal{K}^0$. Since the representation $L^{\varepsilon} \to \mathcal{B}(\mathcal{K})$ is normal, this is equivalent to $z_{\varepsilon}(\mathcal{K}^0) = \{0\}$ and hence, we get the required result.

From the above theorem, we wonder whether every $*$-affine P-module V which is generated by $V_{\varepsilon 0}$, can be obtained in this way of extending an L^{ε}-module. The trivial module P is the extension of the one-dimensional $AP_{\varepsilon 0,\varepsilon 0}$-module given by the dimension function. Another question along this line is whether we can do similar analysis for $AP_{\varepsilon k, \varepsilon k}$ where $k > 0$.

REMARK 4.7. Note that the spaces of affine morphisms and 'annular morphisms' (see [10]) with the color of internal or external rectangle being ± 0, are canonically isomorphic (because there will not be any difference between affine isotopy and the usual planar isotopy in such cases). So, all results on affine category over P and affine P-modules, obtained in Sections 3 and 4, also hold for annular category over P and annular representations.

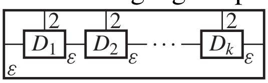
5. Affine modules of irreducible depth two planar algebras

This section deals with irreducible depth two subfactor planar algebras. Such planar algebras are the ones associated to the subfactors arising from action of finite dimensional Kac algebras. We will establish an equivalence between the category of affine modules over such a planar algebra and the representation category of the quantum double of the corresponding Hopf algebra, and thereby, confirming Jones-Walker conjecture in this case.

Throughout this section, P will denote an irreducible depth two subfactor planar algebra and $\varepsilon = \pm$.

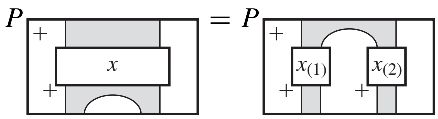
5.1. Affine morphisms at level one

By [19], $H_\varepsilon := P_{\varepsilon 2}$ has a Kac algebra structure. We will first briefly recall this structure in the language of planar algebras (see [15], [4] for details). Suppose

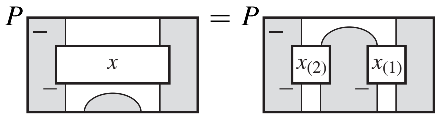
$G_{\varepsilon k} =$ .

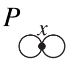
REMARK 5.1. Since depth of P is 2, Range $P_{G_{\varepsilon k}} = P_{\varepsilon(k+1)}$ for all $k \geq 1$. This along with irreducibility of P gives $\dim_{\mathbb{C}}(P_{\varepsilon(k+1)}) = [\dim_{\mathbb{C}}(P_{\varepsilon 2})]^k$ which implies that $P_{G_{\varepsilon k}} : (P_{\varepsilon 2} \otimes P_{\varepsilon 2} \otimes \cdots k \text{ factors}) \to P_{\varepsilon(k+1)}$ is an isomorphism for all $k \geq 1$.

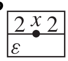
We already know the C^*-algebra structure on H_ε. We now define the comultiplication map $\Delta_\varepsilon : H_\varepsilon \to H_\varepsilon \otimes H_\varepsilon$; we will use Sweedler's notation, namely, $\Delta_\varepsilon(x) = x_{(1)} \otimes x_{(2)}$ which is determined by the equations



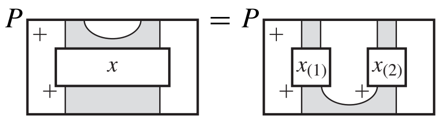
and



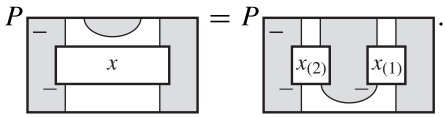
for $x \in H_\varepsilon$. The counit is given by $H_\varepsilon \ni x \overset{\chi_\varepsilon}{\longmapsto} \delta^{-1} P$  $\in P_{\varepsilon 0} \cong \mathbb{C}$ and

the antipode is $H_\varepsilon \ni x \overset{S_\varepsilon}{\longmapsto} P$  $\in H_\varepsilon$. With these structural maps, H_ε

becomes a finite dimensional Kac algebra. The following two relations will be very useful:

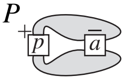


and



LEMMA 5.2. *$H_- \cong (H_+^{\mathrm{op}})^*$ as Kac algebras.*

PROOF. Define a bilinear form

$$H_+ \times H_- \ni (p,a) \overset{\langle\cdot,\cdot\rangle}{\longmapsto} \langle p,a\rangle := \delta^{-1} P$$


$$\in P_{+0} \cong \mathbb{C}.$$

Non-degeneracy of the actions of the trace tangles, implies that $\langle\cdot,\cdot\rangle$ is non-degenerate. From the definition of the structural maps and the above formulae, it is easy to verify

$$\begin{aligned}\langle p, ab\rangle &= \langle p_{(1)}, a\rangle\langle p_{(2)}, b\rangle,\\ \langle qp, a\rangle &= \langle p, a_{(1)}\rangle\langle q, a_{(2)}\rangle,\\ \langle p, a^*\rangle &= \overline{\langle S_+(p^*), a\rangle}\end{aligned}$$

where $p, q \in H_+$ and $a, b \in H_-$.

Next, we recall the definition of the quantum double from [14]. Let H be a finite dimensional Hopf algebra. The quantum double of H is the Hopf algebra $(H^{\mathrm{op}})^* \bowtie H$ (also denoted by DH) which is $(H^{\mathrm{op}})^* \otimes H$ as a vector space with structural maps given by:

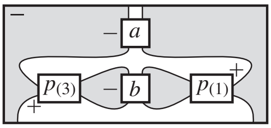
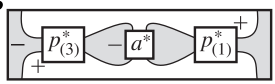
- *Multiplication*: $(f_1 \bowtie h_1)(f_2 \bowtie h_2) = f_1[f_2(S^{-1}((h_1)_{(3)}) \cdot (h_1)_{(1)})] \bowtie (h_1)_{(2)} h_2$,
- *Unit*: $\chi_H \bowtie 1$ (χ_H is the counit of H),
- *Comultiplication*: $\Delta(f \bowtie h) = f_{(1)} \bowtie h_{(1)} \otimes f_{(2)} \bowtie h_{(2)}$,
- *Counit*: $\chi(f \bowtie h) = f(1)\chi_H(h)$,
- *Antipode*: $S(f \bowtie h) = f(h_{(3)} S^{-1}(\cdot) S^{-1}(h_{(1)})) \bowtie S(h_{(2)})$.

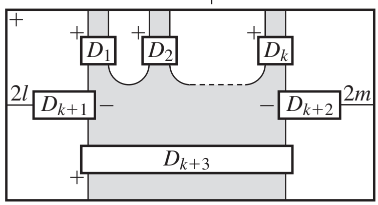
Moreover, if H is a Hopf $*$-algebra, then DH also has a $*$-structure given by

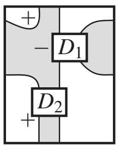
$$(f \bowtie h)^* = \overline{f}(h_{(3)}[S \circ *(\cdot)] S(h_{(1)})) \bowtie h^*_{(2)} = f^*(S^{-1}(h^*_{(3)}) \cdot h^*_{(1)}) \bowtie h^*_{(2)}.$$

Getting back to our context, by Lemma 5.2, DH_+ can be considered as $H_- \bowtie H_+$.

REMARK 5.3. Using the duality defined in the proof of Lemma 5.2, the $*$-algebra structure of DH_+ can be expressed as:

- Multiplication: $(a \bowtie p)(b \bowtie q) = P$  $\bowtie p_{(2)}q$,
- Unit: $1 \bowtie 1$,
- $*$-structure: $(a \bowtie p)^* = P$  $\bowtie p^*_{(2)}$.

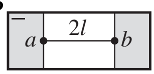
In order to establish a link between the quantum double of H_+ and the affine morphisms, we consider the tangles $T^k_{l,m} :=$ 

and $U :=$  where $k, l, m \geq 1$. When l (resp., m) is zero, then the $T^k_{l,m}$ denotes the tangle obtained by composing the above tangle with 1_{-1} (defined in Figure 1) over the internal disc D_{k+2} (resp., D_{k+1}). Note that $P_{T^1_{1,1}}(1_{H_+}, \cdot, P_{E_{-1}}, \cdot) = P_U$.

PROPOSITION 5.4. *The map* $DH_+ \ni (a \bowtie p) \overset{\Gamma}{\longmapsto} \psi^2_{+1,+1}(P_U(a, p)) \in AP_{+1,+1}$ *is a surjective $*$-algebra homomorphism.*

PROOF. Using the structural maps of $H_\pm$ and DH_+, and affine isotopy, it is completely routine to check that Γ preserves multiplication and $*$.

Surjectivity of Γ: Consider an element $\psi^{2l}_{+1,+1}(x) \in AP_{+1,+1}$ for $x \in P_{+2(l+1)}$. Now, Remark 5.1 implies that Range $P_{T^1_{l,l}}(1_{H_+}, \cdot, \cdot, \cdot) = P_{+2(l+1)}$; so, without loss of generality, we can assume $x = P_{T^1_{l,l}}(1_{H_+}, a, b, p)$ for $p \in H_+$ and $a, b \in P_{-(l+1)}$. Applying affine isotopy, we can write $\psi^{2l}_{+1,+1}(x) = \psi^2_{+1,+1}(P_U((a \odot b), p))$ where

$$P_{-(l+1)} \times P_{-(l+1)} \ni (a, b) \overset{\odot}{\longmapsto} P$$  $$\in H_-.$$

Next, we proceed towards proving injectivity of Γ. Set $V := \text{Range } P_U$ which (by Remark 5.1) is isomorphic to $H_- \otimes H_+$. Proposition 5.4 implies $\dim_{\mathsf{C}}(AP_{+1,+1}) \leq \dim_{\mathsf{C}}(H_+) \dim_{\mathsf{C}}(H_-)$. So, it is enough to construct a surjective linear map from $AP_{+1,+1}$ to V.

For all $l \in \mathsf{N}$, consider the maps

$$P_{-(l+1)} \otimes P_{-(l+1)} \otimes H_+ \ni (a \otimes b \otimes p) \xmapsto{\sigma_l} P_U((a \odot b), p) \in V$$

and

$$P_{-(l+1)} \otimes P_{-(l+1)} \otimes H_+ \ni (a \otimes b \otimes p) \xmapsto{\tau_l} P_{T^1_{l,l}}(1_{H_+}, a, b, p) \in P_{+2(l+1)}.$$

By Remark 5.1, τ_l is an isomorphism and σ_l is surjective. Define the linear maps

$$\mathcal{P}_{+2(l+1)}(P) \ni X \xmapsto{\gamma_l} \sigma_l(\tau_l^{-1}(P_X)) \in V \qquad \text{for} \quad l \geq 1$$

and

$$\mathcal{P}_{+2}(P) \ni X \xmapsto{\gamma_0} P_U(1_{H_-}, P_X) \in V.$$

We construct a map $\mathcal{AT}_{+1,+1}(P) \ni A \xmapsto{\tilde{\gamma}} \gamma_l(T) \in V$ where $A = \Psi^{2l}_{+1,+1}(T)$ for some $T \in \mathcal{T}_{+2(l+1)}(P)$, $l \geq 0$. Then, the obvious question is whether $\tilde{\gamma}$ is well-defined. If so, then we extend it linearly to $\tilde{\gamma} : \mathcal{A}_{+1,+1}(P) \to V$ which also becomes surjective and satisfies $\mathcal{W}_{+1,+1} \subset \ker \tilde{\gamma}$. Thus, $\tilde{\gamma}$ factors through the quotient $AP_{+1,+1}$ and thereby, Γ becomes injective.

Well-definedness of $\tilde{\gamma}$: We will follow the treatment as in Section 3.2. Set $\mathcal{T} := \bigsqcup_{l \in \mathsf{N}_0} \mathcal{T}_{+2(l+1)}(P)$. We define an equivalence relation $\sim$ on $\mathcal{T}$ in Figure 7.

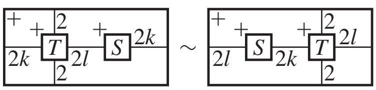


FIGURE 7. $S \in \mathcal{T}_{+(k+l)}(P)$, $T \in \mathcal{T}_{+(k+l+2)}(P)$, $k, l \in \mathsf{N}_0$

Analogous to Lemma 3.7, we have the following useful, straight-forward adaptation of [6, Proposition 2.8] to the setting of morphisms in the affine category over a planar algebra.

LEMMA 5.5. *For* $X \in \mathcal{T}_{+2(k+1)}(P)$, $Y \in \mathcal{T}_{+2(l+1)}(P)$, $k, l \in \mathsf{N}_0$, *we have:*

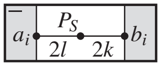
(i) $\Psi^{2k}_{+1,+1}(X) = \Psi^{2l}_{+1,+1}(Y)$ *if and only if* $X \sim Y$, *and*

(ii) $X \sim Y$ *implies* $\gamma_k(X) = \gamma_l(Y)$.

PROOF. (i) The 'if' part can easily be seen using affine isotopy. The 'only if' part can be proved by following the arguments used in the proof of the 'only if' part in Lemma 3.7.

(ii) Suppose X and Y are the tangles on the left and right sides of $\sim$ in Figure 7 respectively. Remark 5.1 implies Range $P_{T^1_{k,l}}(1_{H_+}, \cdot, \cdot, \cdot) = P_{+(k+l+2)}$; so, there exists $\{a_i\}_i \subset P_{-(l+1)}$, $\{b_i\}_i \subset P_{-(k+1)}$ and $\{p_i\}_i \subset H_+$ such that $P_T = \sum_i P_{T^1_{k,l}}(1_{H_+}, a_i, b_i, p_i)$. Now, if $k, l > 0$, then

$$\gamma_k(X) = \sum_i P_U \left(P_{\square}, p_i \right) = \gamma_l(Y).$$



The case when $k = 0 = l$, the equation holds trivially.

Suppose $k = 0 < l$. Again, by Remark 5.1, there exists $\{a_i\}_i \subset P_{-(l+1)}$ and $\{p_i\}_i \subset H_+$ such that $P_T = \sum_i P_{T^1_{0,l}}(1_{H_+}, a_i, p_i)$. Note that $P_Y = \sum_i P_{T^1_{l,l}}(1_{H_+}, a_i, P_{LI_{+l}}(P_S), p_i)$. Thus, $\gamma_l(Y) = \sum_i P_U(a_i \odot P_{LI_{+l}}(P_S), p_i)$. Since P is irreducible, there exists $c_i \in \mathbb{C}$ such that $a_i \odot P_{LI_{+l}}(P_S) = c_i 1_{H_-}$ which also implies $P_X = \sum_i c_i\, p_i$. Hence, $\gamma_0(X) = \sum_i c_i P_U(1_{H_-}, p_i) = \gamma_l(Y)$. Similar arguments yeild the case $k > 0 = l$.

Hence, we have proved the following proposition.

Proposition 5.6. *Γ, as in Proposition 5.4, is an isomorphism.*

5.2. The affine modules of P

Let $t_\varepsilon : H_\varepsilon \to \mathbb{C}$ denote the normalized action of the trace tangle on H_ε, that is, $t_\varepsilon = \delta^{-2} P_{TR^r_{\varepsilon 2}}$. Consider the linear functional $DH_+ \ni a \bowtie p \overset{t}{\longmapsto} t_-(a)t_+(p) \in \mathbb{C}$. From Remark 5.3 and the structural maps in the beginning of Section 5.1, it easily follows $t((a \bowtie p)^*(a \bowtie p)) = t_-(a^*a)t_+(p^*p)$; thus, $t_- \bowtie t_+$ is positive definite and DH_+ becomes a finite dimensional C^*-algebra. Set $\tilde{t} := t \circ \Gamma^{-1} : AP_{+1,+1} \to \mathbb{C}$.

Theorem 5.7. *If $N \subset M$ is an irreducible subfactor with depth two and planar algebra P, then the category of Hilbert affine P-modules is equivalent to the center of the category of N-N-bimodules generated by ${}_N L^2(M)_M$ as additive categories.*

Proof. From [19], one can deduce that the category of N-N-bimodules (appearing in the standard invariant) is contravariantly equivalent to the representation category of the Kac algebra H_+; thus, its center then becomes contravariantly equivalent to the representation category of DH_+ (see [14], Theorem XIII.5.1). So, using Remark 2.16, it is enough to establish a one-to-one correspondence between the isomorphism classes of irreducible Hilbert $+$-affine P-modules and that of irreducible DH_+-modules. The key step towards this will be given by the following construction of an Hilbert $+$-affine P-module generated by $AP_{+1,+1}$.

Set $V_k := AP_{+1,+k}$ for all $k \geq 1$ and $V_{\varepsilon 0} := AP_{+1,\varepsilon 0}$. Note that by [7, Proof of Theorem 6.11], V_k's are all finite dimensional. We define a sesquilinear form $\langle v, w\rangle := \tilde{t}(v^* \circ w)$ for all $v, w \in V_k$, $k \in \{\pm 0\} \cup \mathsf{N}$.

Positivity of $\langle\cdot,\cdot\rangle$: The case $k = 1$ is already covered by Proposition 5.6.

Case 1: Suppose $k > 1$. Note that $V_k = \bigcup_{l\in \mathsf{N}} \psi^{2l}_{+1,+k}(P_{+(2l+k+1)})$. Now, Remark 5.1 implies Range $P_{T^k_{l,l}}(1_{H_+}, \cdot, \dots, \cdot) = P_{+(2l+k+1)}$. Applying affine isotopy, we get

$$\begin{aligned}\psi^{2l}_{+1,+k}&\big(P_{T^k_{l,l}}(1_{H_+}, x_2, \dots, x_k, a, b, p)\big)\\ &= \psi^{2}_{+1,+k}\big(P_{T^k_{1,1}}(1_{H_+}, x_2, \dots, x_k, a\odot b, P_{E_{-1}}, p)\big)\\ &= \psi^{0}_{+1,+k}\big(P_{T^k_{0,0}}(1_{H_+}, x_2, \dots, x_k, 1_{H_+})\big)\circ \psi^{2}_{+1,+1}(P_U(a\odot b, p))\end{aligned}$$

(which is independent of l) for all $x_2, \dots x_k \in H_+$, $a, b \in P_{-(l+1)}$ and $p \in H_+$. Thus, the linear map defined by

$$\begin{aligned}[(H_+)^{\otimes(k-1)} \otimes AP_{+1,+1}] \ni x_2 \otimes \cdots \otimes x_k \otimes w&\\ \overset{\zeta}{\longmapsto} \psi^{2}_{+1,+k}\big(P_{T^k_{0,0}}(1_{H_+}, x_2, \dots, x_k, 1_{H_+})\big)\circ w \in V_k&\end{aligned}$$

is surjective. Since P is irreducible, we have

$$\begin{aligned}\big[\psi^{2}_{+1,+k}&\big(P_{T^k_{0,0}}(1_{H_+}, x_2, \dots, x_k, 1_{H_+})\big)\big]^*\\ &\circ \psi^{2}_{+1,+k}\big(P_{T^k_{0,0}}(1_{H_+}, x_2, \dots, x_k, 1_{H_+})\big) = \delta^k\Big[\prod_{2\leq n\leq k} t_+(x_n^* x_n)\Big]1_{AP_{+1,+1}}\end{aligned}$$

which implies

$$\langle \zeta(x_2\otimes\cdots\otimes x_k\otimes w), \zeta(y_2\otimes\cdots\otimes y_k\otimes v)\rangle = \delta^k\Big[\prod_{2\leq n\leq k} t_+(x_n^* y_n)\Big]t(w^*\circ v)$$

for all $x_2, \dots, x_k, y_2, \dots, y_k \in H_+$ and $w, v \in AP_{+1,+1}$. Hence $\langle\cdot,\cdot\rangle$ is positive definite on V_k.

Case 2: Suppose $k = \varepsilon 0$. Consider the affine morphism $c_\varepsilon \in AP_{\varepsilon 0,+1}$ given by the affine tangle with a single string attached to the two marked points on the boundary of the external rectangle. Now, since $c_\varepsilon \circ v \neq 0$ for all $0 \neq v \in V_{\varepsilon 0}$, we have $\langle v, v\rangle_{V_{\varepsilon 0}} = \delta^{-1}\langle c_\varepsilon \circ v, c_\varepsilon \circ v\rangle_{V_1} > 0$.

Hence, V is a Hilbert affine P-module. Now, V_1 is the regular $AP_{+1,+1}$-module; so, it contains every irreducible $AP_{+1,+1}$-module as a submodule. By Remark 2.14, the affine submodule $\tilde{W}$ of V generated by each of these irreducible $AP_{+1,+1}$-submodule W of V_1, will be irreducible; moreover, $W_1 \cong W_2$ if and only if $\tilde{W}_1 \cong \tilde{W}_2$. On the other hand, if we start with an irreducible

Hilbert affine P-module U, then U_1 is nonzero (since weight of every affine P-modules cannot exceed 1 by [7, Theorem 6.10]) and is an irreducible $AP_{+1,+1}$-module (see Remark 2.14). So, there exists a submodule W of V_1, which is isomorphic to U_1. Using Remark 2.15, we may conclude $U \cong \tilde{W}$.

Hence, we have established a one-to-one correspondence between the isomorphism class of irreducible Hilbert affine P-modules and that of irreducible $AP_{+1,+1}$-modules. This ends the proof.

Note that Theorem 5.7 confirms Jones-Walker conjecture (stated in the introduction) for the case of irreducible depth two subfactors.

Some questions

In Section 4, we provided an explicit way of constructing a large class of Hilbert affine P-modules generated by their zero spaces for any subfactor planar algebra P. The natural question to ask is whether all $*$-affine P-modules (not necessarily bounded) generated by their zero spaces, arise in this way for infinite depth P's. It will also be interesting to analyze the affine P-modules with weight greater than zero.

In Section 5, we used irreducibility of P quite crucially in affirming the Jones-Walker conjecture. The next obvious thing to check will be whether we can make this work in the absence of irreducibility, that is, the 'weak Hopf algebra' case. An important drawback of the category of the Hilbert affine P-modules, is the lack of a monoidal structure, let alone braiding; note that the equivalence established in Theorem 5.7, is an equivalence of additive categories. One would guess some kind of comultiplication structure on the affine category, might yeild an appropriate monoidal structure on the category of affine P-modules.

We will address and answer some of these questions in a forthcoming article.

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