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REGULARITY AND FREE RESOLUTION OF IDEALS WHICH ARE MINIMAL TO d-LINEARITY

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Abstract

For given positive integers $n \geq d$, a d-uniform clutter on a vertex set $[n] = \{1, \dots, n\}$ is a collection of distinct d-subsets of $[n]$. Let C be a d-uniform clutter on $[n]$. We may naturally associate an ideal $I(C)$ in the polynomial ring $S = k[x_1, \dots, x_n]$ generated by all square-free monomials $x_{i_1} \cdots x_{i_d}$ for $\{i_1, \dots, i_d\} \in C$. We say a clutter C has a d-linear resolution if the ideal $I(\overline{C})$ has a d-linear resolution, where $\overline{C}$ is the complement of C (the set of d-subsets of $[n]$ which are not in C).

In this paper, we introduce some classes of d-uniform clutters which do not have a linear resolution, but every proper subclutter of them has a d-linear resolution. It is proved that for any two d-uniform clutters C_1, C_2 the regularity of the ideal $I(\overline{C_1 \cup C_2})$, under some restrictions on their intersection, is equal to the maximum of the regularities of $I(\overline{C}_1)$ and $I(\overline{C}_2)$.

As applications, alternative proofs are given for Fröberg's Theorem on linearity of edge ideals of graphs with chordal complement as well as for linearity of generalized chordal hypergraphs defined by Emtander. Finally, we find minimal free resolutions of the ideal of a triangulation of a pseudo-manifold and a homology manifold explicitly.

1. Introduction

Although the problem of classification of monomial ideals with d-linear resolution is solved for $d = 2$, it is still open for $d > 2$. Passing via polarization, it is enough to solve the problem for square-free monomial ideals. An ideal generated by square-free monomials of degree 2 can be assumed to be an edge ideal of a graph and more generally, an ideal generated by square-free monomials of degree d is the circuit ideal of a d-uniform clutter. R. Fröberg [6] proved that the edge ideal of a graph G has a 2-linear resolution if and only if in the complement graph of G every cycle of length greater than 3 has a chord. In this case, linearity of the resolution does not depend on the characteristic of the ground field. To generalize Fröberg's result to higher dimensional clutters, we face the problem that linearity of resolutions of a circuit ideal of a d-uniform clutter for $d > 2$ depends on the characteristic of the ground field. For instance, the ideal corresponding to triangulation of the projective plane has a linear resolution in characteristic zero while it does not have a linear resolution

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in characteristic 2. In a new proof of Fröberg's Theorem in [13], the notion of cycle plays a key role. That means:

(1) Cycles are exactly those graphs that are minimal to 2-linearity.
(2) The edge ideal of $\overline{G}$ does not have a 2-linear resolution if and only if G contains a cycle of length > 3, as an induced subgraph.

Trying to find a similar notion for cycles, we introduce the notion of minimal to d-linearity in arbitrary d-uniform clutters. By Proposition 6.5, pseudo-manifolds have the property of minimal to d-linearity. Also we know that, if C is a d-uniform clutter which has an induced subclutter isomorphic to a d-dimensional pseudo-manifold, then the ideal $I(\overline{C})$ does not have a linear resolution. But, Example 6.6, shows that the class of pseudo-manifolds is strictly contained in the class of minimal to linearity clutters. Another difficulty for generalizing Fröberg's Theorem, is the term 'induced' in point (2) above. That is, there are clutters which do not have a linear resolution and do not have any induced subclutter minimal to d-linearity. For instance, consider C is a triangulation of the sphere (with large enough number of vertices), which is a pseudo-manifold, let v_1, v_2, v_3 be vertices such that $\{v_1, v_2\}$ belongs to a circuit of C and neither $\{v_1, v_3\}$ nor $\{v_2, v_3\}$ belong to any circuit. Then add a new circuit $\{v_1, v_2, v_3\}$ to C. The new clutter does not have any induced subclutter which is minimal to d-linearity, however its circuit ideal does not have d-linear resolution.

In [4], [7], [16], [17] the authors have partially generalized Fröberg's Theorem. They have introduced several definitions of chordal clutters and proved that corresponding circuit ideals have linear resolution. In [12], the notion of simplicial submaximal circuit is introduced and proved that removing such submaximal circuits does not change the regularity of the circuit ideal. This proves linearity of resolutions of a large class of clutters (Remark 3.10 in [12]). To attack this problem from another direction, in the present paper we investigate clutters which do not have a linear resolution, but any proper subclutter of them has a linear resolution.

Section 2 is devoted to collect prerequisites and basic definitions which we need in the next chapters. In Section 3, some homological behaviours of the Stanley-Reisner ideal of a simplicial complex Δ with $\operatorname{indeg}(I_\Delta) \geq 1 + \dim \Delta$ are investigated and some minor extensions are made for results of Terai and Yoshida in [15].

Sections 4 and 5 contain the main results of this paper. Section 4 is about uniform clutters and their circuit ideals. In this section, we prove that for two d-uniform clutters C_1, C_2, the Castelnuovo-Mumford regularity of the ideal $I(\overline{C_1 \cup C_2})$, is the maximum of the regularities of these two components, whenever $V(C_1) \cap V(C_2)$ is a clique or $\mathrm{SC}(C_1) \cap \mathrm{SC}(C_2) = \varnothing$ (See

Definition 4.1). In Section 5, we define notions of obstruction to d-linearity, minimal to d-linearity and almost tree clutters. These are clutters such that their circuit ideals do not have a d-linear resolution but any proper subclutter of them has a d-linear resolution. We compare these classes and then, compute explicitly the minimal free resolution of clutters which are minimal to d-linearity.

In Section 6, as some applications to the results of previous sections, we give an alternative proof for Fröberg's theorem. Also a proof for linearity of resolution of generalized chordal hypergraphs defined by Emtander in [4] is given. Finally, we find minimal free resolutions of circuit ideals of triangulations of pseudo-manifolds and homology manifolds.

2. Preliminaries

Let K be a field and R be a standard graded K-algebra with irredundant homogeneous maximal ideal $\mathfrak{m}$. Let M be a finitely generated graded R-module and

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

a graded minimal free resolution of M with $F_i = \oplus_j R(-j)^{\beta^K_{i,j}}$ for all i.

The numbers $\beta^K_{i,j}(M) = \dim_K \operatorname{Tor}^R_i(K, M)_j$ are called the *graded Betti numbers* of M and

$$\operatorname{proj}\dim(M) = \sup\{i : \operatorname{Tor}^R_i(K, M) \neq 0\}$$

is called the projective dimension of M. Throughout this paper, we fix the field K and for convenience we write simply $\beta_{i,j}$ instead of $\beta^K_{i,j}$. The Auslander-Buchsbaum Theorem enables us to find the projective dimension in terms of depth.

THEOREM 2.1 (Auslander-Buchsbaum [3, Exercise 19.8]). *Let K be a field and R be a standard graded K-algebra with irredundant homogeneous maximal ideal $\mathfrak{m}$. Let M be a finitely generated graded R-module with finite projective dimension. Then,*

$$\operatorname{proj}\dim M + \operatorname{depth}(\mathfrak{m}, M) = \operatorname{depth}(\mathfrak{m}, R).$$

The *Castelnuovo-Mumford regularity* $\operatorname{reg}(M)$ of $M \neq 0$ is given by

$$\operatorname{reg}(M) = \sup\{j - i : \beta_{i,j}(M) \neq 0\}.$$

The *initial degree* $\operatorname{indeg}(M)$ of M is given by

$$\operatorname{indeg}(M) = \inf\{i : M_i \neq 0\}.$$

We say that a finitely generated graded R-module M has a *d-linear resolution* if its regularity is equal to $d = \operatorname{indeg}(M)$.

A *simplicial complex* Δ over a set of vertices $V = \{v_1, \dots, v_n\}$ is a collection of subsets of V, such that $\{v_i\} \in \Delta$ for all i, and if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set). An element of Δ is called a *face* of Δ, and the *dimension* of a face F of Δ is $|F| - 1$, where $|F|$ is the number of elements of F. The maximal faces of Δ under inclusion are called *facets* of Δ. The *dimension* of Δ, $\dim \Delta$, is the maximum dimension of its facets. Let $\mathcal{F}(\Delta) = \{F_1, \dots, F_q\}$ be the facet set of Δ. A simplicial complex Γ is called a *subcomplex* of Δ if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of $S = K[x_1, \dots, x_n]$ generated by square-free monomials $\{x_{i_1} \cdots x_{i_r} \mid \{v_{i_1}, \dots, v_{i_r}\} \notin \Delta\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ. We have

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\bar{F}},$$

where $P_{\bar{F}}$ denotes the (prime) ideal generated by $\{x_i \mid v_i \notin F\}$. In particular, $\dim K[\Delta] = 1 + \dim \Delta$.

For a simplicial complex Δ of dimension d, let $f_i = f_i(\Delta)$ denote the number of faces of Δ of dimension i; by convention $f_{-1} = 1$. The sequence $\mathbf{f}(\Delta) = (f_{-1}, f_0, \dots, f_d)$ is called the $\mathbf{f}$*-vector* of Δ.

Let Δ be a simplicial complex with vertex set V. An *orientation* on Δ is a linear order on V. A simplicial complex together with an orientation is an *oriented simplicial complex*.

Suppose Δ is an oriented simplicial complex of dimension d, and $F \in \Delta$ a face of dimension i. We write $F = [v_0, \dots, v_i]$ if $F = \{v_0, \dots, v_i\}$ and $v_0 < \dots < v_i$, and $F = [\ ]$ if $F = \varnothing$. With this notation, we define the *augmented oriented chain complex of* Δ,

$$\tilde{\mathscr{C}}(\Delta)\colon 0 \xrightarrow{\partial_{d+1}} C_d \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow 0,$$

by setting

$$C_i = \bigoplus_{\substack{F \in \Delta \\ \dim F = i}} KF \qquad \text{and} \qquad \partial_i(F) = \sum_{j=1}^{i} (-1)^j F_j$$

for all $F \in \Delta$; here $F_j = [v_0, \dots, \hat{v}_j, \dots, v_i]$ for $F = [v_0, \dots, v_i]$. A straightforward computation shows that $\partial_i \circ \partial_{i+1} = 0$. We set

$$\tilde{H}_i(\Delta; K) = H_i\big(\tilde{\mathscr{C}}(\Delta)\big) = \frac{\ker \partial_i}{\operatorname{Im} \partial_{i+1}}, \qquad i = -1, \dots, d,$$

and call $\tilde{H}_i(\Delta; K)$ the *i-th reduced simplicial homology of* Δ. Since $C_i \otimes K$ is a vector space of dimension f_i, elementary linear algebra yields

$$-1 + \sum_{i=0}^{d} (-1)^i f_i = \sum_{i=-1}^{d} (-1)^i \dim_K \tilde{H}_i(\Delta; K). \tag{1}$$

If Δ is a simplicial complex and Δ_1 and Δ_2 are subcomplexes of Δ, then there is an exact sequence

$$\begin{aligned}\cdots \longrightarrow \tilde{H}_j(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_j(\Delta_1; K) \oplus \tilde{H}_j(\Delta_2; K) \\ \longrightarrow \tilde{H}_j(\Delta_1 \cup \Delta_2; K) \longrightarrow \tilde{H}_{j-1}(\Delta_1 \cap \Delta_2; K) \longrightarrow \cdots,\end{aligned} \tag{2}$$

with all coefficients in K, called the *reduced Mayer-Vietoris sequence* of Δ_1 and Δ_2.

Hochster's formula describes the Betti number of a square-free monomial ideal I in terms of the dimension of reduced homology of Δ, when $I = I_\Delta$.

THEOREM 2.2 (Hochster formula, [8, Theorem 8.1.1]). *Let Δ be a simplicial complex on $[n]$. Then,*

$$\beta^K_{i,j}(I_\Delta) = \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W; K),$$

where Δ_W is the simplicial complex with vertex set W and all faces of Δ with vertices in W.

The following theorem, extends the well-known Herzog-Kühl equations [9] in the case of $\beta_{i,d_{i+1}}(M) = 0$ for all $i \geq 0$.

THEOREM 2.3 ([1]). *Let M be a N-graded S-module, and let ρ be its projective dimension. Suppose $\mathbf{d} = (d_0 < d_1 < \cdots < d_\rho < d_{\rho+1}) \in \mathsf{N}^{\rho+2}$ is such that M has a free resolution of the following form:*

$$\begin{aligned}0 \to S(-d_{\rho+1})^{\beta_{\rho,d_{\rho+1}}} \oplus S(-d_\rho)^{\beta_{\rho,d_\rho}} \to S(-d_\rho)^{\beta_{\rho-1,d_\rho}} \oplus S(-d_{\rho-1})^{\beta_{\rho-1,d_{\rho-1}}} \to \\ \cdots \to S(-d_2)^{\beta_{2,d_2}} \oplus S(-d_1)^{\beta_{1,d_1}} \to S(-d_1)^{\beta_{0,d_1}} \oplus S(-d_0)^{\beta_{0,d_0}} \to M \to 0.\end{aligned}$$

For $1 \leq i \leq \rho$, put $\beta'_i = \beta_{i,d_i} - \beta_{i-1,d_i}$. Then we have:

(i) *If* $\operatorname{depth}(M) = \dim M$ *and $\beta_{\rho,d_{\rho+1}} = 0$, then for all $1 \leq i \leq \rho$,*

$$\beta'_i = \beta_0 (-1)^i \prod_{\substack{k=1 \\ k \neq i}}^{\rho} \left(\frac{d_k - d_0}{d_k - d_i} \right).$$

(ii) *If* $\operatorname{depth}(M) = \dim M$, $\beta_{\rho,d_{\rho+1}} \neq 0$ *and* $d_0 = 0$, *then for all* $1 \leq i \leq \rho + 1$,

$$\beta_i' = (-1)^{i-1} \frac{\beta_0\left(\prod_{k=1,k\neq i}^{\rho+1} d_k\right) - \rho!\, e(M)}{\prod_{k=1,k\neq i}^{\rho+1} (d_k - d_i)}.$$

(iii) *If* $\operatorname{depth}(M) = \dim M - 1$, $\beta_{\rho,d_{\rho+1}} = 0$ *and* $d_0 = 0$, *then for all* $1 \leq i \leq \rho$,

$$\beta_i' = (-1)^{i-1} \frac{\beta_0\left(\prod_{k=1,k\neq i}^{\rho} d_k\right) - (\rho - 1)!\, e(M)}{\prod_{k=1,k\neq i}^{\rho} (d_k - d_i)}.$$

3. Simplicial complexes Δ with $\operatorname{indeg}(I_\Delta) \geq 1 + \dim \Delta$

As we shall see later, the ideals which are minimal to linearity are located in the class of square-free monomial ideals I_Δ, with $\operatorname{indeg}(I_\Delta) = 1 + \dim \Delta$ (see Definition 5.1). For square-free monomial ideal I with $\operatorname{indeg}(I) \geq d$, we have the following proposition.

PROPOSITION 3.1. *Let* Δ *be a simplicial complex on* $[n]$ *and* d *be an integer such that* $\operatorname{indeg}(I_\Delta) \geq d$. *Then,*

(i) $\tilde{H}_i(\Delta_W; K) = 0$, *for all* $i < d - 2$ *and* $W \subset [n]$.

(ii) *If* $\beta_{i,j}(I_\Delta) \neq 0$, *then* $1 \leq j \leq n$ *and* $d \leq j - i \leq \dim \Delta + 2$.

PROOF. (i) Let $\dim \Delta = r$ and

$$\begin{aligned} \tilde{\mathscr{C}}(\Delta)\colon 0 \longrightarrow C_r \xrightarrow{\partial_r} \cdots \xrightarrow{\partial_{d+1}} C_d \xrightarrow{\partial_d} C_{d-1} \\ \xrightarrow{\partial_{d-1}} C_{d-2} \xrightarrow{\partial_{d-2}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow 0 \end{aligned}$$

be the augmented chain complex of Δ. Let $\Delta^{(d-2)}$ be the pure $(d-2)$-skeleton of Δ, that is $\Delta^{(d-2)} = \{F \in \Delta \mid \dim F \leq d - 2\}$. Then the augmented chain complex of $\Delta^{(d-2)}$ is:

$$\tilde{\mathscr{C}}(\Delta^{(d-2)})\colon 0 \longrightarrow C_{d-2} \xrightarrow{\partial_{d-2}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow 0.$$

So that $\tilde{H}_i(\Delta; K) = \tilde{H}_i(\Delta^{(d-2)}; K)$ for $i < d - 2$. Since, $\operatorname{indeg}(I_\Delta) \geq d$, the facet set of the complex $\Delta^{(d-2)}$ is all $(d-1)$-subsets of $[n]$. Hence $\tilde{H}_i(\Delta; K) = \tilde{H}_i(\Delta^{(d-2)}; K) = 0$ for $i < d - 2$.

Moreover, if $W \subset [n]$ and $|W| \geq d$, then all $(d-1)$-subsets of W are again in Δ_W. This implies that $\operatorname{indeg}(I_{\Delta_W}) \geq d$. Hence by what we have already proved, we conclude that $\tilde{H}_i(\Delta_W; K) = 0$ for all $i < d - 2$. This completes the proof.

(ii) If $\beta_{i,j}(I_\Delta) \neq 0$, then by Theorem 2.2, there exists $W \subset [n]$ with $|W| = j$ and $\tilde{H}_{j-i-2}(\Delta_W; K) \neq 0$. So that, $1 \leq j = |W| \leq n$ and $j - i - 2 \leq \dim \Delta$. Moreover, by part (i), we have $j - i - 2 \geq d - 2$.

REMARK 3.2. Let Δ be a $(d-1)$-dimensional simplicial complex such that $\operatorname{indeg}(I_\Delta) \geq d$. The main property of Δ is that it contains all faces of dimension $d-2$. Hence Δ contains all faces of dimension $-1, 0, \ldots, d-2$. So that

$$f_i = \binom{n}{i+1}, \qquad i = -1, \ldots, d-2. \tag{3}$$

For a monomial ideal I, let $\mu(I)$ denote the cardinality of a minimal set of generators of I and $e(I)$ denotes the multiplicity of I. As a consequence of Proposition 3.1, we have:

COROLLARY 3.3. *Let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$ such that $\operatorname{indeg}(I_\Delta) \geq d$. Then,*

$$\dim_K \tilde{H}_{d-2}(\Delta; K) - \dim_K \tilde{H}_{d-1}(\Delta; K) = \sum_{i=0}^{d-1} (-1)^{d+i-1} \binom{n}{i} - e(S/I_\Delta). \tag{4}$$

PROOF. Using (1), Proposition 3.1 and (3), we have:

$$(-1)^{d-2} \dim_K \tilde{H}_{d-2}(\Delta; K) + (-1)^{d-1} \dim_K \tilde{H}_{d-1}(\Delta; K)$$
$$= -1 + (-1)^{d-1} f_{d-1} + \sum_{i=0}^{d-2} (-1)^i \binom{n}{i+1}.$$

Since $e(S/I_\Delta) = f_{d-1}$, we get the conclusion.

The following theorems extend some results of Terai and Yoshida (cf. [15]).

THEOREM 3.4. *Let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$ such that $\operatorname{indeg}(I_\Delta) \geq d$. Then,*

(i) *if $\beta_{i,j}(I_\Delta) \neq 0$, then $1 \leq j \leq n$ and $d \leq j - i \leq d + 1$,*
(ii) *$d \leq \operatorname{reg}(I_\Delta) \leq d + 1$,*
(iii) *$\operatorname{indeg} I_\Delta \leq d + 1$, and equality holds if and only if I_Δ has $(d+1)$-linear resolution,*
(iv) *$(n - d) - 1 \leq \operatorname{proj\,dim}(I_\Delta) \leq n - d$.*

PROOF. (i) If $\beta_{i,j}(I_\Delta) \neq 0$, then by Theorem 2.2, there exists $\varnothing \neq W \subset [n]$, such that $|W| = j$ and $\tilde{H}_{j-i-2}(\Delta_W; K) \neq 0$. So that $1 \leq j \leq n$ and by Proposition 3.1, $d - 2 \leq j - i - 2 \leq d - 1$. That is, $d \leq j - i \leq d + 1$.

(ii) By part (i), we have

$$d \le \operatorname{indeg}(I_\Delta) \le \operatorname{reg}(I_\Delta) = \max\{j - i : \beta_{i,j} \neq 0\} \le d + 1.$$

(iii) If $x_{i_1} \cdots x_{i_j} \in I_\Delta$, then $\beta_{0,j} \neq 0$. So that by (i), $j \le d+1$. In particular, $\operatorname{indeg}(I_\Delta) \le d + 1$.

If $\operatorname{indeg}(I_\Delta) = d+1$, then $\operatorname{reg}(I_\Delta) \ge d+1$ and by (ii), I_Δ has $(d+1)$-linear resolution. On the other hand, if I_Δ has $(d + 1)$-linear resolution, then each generator has degree $d + 1$. So that $\operatorname{indeg}(I_\Delta) = d + 1$.

(iv) Let $\rho = \operatorname{proj\,dim}(I_\Delta)$. By Theorem 2.1,

$$\rho + 1 = \operatorname{proj\,dim} \frac{S}{I_\Delta} = n - \operatorname{depth} \frac{S}{I_\Delta} \ge n - \dim \frac{S}{I_\Delta} = n - d.$$

Hence $\rho \ge (n - d) - 1$.

On the other hand, $\beta_\rho(I_\Delta) \neq 0$. Hence, there exists $1 \le j \le n$, such that $\beta_{\rho,j} \neq 0$. So, by (i), $j - \rho \ge d$. This implies that $\rho \le j - d \le n - d$.

THEOREM 3.5. *Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and let Δ be a $(d - 1)$-dimensional simplicial complex on $[n]$ such that* $\operatorname{indeg}(I_\Delta) \ge d$. *Then, S/I_Δ is Cohen-Macaulay if and only if $\tilde{H}_{d-2}(\Delta; K) = 0$.*

PROOF. We know that $\dim S/I_\Delta = d$. So that Theorem 2.1, implies that

$$S/I_\Delta \text{ is Cohen-Macaulay if and only if } \operatorname{proj\,dim} S/I_\Delta = (n - d).$$

In view of Theorem 3.4(iv), it is enough to prove that

$$\operatorname{proj\,dim} S/I_\Delta = (n - d) + 1 \Longleftrightarrow \tilde{H}_{d-2}(\Delta; K) \neq 0.$$

($\Leftarrow$) If $\tilde{H}_{d-2}(\Delta; K) \neq 0$, then by Theorem 2.2, $\beta_{(n-d)+1,n}(S/I_\Delta) \neq 0$. So that $\operatorname{proj\,dim} S/I_\Delta \ge (n-d)+1$. Hence by Theorem 3.4(iv), $\operatorname{proj\,dim} S/I_\Delta = (n - d) + 1$.

($\Rightarrow$) If $\operatorname{proj\,dim} S/I_\Delta = (n - d) + 1$, then $\beta_{(n-d)+1}(S/I_\Delta) \neq 0$. Hence there exists $1 \le j \le n$ such that $\beta_{(n-d)+1,j}(S/I_\Delta) \neq 0$. Using Theorem 3.4(i), $j \ge n$. Hence $j = n$. Thus,

$$\begin{aligned} 0 \neq \beta_{(n-d)+1}\left(\frac{S}{I_\Delta}\right) &= \sum_{j=1}^{n} \beta_{(n-d)+1,j}\left(\frac{S}{I_\Delta}\right) \\ &= \beta_{(n-d)+1,n}\left(\frac{S}{I_\Delta}\right) = \dim \tilde{H}_{d-2}(\Delta; K), \end{aligned}$$

by Theorem 2.2.

Now, let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$ such that $\operatorname{indeg}(I_\Delta) = d$. As a consequence of Theorem 3.4, we conclude that:

COROLLARY 3.6. *Let Δ be a $(d-1)$-dimensional simplicial complex on $[n]$ such that* $\operatorname{indeg}(I_\Delta) = d$. *Then, $I = I_\Delta$ has a d-linear resolution if and only if $\tilde{H}_{d-1}(\Delta; K) = 0$.*

PROOF. If I has a d-linear resolution, then by Theorem 2.2, we have:

$$0 = \beta_{n-d-1,n}(I_\Delta) = \dim_K \tilde{H}_{d-1}(\Delta; K).$$

Assume that I does not have d-linear resolution, by Theorem 3.4(ii), we have:

$$d + 1 = \operatorname{reg}(I) = \max\{j - i : \beta_{i,j}(I_\Delta) \neq 0\}.$$

Let $d + 1 = j_0 - i_0$ and $\beta_{i_0 j_0}(I_\Delta) \neq 0$. Then by Theorem 2.2, there exists $W \subset [n]$ with $|W| = j_0$ and $\tilde{H}_{d-1}(\Delta_W; K) \neq 0$. This in particular implies that $\tilde{H}_{d-1}(\Delta; K) \neq 0$, for $\tilde{H}_{d-1}(\Delta_W; K) \subset \tilde{H}_{d-1}(\Delta; K)$.

4. Clutters and clique complexes

DEFINITION 4.1. A *clutter* C on a vertex set $[n]$ is a set of subsets of $[n]$ (called *circuits* of C) such that if e_1 and e_2 are distinct circuits of C then $e_1 \nsubseteq e_2$. A *d-circuit* is a circuit consisting of exactly d vertices, and a clutter is *d-uniform* if every circuit has d vertices. A $(d-1)$-subset $e \subset [n]$ is called an *submaximal circuit* of C if there exists $F \in C$ such that $e \subset F$. The set of all submaximal circuits of C is denoted by $\mathrm{SC}(C)$. For $e \in \mathrm{SC}(C)$, we denote by $\deg_C(e)$, the *degree* of e to be

$$\deg_C(e) = |\{F \in C : e \subset F\}|.$$

For a subset $W \subset [n]$, the *induced subclutter* of C on W, C_W, is a clutter with vertices W and those circuits of C for which their vertices are in W.

For a non-empty clutter C on vertex set $[n]$, we define the ideal $I(C)$, as follows:

$$I(C) = (\mathbf{x}_T : T \in C),$$

where $\mathbf{x}_T = x_{i_1} \cdots x_{i_t}$ for $T = \{i_1, \ldots, i_t\}$, and we define $I(\varnothing) = 0$.

Let $n \geq d$ be positive integers. We define $C_{n,d}$, the *maximal d-uniform clutter on* $[n]$, as following:

$$C_{n,d} = \{F \subset [n] : |F| = d\}.$$

One can check that $I(C_{n,d})$ has d-linear resolution (see also [12, Example 2.12]).

If C is a d-uniform clutter on $[n]$, we define $\overline{C}$, the *complement* of C, to be

$$\overline{C} = C_{n,d} \setminus C = \{F \subset [n] : |F| = d,\ F \notin C\}.$$

Frequently in this paper, we take a d-uniform clutter C and we consider the square-free ideal $I = I(\overline{C})$ in the polynomial ring $S = K[x_1, \dots, x_n]$. We call $I = I(\overline{C})$ the *circuit ideal* of C.

DEFINITION 4.2. Let C be a d-uniform clutter on $[n]$. A subset $V \subset [n]$ is called a *clique* in C, if all d-subsets of V belongs to C. Note that a subset of $[n]$ with less than d elements is supposed to be a clique. The simplicial complex generated by cliques of C is called *clique complex* of C and is denoted by $\Delta(C)$.

REMARK 4.3. Let C be a d-uniform clutter on $[n]$ and $\Delta = \Delta(C)$ be its clique complex. Then by our definition, all the subsets of $[n]$ with less than d elements are also in $\Delta(C)$. In particular, this implies that indeg $I_\Delta \geq d$. So that by Proposition 3.1, we have:

$$\tilde{H}_i(\Delta_W; K) = 0, \qquad \text{for all} \quad i < d-2 \text{ and } W \subset [n]. \tag{5}$$

PROPOSITION 4.4. *Let C be a d-uniform clutter on $[n]$ with $I = I(\overline{C}) \subset K[x_1, \dots, x_n]$ the circuit ideal. Let $\Delta = \Delta(C)$ be the clique complex of C. Then,*

(i) $C = \mathcal{F}\left(\Delta^{(d-1)}\right)$,

(ii) *for all $u \in G(I_\Delta)$,* $\deg(u) = d$, *and*

(iii) $I_\Delta = I$.

PROOF. We know that,

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\bar{F}}.$$

So that,

$$\mathbf{x}_T \in I_\Delta \Longleftrightarrow T \cap ([n] \setminus F) \neq \varnothing, \qquad \text{for all} \quad F \in \mathcal{F}(\Delta). \tag{6}$$

(i) Clear.

(ii) Let $u = \mathbf{x}_T \in G(I_\Delta)$. By Remark 4.3, we know that $\deg(u) = |T| \geq d$. If $\deg(u) = |T| > d$, then for all d-subset T' of T, $\mathbf{x}_{T'} \notin I_\Delta$. This means that $T' \in \Delta$ for all d-subset T' of T (i.e. T is a clique in C). So that $T \in \Delta$ which is contradiction to the fact that $u = \mathbf{x}_T \in G(I_\Delta)$.

(iii) Let $T \in \overline{C}$ and $\mathbf{x}_T \notin I_\Delta$. Then, by (6), there exist $F \in \mathcal{F}(\Delta)$ such that $T \subset F$. Since T is a d-subset of F, so $T \in C$ which is contradiction. So that $I(\overline{C}) \subset I_\Delta$.

For the converse, let $\mathbf{x}_T \in G(I_\Delta)$. Then, $T \notin \Delta$. Using part (i), $T \notin C$. Moreover, by (ii), we have $|T| = d$. Since $|T| = d$ and $T \notin C$, one can say $T \in \overline{C}$. This means that $I_\Delta \subset I(\overline{C})$. This completes the proof.

DEFINITION 4.5. A d-uniform clutter C is called *decomposable* if there exist proper d-uniform subclutters C_1 and C_2 such that $C = C_1 \cup C_2$ and either $V(C_1) \cap V(C_2)$ is a clique or $\mathrm{SC}(C_1) \cap \mathrm{SC}(C_2) = \varnothing$.

In this case, we write $C = C_1 \uplus C_2$. A d-uniform clutter is said to be *indecomposable* if it is not decomposable. For $d = 2$, this definition coincides with the definition of decomposable graphs in [8].

Below we will find the regularity of the circuit ideal of C in terms of circuit ideals of C_1 and C_2, whenever $C = C_1 \uplus C_2$. First we need the following lemma.

LEMMA 4.6. *Let C_1 and C_2 be d-uniform clutters on two vertex sets V_1 and V_2 and put $C = C_1 \cup C_2$. Let Δ (resp. Δ_1, Δ_2) be the clique complex of C (resp. C_1, C_2).*

(i) *If $G \subset V_1 \cup V_2$ with $G \cap (V_1 \setminus V_2) \neq \varnothing$ and $G \cap (V_2 \setminus V_1) \neq \varnothing$, then $G \in \Delta \iff |G| \leq d - 1$.*

(ii) *$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta_1 \cup \Delta_2; K)$, for all $i > d - 2$.*

PROOF. (i) Let G be a subset of $V_1 \cup V_2$, as in (ii). If $|G| \leq d - 1$, then by definition, G is a clique in C and $G \in \Delta$.

Now, let $|G| \geq d$ and $x \in G \cap (V_1 \setminus V_2)$, $y \in G \cap (V_2 \setminus V_1)$. If F be a d-subset of G which contains x and y, then by Proposition 4.4(i), $F \notin C_1 \cup C_2 = C$. Hence $G \notin \Delta$.

(ii) First note that for $F \in \Delta$, we have for $i = 1, 2$:

$$F \in \Delta_i \iff F \subset V_i. \tag{7}$$

Now, let

$$\Delta_3 = \big\langle G \in \Delta : G \cap (V_1 \setminus V_2) \neq \varnothing, G \cap (V_2 \setminus V_1) \neq \varnothing \big\rangle.$$

Then (i) and (7), imply that:

$$\dim \Delta_3 = d - 2, \qquad \Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3.$$

It is clear that $\dim(\Delta_1 \cap \Delta_3) = \dim(\Delta_2 \cap \Delta_3) = d - 3$. In particular,

$$\tilde{H}_i\left((\Delta_1 \cup \Delta_2) \cap \Delta_3; K\right) = 0, \qquad \text{for all} \quad i > d - 3.$$

Hence from (2), for all $i > d-2$, we have:

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta_1 \cup \Delta_2; K) \oplus \tilde{H}_i(\Delta_3; K) = \tilde{H}_i(\Delta_1 \cup \Delta_2; K).$$

COROLLARY 4.7. *Let $C = C_1 \cup C_2$ be a d-uniform clutter and Δ (resp. Δ_1, Δ_2) be the clique complex of C (resp. C_1, C_2). If $V(C_1) \cap V(C_2)$ is a clique in C, then*

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K), \qquad \text{for all } i > d-2.$$

PROOF. By our assumption, $\Delta_1 \cap \Delta_2$ is a simplex. So that $\tilde{H}_i(\Delta_1 \cap \Delta_2; K) = 0$ for all i. Using (2), for all $i > 0$, we have:

$$\tilde{H}_i(\Delta_1 \cup \Delta_2; K) \cong \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K).$$

Combining with Lemma 4.6(ii), we get the conclusion.

COROLLARY 4.8. *Let $C = C_1 \cup C_2$ be a d-uniform clutter and Δ (resp. Δ_1, Δ_2) be the clique complex of C (resp. C_1, C_2). If $\mathrm{SC}(C_1) \cap \mathrm{SC}(C_2) = \varnothing$, then*

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K), \qquad \text{for all} \quad i > d-2.$$

PROOF. By our assumption, $\dim(\Delta_1 \cap \Delta_2) \leq d-2$. So that $\tilde{H}_i(\Delta_1 \cap \Delta_2; K) = 0$ for all $i > d-2$. Using (2), for all $i > d-1$, we have:

$$\tilde{H}_i(\Delta_1 \cup \Delta_2; K) \cong \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K)$$

and $\tilde{H}_{d-1}(\Delta_1; K) \oplus \tilde{H}_{d-1}(\Delta_2; K) \hookrightarrow \tilde{H}_{d-1}(\Delta_1 \cup \Delta_2; K)$.

We claim that $\tilde{H}_{d-1}(\Delta_1; K) \oplus \tilde{H}_{d-1}(\Delta_2; K) \cong \tilde{H}_{d-1}(\Delta_1 \cup \Delta_2; K)$.

PROOF OF CLAIM. Let $\mathscr{C}(\Delta, \partial)$ (resp. $\mathscr{C}(\Delta_1, \partial^{(1)}), \mathscr{C}(\Delta_2, \partial^{(2)})$) be the chain complex of Δ (resp. Δ_1, Δ_2). Since $\mathrm{SC}(C_1) \cap \mathrm{SC}(C_2) = \varnothing$, we have:

$$\bigoplus_{\substack{F \in \Delta \\ \dim F = d-1}} KF = \Bigg(\bigoplus_{\substack{F \in \Delta_1 \\ \dim F = d-1}} KF\Bigg) \oplus \Bigg(\bigoplus_{\substack{F \in \Delta_2 \\ \dim F = d-1}} KF\Bigg). \tag{8}$$

Take $0 \neq F + \mathrm{Im}\, \partial_d \in \tilde{H}_{d-1}(\Delta; K)$. Then by (8), we can separate F as $F = (c_1 F_1 + \cdots + c_r F_r) + (c_1' G_1 + \cdots + c_s' G_s)$, where $c_i, c_i' \in K$ and $F_i \in C_1, G_i \in C_2$. Let

$$\partial_{d-1}(c_1 F_1 + \cdots + c_r F_r) = (d_1 e_1 + \cdots + d_{r'} e_{r'}),$$
$$\partial_{d-1}(c_1' G_1 + \cdots + c_s' G_s) = (d_1' f_1 + \cdots + d_{s'}' f_{s'}),$$

where $d_i, d'_i \in K$ and $e_i \in \mathrm{SC}(\mathcal{C}_1)$, $f_i \in \mathrm{SC}(\mathcal{C}_2)$. Since

$$\begin{aligned} 0 = \partial_d(F) &= \partial_{d-1}(c_1F_1 + \cdots + c_rF_r) + \partial_{d-1}(c'_1G_1 + \cdots + c'_sG_s) \\ &= (d_1e_1 + \cdots + d_{r'}e_{r'}) + (d'_1f_1 + \cdots + d'_{s'}f_{s'}) \end{aligned}$$

and $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$, we conclude that

$$\partial_{d-1}(c_1F_1 + \cdots + c_rF_r) = \partial_{d-1}(c'_1G_1 + \cdots + c'_sG_s) = 0.$$

This means that the natural map

$$\tilde{H}_{d-1}(\Delta_1; K) \oplus \tilde{H}_{d-1}(\Delta_2; K) \hookrightarrow \tilde{H}_{d-1}(\Delta_1 \cup \Delta_2; K)$$

is onto too, showing the claim.

By what we have already proved, we have:

$$\tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \cong \tilde{H}_i(\Delta_1 \cup \Delta_2; K), \qquad \text{for all} \quad i > d-2.$$

In combination with Lemma 4.6(ii), we get the conclusion.

REMARK 4.9. Let $\mathcal{C}_1, \mathcal{C}_2$ be d-uniform clutters on vertex set V_1, V_2 with $V_1 \cup V_2 = [n]$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. For all $W \subset [n]$, one can easily check that:

(i) $\mathcal{C}_W = (\mathcal{C}_1)_W \cup (\mathcal{C}_2)_W$,

(ii) $\Delta_W = \Delta(\mathcal{C}_W)$,

(iii) $\mathrm{SC}((\mathcal{C}_1)_W) \cap \mathrm{SC}((\mathcal{C}_2)_W) \subset \mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2)$.

Hence, if $V_1 \cap V_2$ is a clique or $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$, then (i)–(iii) and Corollaries 4.7 and 4.8, imply that

$$\tilde{H}_i(\Delta_W; K) \cong \tilde{H}_i((\Delta_1)_W; K) \oplus \tilde{H}_i((\Delta_2)_W; K), \qquad \text{for all} \quad i > d-2. \tag{9}$$

Now we present the main theorem of this section.

THEOREM 4.10. *Let $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ be a d-uniform clutter and let I (resp. I_1, I_2) be the circuit ideals of $\mathcal{C}$ (resp. $\mathcal{C}_1, \mathcal{C}_2$). Then,*

(i) *$\beta_{i,j}(I) \geq \beta_{i,j}(I_1) + \beta_{i,j}(I_2)$, for $j - i > d$.*

(ii) *If I_1 and I_2 are non-zero ideals, then* $\mathrm{reg}(I) = \max\{\mathrm{reg}(I_1), \mathrm{reg}(I_2)\}$.

PROOF. (i) Let Δ (resp. Δ_1, Δ_2) be the clique complex of $\mathcal{C}$ (resp. $\mathcal{C}_1, \mathcal{C}_2$).

Then, by (9) and Theorem 2.2, for $j - i > d$, we have:

$$\begin{aligned}
\beta_{i,j}(I_\Delta) &= \sum_{\substack{W\subset[n]\\|W|=j}} \dim_K \tilde{H}_{j-i-2}(\Delta_W; K)\\
&= \sum_{\substack{W\subset[n]\\|W|=j}} \left[\dim_K \tilde{H}_{j-i-2}((\Delta_1)_W; K) + \dim_K \tilde{H}_{j-i-2}((\Delta_2)_W; K)\right]\\
&= \sum_{\substack{W\subset[n]\\|W|=j}} \dim_K \tilde{H}_{j-i-2}((\Delta_1)_W; K) + \sum_{\substack{W\subset[n]\\|W|=j}} \dim_K \tilde{H}_{j-i-2}((\Delta_2)_W; K)\\
&\geq \beta_{i,j}(I_{\Delta_1}) + \beta_{i,j}(I_{\Delta_2}).
\end{aligned}$$

Hence by Proposition 4.4(iii), $\beta_{i,j}(I) \geq \beta_{i,j}(I_1)+\beta_{i,j}(I_2)$, whenever $j-i > d$.

(ii) If I has a d-linear resolution, $\beta_{i,j}(I) = 0$ for all $j - i > d$. So that (i) implies that $\beta_{i,j}(I_1) = \beta_{i,j}(I_2) = 0$, for all $j - i > d$. This means that, both ideals I_1 and I_2 have a d-linear resolution and the equality $\operatorname{reg}(I) = \max\{\operatorname{reg}(I_1), \operatorname{reg}(I_2)\}$ holds.

Assume that, I does not have d-linear resolution. Let

$$r = \operatorname{reg}(I) = \max\{j - i : \beta_{i,j}(I) \neq 0\}$$

and j_0, i_0 be such that $r = j_0 - i_0$ with $\beta_{i_0,j_0}(I) \neq 0$. By Theorem 2.2, there exists a $W \subset [n]$, with $|W| = j_0$ and $\tilde{H}_{r-2}(\Delta_W; K) \neq 0$. Since $r-2 > d-2$, from (9) we conclude that

$$\begin{aligned}
\text{either} \qquad & \tilde{H}_{r-2}((\Delta_1)_W; K) \neq 0\\
\text{or} \qquad & \tilde{H}_{r-2}((\Delta_2)_W; K) \neq 0.
\end{aligned}$$

Without loss of generality, we may assume that $\tilde{H}_{r-2}((\Delta_1)_W; K) \neq 0$ and we put $W' = W \cap V(\Delta_1)$. Then, W' is a subset of the vertex set of Δ_1 with the property that $\tilde{H}_{r-2}((\Delta_1)_{W'}; K) \neq 0$. Using Theorem 2.2 once again, we have:

$$\begin{aligned}
\beta_{|W'|-r,|W'|}(I_1) &= \sum_{\substack{T\subset V(\Delta_1)\\|T|=|W'|}} \dim_K \tilde{H}_{r-2}((\Delta_1)_T; K)\\
&\geq \dim_K \tilde{H}_{r-2}((\Delta_1)_{W'}; K) > 0.
\end{aligned}$$

Hence, $\beta_{|W'|-r,|W'|}(I_1) \neq 0$ and,

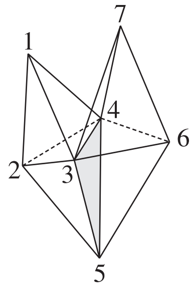
$$\begin{aligned}
\max\{\operatorname{reg}(I_1), \operatorname{reg}(I_2)\} \geq \operatorname{reg}(I_1) &= \max\{j - i : \beta_{i,j}(I_1) \neq 0\}\\
&\geq (|W'|) - (|W'| - r) = r.
\end{aligned}$$

The inequality, $\max\{\mathrm{reg}(I_1), \mathrm{reg}(I_2)\} \leq r$ comes from (i). Putting together these inequalities, we get the conclusion.

The following example shows that, the inequality $\beta_{i,j}(I) \geq \beta_{i,j}(I_1) + \beta_{i,j}(I_2)$, for $j - i > d$ in Theorem 4.10, may be strict.

EXAMPLE 4.11. Consider the 3-uniform clutter

$$\mathcal{C} = \{123, 124, 134, 235, 245, 345, 347, 367, 467, 356, 456\}.$$



$\mathcal{C}$

Let $\mathcal{C}_1 = \{123, 124, 134, 235, 245, 345\}$ and $\mathcal{C}_2 = \{345, 347, 367, 467, 356, 456\}$. Then, $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ and a direct computation using CoCoA, shows that the minimal free resolution of the ideal $I(\overline{\mathcal{C}})$ is

$$0 \longrightarrow S^6(-7) \longrightarrow S^{30}(-6) \oplus S^2(-7) \longrightarrow S^{62}(-5) \oplus S^4(-6) \longrightarrow S^{61}(-4) \oplus S^2(-5) \longrightarrow S^{24}(-3) \longrightarrow I \longrightarrow 0.$$

Note that $\beta^K_{2,6}(I(\overline{\mathcal{C}}_1)) = \beta^K_{2,6}(I(\overline{\mathcal{C}}_2)) = 0$, while $\beta^K_{2,6}(I(\overline{\mathcal{C}})) = 4$.

REMARK 4.12. Let $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ be a d-uniform clutter on $[n]$ with I (resp. I_1, I_2) be the circuit ideals of $\mathcal{C}$ (resp. $\mathcal{C}_1, \mathcal{C}_2$). Let Δ (resp. Δ_1, Δ_2) be the clique complex of $\mathcal{C}$ (resp. $\mathcal{C}_1, \mathcal{C}_2$).

- If both of I_1 and I_2 are zero ideals, then Δ_1 and Δ_2 are simplexes and they have zero reduced homologies in all degrees. So that $\tilde{H}_i(\Delta_W; K) = 0$ for all $W \subset [n]$ and $i > d - 2$ by (9). So that $\beta_{i,j}(I) = 0$ for all $j - i > d$. That is, the ideal I has a d-linear resolution.
- If only one of the ideals I_1 or I_2 is a zero ideal, say I_1, then Δ_1 is a simplex and all the reduced homologies of Δ_1 is zero. Using (9), we conclude that $\tilde{H}_i(\Delta_W; K) \cong \tilde{H}_i((\Delta_2)_W; K)$ for all $W \subset [n]$ and $i > d - 2$. This implies that $\mathrm{reg}(I) = \mathrm{reg}(I_2)$.
- If I_1 and I_2 are non-zero ideals, then Theorem 4.10(ii) implies that

$$\mathrm{reg}(I) = \max\{\mathrm{reg}(I_1), \mathrm{reg}(I_2)\}.$$

5. Minimal to d-linearity

In this section, we define three classes of clutters for which their circuit ideals do not have d-linear resolution but the circuit ideal of any proper subclutter of them has a d-linear resolution.

A clutter C is said to be *connected* if for each two vertices v_1 and v_2, there is a sequence of circuits $F_1, \dots, F_r$ such that $v_1 \in F_1$, $v_2 \in F_r$ and $F_i \cap F_{i+1} \neq \varnothing$. A connected d-uniform clutter C is called a *tree* if any subclutter of C has a submaximal circuit of degree one. A union of trees is called a *forest*. By Remark 3.10 of [12], the circuit ideal of any d-uniform forest has a d-linear resolution.

Definition 5.1. Let C be a d-uniform clutter on $[n]$, $\Delta = \Delta(C)$ its clique complex. Suppose that $I = I(\overline{C}) \subset K[x_1, \dots, x_n]$, the circuit ideal of C, does not have d-linear resolution.

(i) The clutter C is called *obstruction to d-linearity* if for every proper subclutter $C' \subsetneq C$, the ideal $I(\overline{C}')$ has a d-linear resolution.

(ii) The clutter C is called *minimal to d-linearity* if it is obstruction to d-linearity and $\dim \Delta = d - 1$.

(iii) The clutter C is called *almost tree* if every proper subclutter of C has a submaximal circuit of degree 1.

Let $\mathscr{C}_d^{\text{obs}}$, $\mathscr{C}_d^{\text{min}}$ and $\mathscr{C}_d^{\text{a.tree}}$ denote the classes of clutters which are obstruction to d-linearity, minimal to d-linearity and almost tree, respectively.

Note that if $C \in \mathscr{C}_d^{\text{min}}$ and $\Delta = \Delta(C)$ is its clique complex, then we have:

$$\operatorname{indeg} I_\Delta = \operatorname{indeg} I(\overline{C}) = d = 1 + \dim \Delta. \tag{10}$$

Lemma 5.2. *Let C be a d-uniform clutter on $[n]$ which is minimal to d-linearity and $\Delta = \Delta(C)$ be the clique complex of C. Then,*

(i) $\dim_K \tilde{H}_{d-1}(\Delta; K) = 1$.

(ii) *If $W \subsetneq [n]$, then $\tilde{H}_{d-1}(\Delta_W; K) = 0$.*

Proof. (i) Let $0 \neq F = c_1F_1 + \dots + c_rF_r \in \tilde{H}_{d-1}(\Delta; K)$ where $c_i \in K$ and $F_i \in C$. Then, $\operatorname{Supp}(F) := \{F_i : c_i \neq 0\}$ is equal to C, because every proper subclutter of C has linear resolution.

If $\dim_K \tilde{H}_{d-1}(\Delta; K) > 1$ and $F = c_1F_1 + \ldots + c_rF_r$, $G = d_1F_1 + \ldots + d_rF_r$ be two basis element of $\tilde{H}_{d-1}(\Delta; K)$, then $0 \neq c_1G - d_1F \in \tilde{H}_{d-1}(\Delta; K)$ and $\operatorname{Supp}(c_1G - d_1F) \subsetneq C$ which is a contradiction.

(ii) One can easily check that $\Delta_W = \Delta(C_W)$ for all $W \subset [n]$. By definition, for all $W \subsetneq [n]$, the induced clutter C_W has linear resolution. So that by Theorem 2.2, $\tilde{H}_{d-1}(\Delta_W; K) = \tilde{H}_{d-1}(\Delta(C_W); K) = 0$.

The following is the main theorem of this section which gives an explicit minimal free resolution for the circuit ideal of a clutter which is minimal to d-linearity.

THEOREM 5.3. *Let C be a d-uniform clutter on $[n]$ which is minimal to d-linearity and $I = I(\overline{C}) \subset K[x_1, \dots, x_n]$ be the circuit ideal. Then the minimal free resolution of I is*

$$0 \longrightarrow S^{\beta_{n-d,n}}(-n) \longrightarrow S(-n) \oplus S^{\beta_{n-d-1,n-1}}(-(n-1)) \longrightarrow S^{\beta_{n-d-2,n-2}}(-(n-2)) \longrightarrow \cdots \longrightarrow S^{\beta_{1,d+1}}(-(d+1)) \longrightarrow S^{\beta_{0,d}}(-d) \longrightarrow I \longrightarrow 0, \tag{11}$$

where

(i) $\beta_{n-d,n}(I) = 1 - e(S/I) + \sum_{i=0}^{d-1} (-1)^{d+i-1} \binom{n}{i}$,

(ii) $\beta_{i,i+d}(I) = \binom{n-d}{i} \left(\frac{d}{d+i} \binom{n}{d} - e(S/I) \right)$, *for* $0 \leq i \leq n-d-1$ *and* $e(S/I) = \binom{n}{d} - \mu(I)$.

PROOF. Let $\Delta = \Delta(C)$ be the clique complex of C. Since $\operatorname{indeg}(I_\Delta) = \operatorname{indeg} I(\overline{C}) = d = 1 + \dim \Delta$, by Theorem 3.4(i) and Lemma 5.2(ii), $\beta_{i,j}(I) = 0$ either $j - i < d$ or $j - i > d + 1$ or $j - i = d + 1$ and $j < n$. Moreover, we have $\beta_{n-(d+1),n} = \dim_K \tilde{H}_{d-1}(\Delta; K) = 1$. Hence the minimal free resolution of I has the form (11). The equation (ii) comes from Theorem 2.3. Using Theorem 2.2 once again, we have $\beta_{n-d,n}(I) = \dim_K \tilde{H}_{d-2}(\Delta; K)$. Hence (i) comes from Corollary 3.3. In order to find the multiplicity, note that $e(S/I) = f_{d-1}(\Delta) = |C| = \binom{n}{d} - \mu(I)$.

Let C be a d-uniform clutter. The clutter C is called *strongly connected* (or *connected in codimension one*) if for any two circuits $F, G \in C$, there exists a chain of circuits $F = F_0, \dots, F_s = G$ in C such that $|F_i \cap F_{i+1}| = d - 1$, for $i = 0, \dots, s - 1$.

Besides the algebraic properties of the clutters $C \in \mathscr{C}_d^{\text{obs}}$, a combinatorial property of such clutters is that they are strongly connected.

PROPOSITION 5.4. *If $C \in \mathscr{C}_d^{\text{obs}}$ be a d-uniform clutter, then*

(i) *C is indecomposable, and*

(ii) *C is strongly connected.*

PROOF. Let $C = C_1 \uplus C_2$ where C_1 and C_2 are proper subclutters of C. By definition, the ideals $I_1 = I(\overline{C}_1)$ and $I_2 = I(\overline{C}_2)$ have d-linear resolutions. In view of Remark 4.12, the ideal $I(\overline{C})$ has d-linear resolution which is a contradiction.

(ii) Let $\mathcal{C}_1 \subset \mathcal{C}$ be the maximal subclutter (with respect to inclusion) of $\mathcal{C}$ which is strongly connected. Clearly, $\mathcal{C}_1 \neq \varnothing$, because every clutter with one circuit is strongly connected.

Assume that $\mathcal{C}_1 \subsetneq \mathcal{C}$ and let $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. By the maximality of $\mathcal{C}_1$, $\mathrm{SC}(\mathcal{C}_1) \cap \mathrm{SC}(\mathcal{C}_2) = \varnothing$, that is $\mathcal{C} = \mathcal{C}_1 \uplus \mathcal{C}_2$ which contradicts to (i). So that $\mathcal{C}_1 = \mathcal{C}$ is strongly connected.

LEMMA 5.5. *Let $\mathcal{C}$ be a d-uniform clutter which is a tree or almost tree and $\Delta = \Delta(\mathcal{C})$ be the clique complex of $\mathcal{C}$. Then,* $\dim \Delta = d - 1$. *In particular,* $\mathscr{C}_d^{\text{a.tree}} \subset \mathscr{C}_d^{\min}$.

PROOF. If $G \in \Delta$ and $|G| > d$ and V is the vertex set of G, then $\mathcal{C}_V = \{F \in \mathcal{C} : F \subset G\}$. Hence for all $e \in \mathrm{SC}(\mathcal{C}_V)$, $\deg_{\mathcal{C}_V}(e) \geq 2$. This contradicts to the fact that $\mathcal{C}_V$ has submaximal circuit of degree 1. So that all faces of $\Delta(\mathcal{C})$ have at most d elements. Since $\mathcal{C} \subset \Delta$, we conclude that $\dim \Delta = d - 1$.

If $\mathcal{C} \in \mathscr{C}_d^{\text{a.tree}}$, then by what we have already proved, we know that $\dim \Delta(\mathcal{C}) = d - 1$. Also, the argument before Definition 5.1 implies that for every proper subclutter $\mathcal{C}' \subsetneq \mathcal{C}$, the ideal $I(\overline{\mathcal{C}'})$ has a linear resolution. Hence $\mathcal{C} \in \mathscr{C}_d^{\min}$.

We have shown that $\mathscr{C}_d^{\text{a.tree}} \subseteq \mathscr{C}_d^{\min} \subseteq \mathscr{C}_d^{\text{obs}}$. All our evidence and computations lead us to make the following conjecture.

CONJECTURE 5.6. $\mathscr{C}_d^{\text{a.tree}} = \mathscr{C}_d^{\min} = \mathscr{C}_d^{\text{obs}}$.

6. Some applications

6.1. Fröberg's Theorem

Let G be a simple graph (2-uniform clutter). Fröberg [6] proved that the ideal $I(\overline{G})$ has 2-linear resolution if and only if G is a chordal graph. A graph is called *chordal* if each cycle in G has a chord, i.e. any minimal induced cycle in G is of length 3. In this section, we will present an alternative proof for this theorem.

Let C_n be a cycle of length $n > 3$. Though the Betti numbers of the circuit ideal of C_n are well-known (see e.g. [5, Proposition 3.1] or [10, Theorem 1]), we can recover them using results of this paper.

Let $\Delta = \Delta(C_n)$ be the clique complex of C_n and $I = I(\overline{C}_n)$ be the circuit ideal. Then $\operatorname{indeg} I_\Delta = 1 + \dim \Delta$ and by Corollary 3.3, $\dim \tilde{H}_1(\Delta; K) = 1$. In particular, I does not have linear resolution (Corollary 3.6) and C_n is minimal to 2-linearity (Lemma 5.5). Moreover, By Theorem 5.3, the minimal free resolution of I is

$$0 \longrightarrow S(-n) \longrightarrow S^{\beta_{n-4,n-2}}(-(n-2)) \longrightarrow \cdots$$
$$\longrightarrow S^{\beta_{1,3}}(-3) \longrightarrow S^{\beta_{0,2}}(-2) \longrightarrow I \longrightarrow 0$$

where $\beta_{i,i+2}(I) = n\binom{n-2}{i}\binom{n-3-i}{2+i}$ for $0 \le i \le n-4$.

Thus, if a graph G has a cycle as an induced subgraph, then by Theorem 2.2, the ideal $I(\overline{G})$ does not have linear resolution. This means that, the ideal $I(\overline{G})$ does not have linear resolution if G is not chordal.

Conversely, if $G \neq \mathcal{C}_{n,2}$ is chordal, then by Dirac's Theorem [2] (see also [8, Lemma 9.2.1]), there exist proper induced subgraphs G_1 and G_2 such that $G = G_1 \uplus G_2$. Since G_1 and G_2 are induced subgraphs of a chordal graph G, we conclude that G_1 and G_2 are chordal. Hence induction and Remark 4.12, implies that the ideal $I(\overline{G})$ has a 2-linear resolution.

6.2. *Generalized chordal clutters*

E. Emtander [4] has defined generalized chordal clutters as the following.

DEFINITION 6.1. A *generalized chordal clutter* is a d-uniform clutter, obtained inductively as follows:

(a) $\mathcal{C}_{n,d}$ is a generalized chordal clutter.

(b) If $\mathcal{G}$ is generalized chordal clutter, then so is $\mathcal{C} = \mathcal{G} \cup_{\mathcal{C}_{i,d}} \mathcal{C}_{n,d}$ for all $0 \le i < n$.

(c) If $\mathcal{G}$ is generalized chordal and $V \subset V(\mathcal{G})$ is a finite set with $|V| = d$ and at least one element of $\{F \subset V : |F| = d-1\}$ is not a subset of any element of $\mathcal{G}$, then $\mathcal{G} \cup V$ is generalized chordal.

Emtander proved that the circuit ideal of generalized chordal clutters have d-linear resolution over any field K (cf. [4, Theorem 5.1]). We can recover this result as a special case of Theorem 4.10.

Let $\mathcal{C}$ be a generalized chordal clutter. If $\mathcal{C}$ has a circuit F, with property (c) in the above definition, then Remark 3.10 of [12] together with induction, implies that $I(\overline{\mathcal{C}})$ has a d-linear resolution. So we may assume that $\mathcal{C} = \mathcal{G} \cup_{\mathcal{C}_{i,d}} \mathcal{C}_{n,d}$. Again, in this case, Remark 4.12 together with induction, implies that the ideal $I(\overline{\mathcal{C}})$ has a d-linear resolution over the field K.

6.3. *Resolution of pseudo-manifolds*

DEFINITION 6.2. A d-uniform clutter $\mathcal{C}$ is called a *pseudo-manifold* if $\mathcal{C}$ is strongly connected and each $e \in \mathrm{SC}(\mathcal{C})$ has degree 2.

For more details on pseudo-manifolds and the concept of orientability, we refer the reader to [11, Chapter IX].

LEMMA 6.3. *Let $\mathcal{C}$ be a d-uniform clutter such that* $\deg_{\mathcal{C}}(e) = 2$ *for all* $e \in \mathrm{SC}(\mathcal{C})$. *Then, every proper subclutter of $\mathcal{C}$ has a submaximal circuit of degree* 1 *if and only if $\mathcal{C}$ is strongly connected. In particular, every proper subclutter of a pseudo-manifold is a forest.*

PROOF. ($\Rightarrow$) Let $F \in \mathcal{C}$ and $\mathcal{C}_1$ be a maximal subclutter of $\mathcal{C}$ which consists of all $G \in \mathcal{C}$ such that there is a chain $F = F_0, F_1, \ldots, F_r = G$ of circuits of $\mathcal{C}$ with $|F_i \cap F_{i+1}| = d - 1$ for $i = 0, \ldots, r - 1$.

If $\mathcal{C}_1 \subsetneq \mathcal{C}$, then $\mathcal{C}_1$ has a submaximal circuit e of degree 1. By the maximality of $\mathcal{C}_1$, we have

$$1 = \deg_{\mathcal{C}_1}(e) = \deg_{\mathcal{C}}(e).$$

This contradicts to our assumption on $\mathcal{C}$.

($\Leftarrow$) Let $\mathcal{C}' \subsetneq \mathcal{C}$ such that $\deg_{\mathcal{C}'}(e) = 2 = \deg_{\mathcal{C}}(e)$ for all $e \in \mathrm{SC}(\mathcal{C}')$. Take $F \in \mathcal{C}'$ and $G \in \mathcal{C} \setminus \mathcal{C}'$. By our assumption, there exist a chain $F = F_0, F_1, \ldots, F_r = G$ of circuits of $\mathcal{C}$ such that $|F_i \cap F_{i+1}| = d - 1$ for $i = 0, \ldots, r - 1$.

Since $F_0 = F \in \mathcal{C}'$ and $|F_0 \cap F_1| = d - 1$, we conclude that $F_0 \cap F_1 \in \mathrm{SC}(\mathcal{C}')$. Hence, by our assumption, $\deg_{\mathcal{C}'}(F_0 \cap F_1) = 2$ which implies that $F_1 \in \mathcal{C}'$. The same argument shows that $F_0, F_1, \ldots, F_r$ are in $\mathcal{C}'$. This is a contradiction by our choice of $F_r = G$.

REMARK 6.4. Let $\mathcal{C}$ be a d-uniform pseudo-manifold and $\Delta = \Delta(\mathcal{C})$ be the clique complex of $\mathcal{C}$. In view of Lemmas 6.3 and 5.5, we have:

(a) every proper subclutter of $\mathcal{C}$ has a submaximal circuit of degree 1,

(b) $\operatorname{indeg}(I_\Delta) = 1 + \dim \Delta$.

Putting these results together, Corollary 3.6 implies that

$$I(\overline{\mathcal{C}}) \text{ is minimal to } d\text{-linearity if and only if } \tilde{H}_{d-1}(\Delta; K) \neq 0.$$

PROPOSITION 6.5. *Let $\mathcal{C}$ be a d-uniform clutter.*

(i) *If $\mathcal{C}$ is oriented pseudo-manifold, then $\mathcal{C}$ is minimal to d-linearity.*

(ii) *If $\mathcal{C}$ is non-oriented pseudo-manifold, then $\mathcal{C}$ is minimal to d-linearity if and only if* $\mathrm{Char}(K) = 2$.

PROOF. Let $\mathcal{C}$ be a d-uniform pseudo-manifold and $\Delta = \Delta(\mathcal{C})$ be its clique complex. In view of Lemma 5.5, we know that $\dim \Delta = d - 1$ and $\mathcal{C} = \mathcal{F}(\Delta)$. In particular, $\tilde{H}_{d-1}(\Delta; K) \cong \tilde{H}_{d-1}(\langle \mathcal{C} \rangle; K)$, where $\langle \mathcal{C} \rangle$ is the simplicial complex generated by $\mathcal{C}$. But we know that (see [11, Chapter X, Exercise 6.5] or [14, §43, Exercise 5]):

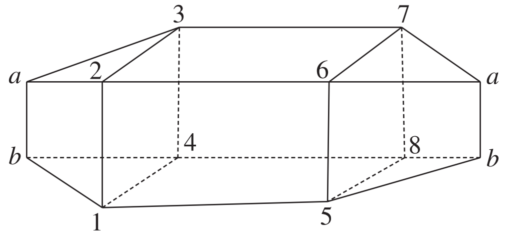
$$\tilde{H}_{d-1}(\langle \mathcal{C} \rangle; K) = \begin{cases} K, & \text{if } \mathcal{C} \text{ is oriented,} \\ \mathrm{Tor}(\mathsf{Z}_2, K), & \text{if } \mathcal{C} \text{ is non-oriented,} \end{cases}$$

where $\mathrm{Tor}(\mathsf{Z}_2, K) = \{a \in K : 2.a = 0\}$. Now, the conclusion follows from Remark 6.4.

Note that if Δ is a triangulation of a connected compact d-manifold (or homology d-manifold), then $C = \mathcal{F}(\Delta)$ is a d-uniform pseudo-manifold (see [14, §43, §63]). So that we may use Theorem 5.3 to find the minimal free resolution of the ideal $I(\overline{C})$. It is worth noting that pseudo-manifolds are strictly contained in $\mathscr{C}_d^{\text{a.tree}}$, as the next example shows.

EXAMPLE 6.6. Let Δ be a triangulation of the following shape and $C = \mathcal{F}(\Delta)$. That is:

$$\Delta = \langle a23, b14, ab1, a12, ab4, a34, 236, 367, 125, 256, \\ 145, 458, 348, 378, a67, b58, ab5, a56, ab8, a78\rangle.$$



Then, C is not a pseudo-manifold, because $\deg_C(ab) = 4$, but C is almost tree and hence minimal to linearity.

EXAMPLE 6.7. Let Δ_1 be a triangulation of a torus and Δ_2 be a triangulation of a projective plane such that they intersect in one triangle. Let $C = \mathcal{F}(\Delta_1) \cup \mathcal{F}(\Delta_2)$ be the corresponding 3-uniform clutter on the vertex set $[n]$.

In view of Theorem 4.10(ii), $\operatorname{reg}(I) = 4$ in any characteristic of the base field.

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