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THE DETERMINANT LINE BUNDLE FOR FREDHOLM OPERATORS: CONSTRUCTION, PROPERTIES, AND CLASSIFICATION

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Abstract

We provide a thorough construction of a system of compatible determinant line bundles over spaces of Fredholm operators, fully verify that this system satisfies a number of important properties, and include explicit formulas for all relevant isomorphisms between these line bundles. We also completely describe all possible systems of compatible determinant line bundles and compare the conventions and approaches used elsewhere in the literature.

1. Introduction

A Fredholm operator between Banach vector spaces X and Y is a bounded homomorphism $D: X \to Y$ such that

$$\operatorname{Im} D \equiv \{Dx : x \in X\}$$

is closed in Y and the dimensions of its kernel and cokernel,

$$\kappa(D) \equiv \{x \in X : Dx = 0\} \quad \text{and} \quad \mathfrak{c}(D) \equiv Y/(\operatorname{Im} D),$$

are finite (the first condition is implied by the other two, but is traditionally stated explicitly.) The space $\mathscr{F}(X, Y)$ of Fredholm operators is an open subspace of the space $\mathscr{B}(X, Y)$ of bounded linear operators $D: X \to Y$ in the normed topology; see [13, Theorem A.1.5(ii)]. Quillen's construction, outlined in [15, Section 2], associates to each Fredholm operator D a $\mathbb{Z}_2$-graded one-dimensional vector space $\lambda(D) = \det D$, called *the determinant line of* D, and topologizes, in a systematic way, the set

$$\det_{X,Y} \equiv \bigsqcup_{D \in \mathscr{F}(X,Y)} \lambda(D)$$

as a line bundle over $\mathscr{F}(X, Y)$ for each pair (X, Y) of Banach vector spaces. There are in fact infinitely many compatible systems of such topologies, all

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of which we describe in Section 3.4; they are isomorphic pairwise. This is contrary to suggestions in many papers that there is a unique way of topologizing determinant line bundles in a systematic way and can be viewed as capturing the essence of the *unique up to a canonical isomorphism* statement in [11, Theorem 1]. We describe some intrinsic and not so-intrinsic ways of narrowing down the choices and of choosing a specific system at the end of Section 2 and at the end of Remark 3.1.

The determinant line bundle plays a prominent role in a number of geometric situations, but unfortunately there appears to be no thorough description of its construction and properties in the literature. The key issue in its construction is the existence of a collection of (set-theoretic) trivializations for $\det_{X,Y}$, such as $\tilde{\mathscr{I}}_{D,T}$ in (2.6) and $\hat{\mathscr{I}}_{\Theta;D}$ in (3.2), that overlap continuously. The justification for the existence of such a collection in [15] consists of an allusion to some unspecified collection of compatible isomorphisms relating the determinant line bundles in the short exact triples

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \mathsf{R}^{k+m} & \longrightarrow & \mathsf{R}^{k+m} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathsf{R}^{c+m} & \longrightarrow & \mathsf{R}^{c+m} & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{1.1}$$

of homomorphisms, where the middle arrow is the projection onto the last m coordinates. Explicit formulas for such a collection of isomorphisms appear in [1, Section (f)], [5, Section 3.2.1], [12, Section 20.2], [13, Appendix A.2], [16, Section 2], and [17, Section (11a)], while [10, Appendix D.2] and [11, Chapter I] describe it more abstractly. The proof of [13, Theorem A.2.2] uses them to describe trivializations for determinant line bundles for Fredholm operators without checking that they overlap continuously, which in fact is not the case, as discovered in [14]; see Section 3.3 for more details. Key properties of such collections of isomorphisms necessary for the construction of the determinant line bundle are specified in [1], [12], [17], and the construction itself is then briefly outlined. The discussion of the relevant considerations from linear algebra is more extensive in [10], but it contains an important deficiency, which is described in Remark 4.9, and does not complete the construction. However, the general approach of [10, Appendix B] is well-suited for an explicit construction of the determinant line bundle and the analysis of its properties. Explicit formulas for the above collection are used directly to topologize determinant line bundles over spaces of Fredholm operators and for Kuranishi structures in [16] and [14], respectively. The latter are closely related to the two-term case of the bounded complexes of vector bundles for which a determinant line bundle is constructed in [11]. As explained in detail in Section 3.2, using [11, Theorem I], which predates [15], is perhaps the most efficient way for constructing the determinant line bundle and verifying its properties and would eliminate the need for most of our Section 4, but at the cost of explicit formulas for important isomorphisms (which may well be useful in specific applications) and of being self-contained. None of the above works explicitly considers most of the non-trivial properties of the determinant line bundle for Fredholm operators listed in Section 2.

This paper provides a comprehensive construction of a system of determinant line bundles and a complete verification of many important properties it satisfies. Section 2 sets up the necessary notation and precisely describes the properties we later show this system satisfies. Section 3.1 outlines the determinant line bundle construction carried out in this paper and three alternative approaches, while Section 3.2 provides more details for the approach based on the results obtained in [11]. Section 3.3 compares several conventions for the determinant line bundle that have appeared in the literature. Section 3.4 establishes Theorem 2, which describes all determinant line bundle systems satisfying the properties in Section 2 and shows that such systems correspond to collections of isomorphisms

$$A_{i,c}\colon \Lambda^c(\mathbb{R}^c) \longrightarrow \mathbb{R}, \quad i \in \mathbb{Z},\ c \in \mathbb{Z}^+,\ c \geq -i; \tag{1.2}$$

in contrast to the viewpoint of the previous paragraph, there are *no* compatibility conditions on the isomorphisms in these collections. By Theorem 2, the compatible systems of topologies on determinant line bundles correspond to the compatible systems of isomorphisms for the exact triples (1.1) and to the compatible collections of isomorphisms for exact triples of Fredholm operators. Section 4, which is motivated by [11, Section 1] and [10, Appendix D.2], deals with the linear algebra preliminaries used in our construction. Section 5 concludes this paper with topological arguments; this section is motivated by the approach in [13, Appendix A.2]. Many of the individual steps that we describe in this paper are not new. However, even the full statement of Theorem 1 in Section 2 does not seem to appear elsewhere.

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2. Properties of the determinant line bundle

All vector spaces we consider are over $\mathbb{R}$. We denote by $\mathfrak{d}(V)$ the dimension of a vector space V and by

$$\lambda(V) \equiv \Lambda^{\mathrm{top}} V \equiv \Lambda^{\mathfrak{d}(V)} V \quad \text{and} \quad \lambda^*(V) \equiv (\lambda(V))^*$$

the top exterior power of V and its dual, whenever $\mathfrak{d}(V)<\infty$. We view $\lambda(V)$ and $\lambda^*(V)$ as graded lines of degrees

$$\deg\lambda(V),\deg\lambda^*(V)=\mathfrak{d}(V)+2\mathbb{Z}\in\mathbb{Z}_2.$$

For any two $\mathbb{Z}_2$-graded lines L_1 and L_2, we define

$$\deg(L_1\otimes L_2)=\deg L_1+\deg L_2,$$

$$R\colon L_1\otimes L_2\longrightarrow L_2\otimes L_1,\qquad R(v_1\otimes v_2)=(-1)^{(\deg L_1)(\deg L_2)}v_2\otimes v_1. \tag{2.1}$$

If $\mathscr{L}_1,\mathscr{L}_2\to\mathscr{F}$ are $\mathbb{Z}_2$-graded line bundles (each fiber has a grading varying continuously over $\mathscr{F}$), the fiberwise isomorphisms R give rise to an isomorphism

$$R\colon\mathscr{L}_1\otimes\mathscr{L}_2\longrightarrow\mathscr{L}_2\otimes\mathscr{L}_1$$

of $\mathbb{Z}_2$-graded line bundles over $\mathscr{F}$. If L is a line and $v\in L-0$, we define $v^*\in L^*$ by $v^*(v)=1$.

For a Fredholm operator $D\colon X\to Y$, we define

$$\lambda(D)=\lambda(\kappa(D))\otimes\lambda^*(\mathfrak{c}(D)) \tag{2.2}$$

with the grading

$$\deg\lambda(D)\equiv\operatorname{ind}D+2\mathbb{Z}\equiv\mathfrak{d}(\kappa(D))-\mathfrak{d}(\mathfrak{c}(D))+2\mathbb{Z}\in\mathbb{Z}_2.$$

This is the same definition as in [12, Section 20.2] and [14, Section 7.4]; we discuss alternative versions of (2.2) in Section 3.

The line bundles $\det_{X,Y}$ satisfy a number of important compatibility properties, which we now describe. A *homomorphism between Fredholm operators* $D\colon X\to Y$ *and* $D'\colon X'\to Y'$ is a pair of homomorphisms $\phi\colon X\to X'$ and $\psi\colon Y\to Y'$ so that $D'\circ\phi=\psi\circ D$; an *isomorphism between Fredholm operators* D *and* D' is a homomorphism $(\phi,\psi)\colon D\to D'$ so that ϕ and ψ are isomorphisms. Such an isomorphism induces isomorphisms

$$\lambda(\phi)\colon\lambda(\kappa(D))\longrightarrow\lambda(\kappa(D')),\qquad\lambda(\psi^{-1})\colon\lambda(\mathfrak{c}(D'))\longrightarrow\lambda(\mathfrak{c}(D)),$$
$$\widetilde{\mathscr{I}}_{\phi,\psi;D}\colon\lambda(D)\longrightarrow\lambda(D'),\qquad x\wedge\alpha\longrightarrow(\lambda(\phi)x)\wedge(\alpha\circ\lambda(\psi^{-1})). \tag{2.3}$$

Isomorphisms $\phi\colon X\to X'$ and $\psi\colon Y\to Y'$ between Banach vector spaces induce a homeomorphism

$$\mathscr{I}_{\phi,\psi}\colon\mathscr{F}(X,Y)\longrightarrow\mathscr{F}(X',Y'),\qquad D\longrightarrow\psi\circ D\circ\phi^{-1};$$

in particular, $(\phi, \psi)\colon D \to \mathscr{I}_{\phi,\psi}(D)$ is an isomorphism of Fredholm operators for each $D \in \mathscr{F}(X, Y)$. Putting the isomorphisms $\widetilde{\mathscr{I}}_{\phi,\psi;D}$ together, we obtain a bundle map

$$\widetilde{\mathscr{I}}_{\phi,\psi}\colon \det_{X,Y} \longrightarrow \mathscr{I}^*_{\phi,\psi} \det_{X',Y'} \tag{2.4}$$

covering the identity on $\mathscr{F}(X, Y)$.

Naturality I: The map $\widetilde{\mathscr{I}}_{\phi,\psi}$ is continuous for every isomorphism

$$(\phi, \psi)\colon (X, Y) \longrightarrow (X', Y')$$

of pairs of Banach vector spaces.

With X, Y as above, we define

$$\mathscr{F}^*(X, Y) = \{D \in \mathscr{F}(X, Y) : \mathfrak{c}(D) = \{0\}\};$$

this is an open subset of $\mathscr{F}(X, Y)$. For each $D \in \mathscr{F}^*(X, Y)$, right inverse $T\colon Y \to X$ of D, and $P \in \mathscr{B}(X, Y)$ sufficiently small, the homomorphism

$$\Phi_{D,T;P}\colon \kappa(D + P) \longrightarrow \kappa(D), \quad x \longrightarrow x - TDx, \tag{2.5}$$

is an isomorphism and thus induces an isomorphism

$$\lambda(D) = \lambda(\kappa(D)) \otimes \mathsf{R}^* \longrightarrow \lambda(\kappa(D + P)) \otimes \mathsf{R}^* = \lambda(D + P).$$

Putting these isomorphisms together, we obtain a bundle map

$$\widetilde{\mathscr{I}}_{D,T}\colon U_{D,T} \times \lambda(D) \longrightarrow \det_{X,Y}\big|_{U_{D,T}} \tag{2.6}$$

covering the identity on an open neighborhood $U_{D,T}$ of D in $\mathscr{F}^*(X, Y)$.

Normalization I: The map $\widetilde{\mathscr{I}}_{D,T}$ is continuous for every $D \in \mathscr{F}^*(X, Y)$, right inverse $T\colon Y \to X$ of D, and sufficiently small open neighborhood $U_{D,T}$ of D in $\mathscr{F}^*(X, Y)$.

For Banach vector spaces X', Y', X'', Y'', the direct sum operation

$$\oplus\colon \mathscr{F}(X', Y') \times \mathscr{F}(X'', Y'') \longrightarrow \mathscr{F}(X' \oplus X'', Y' \oplus Y''),$$
$$(D', D'') \longrightarrow D' \oplus D'',$$

is a continuous map. Let

$$R_{X',X''}\colon X' \oplus X'' \longrightarrow X'' \oplus X'$$

and

$$\mathscr{R}_{\mathscr{F}} : \mathscr{F}(X', Y') \times \mathscr{F}(X'', Y'') \longrightarrow \mathscr{F}(X'', Y'') \times \mathscr{F}(X', Y')$$

be the maps interchanging the factors and

$$\pi_1, \pi_2 : \mathscr{F}(X', Y') \times \mathscr{F}(X'', Y'') \longrightarrow \mathscr{F}(X', Y'), \mathscr{F}(X'', Y'')$$

be the projection maps. We denote by

$$\oplus' : \mathscr{F}(X', Y') \times \mathscr{F}(X'', Y'') \longrightarrow \mathscr{F}(X'' \oplus X', Y'' \oplus Y')$$

and

$$\oplus : \mathscr{F}(X', Y') \times \mathscr{F}(X'', Y'') \times \mathscr{F}(X''', Y''') \longrightarrow \mathscr{F}(X' \oplus X'' \oplus X''', Y' \oplus Y'' \oplus Y''')$$

the compositions

$$\oplus \circ \mathscr{R}_{\mathscr{F}} = \mathscr{I}_{R_{X',X''}, R_{Y',Y''}} \circ \oplus \tag{2.7}$$

and

$$\oplus \circ (\oplus \times \mathrm{id}_{\mathscr{F}(X''',Y''')}) = \oplus \circ (\mathrm{id}_{\mathscr{F}(X',Y')} \times \oplus), \tag{2.8}$$

respectively.

For Banach vector spaces X_1, X_2, X_3, the composition map

$$\mathscr{C}_{X_2} : \mathscr{F}(X_1, X_2) \times \mathscr{F}(X_2, X_3) \longrightarrow \mathscr{F}(X_1, X_3), \quad (D_1, D_2) \longrightarrow D_2 \circ D_1,$$

is continuous as well. If X_4 is another Banach vector space, let

$$\mathscr{C}_{X_2,X_3} : \mathscr{F}(X_1, X_2) \times \mathscr{F}(X_2, X_3) \times \mathscr{F}(X_3, X_4) \longrightarrow \mathscr{F}(X_1, X_4)$$

denote the compositions

$$\mathscr{C}_{X_3} \circ \{\mathscr{C}_{X_2} \times \mathrm{id}_{\mathscr{F}(X_3,X_4)}\} = \mathscr{C}_{X_2} \circ \{\mathrm{id}_{\mathscr{F}(X_1,X_2)} \times \mathscr{C}_{X_3}\}. \tag{2.9}$$

For Banach vector spaces X, Y, X', Y', X'', Y'', let

$$\mathscr{T}(X, Y; X', Y'; X'', Y'') \subset \mathscr{F}(X, Y) \times \mathscr{F}(X', Y') \times \mathscr{F}(X'', Y'') \times \mathscr{B}(X', X) \times \mathscr{B}(X, X'') \times \mathscr{B}(Y', Y) \times \mathscr{B}(Y, Y'') \tag{2.10}$$

be the space of *exact triples of Fredholm operators*, i.e. the subspace of tuples $(D, D', D'', \mathfrak{i}_X, \mathfrak{j}_X, \mathfrak{i}_Y, \mathfrak{j}_Y)$ corresponding to the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{\mathfrak{i}_X} & X & \xrightarrow{\mathfrak{j}_X} & X'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle D'} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D''} & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\mathfrak{i}_Y} & Y & \xrightarrow{\mathfrak{j}_Y} & Y'' & \longrightarrow & 0 \end{array} \tag{2.11}$$

with exact rows. For $\star = {}', {}''$, denote by

$$\mathscr{T}^\star(X, Y; X', Y'; X'', Y'') \subset \mathscr{T}(X, Y; X', Y'; X'', Y'')$$

the subspace of diagrams (2.11) so that $D^\star$ is an isomorphism. If

$$\mathfrak{t} \in \mathscr{T}'(X, Y; X', Y'; X'', Y'')$$

is as in (2.11), the homomorphisms

$$\mathfrak{j}_X : \kappa(D) \longrightarrow \kappa(D''), \quad \mathfrak{j}_Y : \mathfrak{c}(D) \longrightarrow \mathfrak{c}(D'')$$

are isomorphisms; let

$$\begin{gathered} \mathscr{I}'_{\mathfrak{t}} : \lambda(D') \otimes \lambda(D'') \longrightarrow \lambda(D), \\ 1 \otimes 1^* \otimes (\lambda(\mathfrak{j}_X)x) \otimes \alpha'' \longrightarrow x \otimes (\alpha'' \circ \lambda(\mathfrak{j}_Y)), \end{gathered} \tag{2.12}$$

be the natural induced isomorphism. If $\mathfrak{t} \in \mathscr{T}''(X, Y; X', Y'; X'', Y'')$ is as in (2.11), the homomorphisms

$$\mathfrak{i}_X : \kappa(D') \longrightarrow \kappa(D), \quad \mathfrak{i}_Y : \mathfrak{c}(D') \longrightarrow \mathfrak{c}(D)$$

are isomorphisms; let

$$\begin{gathered} \mathscr{I}''_{\mathfrak{t}} : \lambda(D') \otimes \lambda(D'') \longrightarrow \lambda(D), \\ x' \otimes (\alpha \circ \lambda(\mathfrak{i}_X)) \otimes 1 \otimes 1^* \longrightarrow (\lambda(\mathfrak{i}_X)x') \otimes \alpha, \end{gathered} \tag{2.13}$$

be the natural induced isomorphism.

With notation as in (2.10), denote by

$$\pi_{\mathrm{C}}, \pi_{\mathrm{L}}, \pi_{\mathrm{R}} : \mathscr{T}(X, Y; X', Y'; X'', Y'') \longrightarrow \mathscr{F}(X, Y), \mathscr{F}(X', Y'), \mathscr{F}(X'', Y'')$$

the restrictions of the projection maps and by

$$\begin{aligned} \mathscr{C}_{\mathscr{T}} : \mathscr{T}(X_1, X_2; X_1', X_2'; X_1'', X_2'') \times \mathscr{T}(X_2, X_3; X_2', X_3'; X_2'', X_3'') \\ \longrightarrow \mathscr{T}(X_1, X_3; X_1', X_3'; X_1'', X_3'') \end{aligned}$$

the continuous map sending commutative diagrams

$$(2.14)\quad \begin{array}{ccccccccc} 0 & \longrightarrow & X_1' & \xrightarrow{\mathfrak{i}_1} & X_1 & \xrightarrow{\mathfrak{j}_1} & X_1'' & \longrightarrow & 0 \\ & & \downarrow D_1' & & \downarrow D_1 & & \downarrow D_1'' & & \\ 0 & \longrightarrow & X_2' & \xrightarrow{\mathfrak{i}_2} & X_2 & \xrightarrow{\mathfrak{j}_2} & X_2'' & \longrightarrow & 0 \\ \\ 0 & \longrightarrow & X_2' & \xrightarrow{\mathfrak{i}_2} & X_2 & \xrightarrow{\mathfrak{j}_2} & X_2'' & \longrightarrow & 0 \\ & & \downarrow D_2' & & \downarrow D_2 & & \downarrow D_2'' & & \\ 0 & \longrightarrow & X_3' & \xrightarrow{\mathfrak{i}_3} & X_3 & \xrightarrow{\mathfrak{j}_3} & X_3'' & \longrightarrow & 0 \end{array}$$

to the commutative diagram

$$(2.15)\quad \begin{array}{ccccccccc} 0 & \longrightarrow & X_1' & \xrightarrow{\mathfrak{i}_1} & X_1 & \xrightarrow{\mathfrak{j}_1} & X_1'' & \longrightarrow & 0 \\ & & \downarrow D_2'\circ D_1' & & \downarrow D_2\circ D_1 & & \downarrow D_2''\circ D_1'' & & \\ 0 & \longrightarrow & X_3' & \xrightarrow{\mathfrak{i}_3} & X_3 & \xrightarrow{\mathfrak{j}_3} & X_3'' & \longrightarrow & 0 \end{array}$$

We note that

$$(2.16)\quad \begin{aligned} (\pi_{\mathrm{C}}, \pi_{\mathrm{L}}, \pi_{\mathrm{R}}) \circ \mathscr{C}_{\mathscr{T}} = \big(&\mathscr{C}_{X_2} \circ (\pi_{\mathrm{C}} \circ \pi_1, \pi_{\mathrm{C}} \circ \pi_2), \\ &\mathscr{C}_{X_2'} \circ (\pi_{\mathrm{L}} \circ \pi_1, \pi_{\mathrm{L}} \circ \pi_2), \mathscr{C}_{X_2''} \circ (\pi_{\mathrm{R}} \circ \pi_1, \pi_{\mathrm{R}} \circ \pi_2)\big), \end{aligned}$$

where

$$\begin{aligned} \pi_1, \pi_2 \colon &\mathscr{T}(X_1, X_2; X_1', X_2'; X_1'', X_2'') \times \mathscr{T}(X_2, X_3; X_2', X_3'; X_2'', X_3'') \\ &\longrightarrow \mathscr{T}(X_1, X_2; X_1', X_2'; X_1'', X_2''), \mathscr{T}(X_2, X_3; X_2', X_3'; X_2'', X_3'') \end{aligned}$$

are the projection maps.

Associating the direct sum $D' \oplus D''$ with the commutative diagram

$$(2.17)\quad \begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{\mathfrak{i}_X} & X' \oplus X'' & \xrightarrow{\mathfrak{j}_X} & X'' & \longrightarrow & 0 \\ & & \downarrow D' & & \downarrow D'\oplus D'' & & \downarrow D'' & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\mathfrak{i}_Y} & Y' \oplus Y'' & \xrightarrow{\mathfrak{j}_Y} & Y'' & \longrightarrow & 0 \end{array} \qquad \begin{aligned} &\mathfrak{i}_X(x') = (x', 0) \\ &\mathfrak{j}_X(x', x'') = x'' \\ \\ &\mathfrak{i}_Y(y') = (y', 0) \\ &\mathfrak{j}_Y(y', y'') = y'', \end{aligned}$$

we obtain an embedding

$$\iota_\oplus \colon \mathscr{F}(X', Y') \times \mathscr{F}(X'', Y'') \longrightarrow \mathscr{T}(X' \oplus X'', Y' \oplus Y''; X', Y'; X'', Y'')$$

subject to

$$\pi_{\mathrm{C}} \circ \iota_{\oplus} = \oplus, \quad \pi_{\mathrm{L}} \circ \iota_{\oplus} = \pi_1, \quad \pi_{\mathrm{R}} \circ \iota_{\oplus} = \pi_2.$$

The isomorphisms (2.12) and (2.13) give rise to canonical identifications

$$\text{(2.18)} \qquad \lambda(\mathrm{id}_Z \oplus D) = \lambda(D) = \lambda(D \oplus \mathrm{id}_Z),$$

$$\begin{aligned}(0, x_1) \wedge \ldots \wedge (0, x_k) \otimes ((0, y_1) \wedge \ldots \wedge (0, y_\ell))^* \\ \longleftrightarrow x_1 \wedge \ldots \wedge x_k \otimes (y_1 \wedge \ldots \wedge y_\ell)^* \\ \longleftrightarrow (x_1, 0) \wedge \ldots \wedge (x_k, 0) \otimes ((y_1, 0) \wedge \ldots \wedge (y_\ell, 0))^*,\end{aligned}$$

for any Fredholm operator $D: X \to Y$ and Banach vector space Z.

Associating the composition $D_2 \circ D_1$ with the commutative diagram

(2.19)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{\mathrm{i}_X} & X_1 \oplus X_2 & \xrightarrow{\mathrm{j}_X} & X_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle D_1} & & \downarrow{\scriptstyle D_2 D_1 \oplus \mathrm{id}_{X_2}} & & \downarrow{\scriptstyle D_2} & & \\ 0 & \longrightarrow & X_2 & \xrightarrow{\mathrm{i}_Y} & X_3 \oplus X_2 & \xrightarrow{\mathrm{j}_Y} & X_3 & \longrightarrow & 0 \end{array}$$

$$\mathrm{i}_X(x_1) = (x_1, D_1 x_1) \qquad \mathrm{j}_X(x_1, x_2) = D_1 x_1 - x_2$$

$$\mathrm{i}_Y(x_2) = (D_2 x_2, x_2) \qquad \mathrm{j}_Y(x_3, x_2) = x_3 - D_2 x_2,$$

we obtain an embedding

$$\iota_{\mathscr{C}}: \mathscr{F}(X_1, X_2) \times \mathscr{F}(X_2, X_3) \longrightarrow \mathscr{T}(X_1 \oplus X_2, X_3 \oplus X_2; X_1, X_2; X_2, X_3)$$

subject to

$$\pi_{\mathrm{C}} \circ \iota_{\mathscr{C}}(D_1, D_2) = \mathscr{C}_{X_2}(D_1, D_2) \oplus \mathrm{id}_{X_2}, \quad \pi_{\mathrm{L}} \circ \iota_{\mathscr{C}} = \pi_1, \quad \pi_{\mathrm{R}} \circ \iota_{\mathscr{C}} = \pi_2.$$

In particular, $\iota_{\mathscr{C}}^* \pi_{\mathrm{C}}^* \det_{X,Y} = \mathscr{C}_{X_2}^* \det_{X_1, X_3}$. If $D \in \mathscr{F}(X, Y)$, the compositions $D \circ \mathrm{id}_X$ and $\mathrm{id}_Y \circ D$ correspond to elements of

$$\mathscr{T}'(X \oplus X, Y \oplus X; X, X; X, Y) \quad \text{and} \quad \mathscr{T}''(X \oplus Y, Y \oplus Y; X, Y; Y, Y),$$

respectively, with the isomorphisms $\mathscr{I}'_{\mathrm{t}}$ and $\mathscr{I}''_{\mathrm{t}}$ of (2.12) and (2.13) given by

$$1 \otimes 1^* \otimes x \otimes \beta \longrightarrow x \otimes \beta \quad \text{and} \quad x \otimes \beta \otimes 1 \otimes 1^* \longrightarrow x \otimes \beta$$

under the identifications (2.18).

Exact Triples: There exists a collection of (continuous) line bundle isomorphisms

$$\text{(2.20)} \qquad \Psi: \pi_{\mathrm{L}}^* \det_{X', Y'} \otimes \pi_{\mathrm{R}}^* \det_{X'', Y''} \longrightarrow \pi_{\mathrm{C}}^* \det_{X, Y}$$

over $\mathscr{T}(X, Y; X', Y'; X'', Y'')$ parametrized by tuples $(X, Y; X', Y'; X'', Y'')$ of Banach vector spaces with the following properties.

Naturality II: The isomorphisms Ψ commute with isomorphisms of exact triples of Fredholm operators, i.e. for each isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D'_{\mathrm{T}} & \longrightarrow & D_{\mathrm{T}} & \longrightarrow & D''_{\mathrm{T}} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \phi',\psi'} & & \big\downarrow{\scriptstyle \phi,\psi} & & \big\downarrow{\scriptstyle \phi'',\psi''} & & \\ 0 & \longrightarrow & D'_{\mathrm{B}} & \longrightarrow & D_{\mathrm{B}} & \longrightarrow & D''_{\mathrm{B}} & \longrightarrow & 0 \end{array} \tag{2.21}$$

of exact triples of Fredholm operators, the diagram

$$\begin{array}{ccc} \lambda(D'_{\mathrm{T}}) \otimes \lambda(D''_{\mathrm{T}}) & \xrightarrow{\quad \Psi_{\mathfrak{t}_{\mathrm{T}}} \quad} & \lambda(D_{\mathrm{T}}) \\ \big\downarrow{\scriptstyle \widetilde{\mathscr{I}}_{\phi',\psi';D'_{\mathrm{T}}} \otimes \widetilde{\mathscr{I}}_{\phi'',\psi'';D''_{\mathrm{T}}}} & & \big\downarrow{\scriptstyle \widetilde{\mathscr{I}}_{\phi,\psi;D_{\mathrm{T}}}} \\ \lambda(D'_{\mathrm{B}}) \otimes \lambda(D''_{\mathrm{B}}) & \xrightarrow{\quad \Psi_{\mathfrak{t}_{\mathrm{B}}} \quad} & \lambda(D_{\mathrm{B}}) \end{array}$$

where $\widetilde{\mathscr{I}}_{\phi^\star,\psi^\star;D^\star_{\mathrm{T}}}$ are the isomorphisms (2.3) and Ψ_{T} and Ψ_{B} are the isomorphisms (2.21) for the top and bottom exact triples in (2.20), commutes.

Naturality III: For each $\star = {}', {}''$ and $\mathfrak{t} \in \mathscr{T}^\star(X, Y; X', Y'; X'', Y'')$, the restriction $\Psi_{\mathfrak{t}}$ of Ψ to the fiber over $\mathfrak{t}$ is the canonical isomorphism $\mathscr{I}^\star_{\mathfrak{t}}$ of (2.12) or (2.13).

Normalization II: For each $\mathfrak{t} \in \mathscr{T}(X, Y; X', Y'; X'', Y'')$ as in (2.10) with $D' \in \mathscr{F}^*(X', Y')$ and $D'' \in \mathscr{F}^*(X'', Y'')$, $\Psi_{\mathfrak{t}}$ is the canonical isomorphism $\wedge_{\kappa(D)}$ of Lemma 4.1 for the short exact sequence

$$0 \longrightarrow \kappa(D') \longrightarrow \kappa(D) \longrightarrow \kappa(D'') \longrightarrow 0$$

of finite-dimensional vector spaces.

Compositions: The isomorphisms

$$\widetilde{\mathscr{C}}_{D_1,D_2} \equiv \Psi_{\iota_{\mathscr{C}}(D_1,D_2)} \colon \lambda(D_1) \otimes \lambda(D_2) \longrightarrow \lambda(D_2 \circ D_1)$$

with $D_1 \in \mathscr{F}(X_1, X_2)$ and $D_2 \in \mathscr{F}(X_2, X_3)$ provide liftings of (2.9) and (2.16) to determinant line bundles, i.e. the diagram

$$\begin{array}{ccc} \lambda(D_1) \otimes \lambda(D_2) \otimes \lambda(D_3) & \xrightarrow{\quad \mathrm{id} \otimes \widetilde{\mathscr{C}}_{D_2,D_3} \quad} & \lambda(D_1) \otimes \lambda(D_3 \circ D_2) \\ \big\downarrow{\scriptstyle \widetilde{\mathscr{C}}_{D_1,D_2} \otimes \mathrm{id}} & & \big\downarrow{\scriptstyle \widetilde{\mathscr{C}}_{D_2 \circ D_1, D_3}} \\ \lambda(D_2 \circ D_1) \otimes \lambda(D_3) & \xrightarrow{\quad \widetilde{\mathscr{C}}_{D_2 \circ D_1, D_3} \quad} & \lambda(D_3 \circ D_2 \circ D_1) \end{array} \tag{2.22}$$

commutes for all $D_1 \in \mathscr{F}(X_1, X_2)$, $D_2 \in \mathscr{F}(X_2, X_3)$, and $D_3 \in \mathscr{F}(X_3, X_4)$ and the diagram

$$
\begin{CD}
\lambda(D_1') \otimes \lambda(D_1'') \otimes \lambda(D_2') \otimes \lambda(D_2'') @>{\Psi_{\mathfrak{t}_1} \otimes \Psi_{\mathfrak{t}_2}}>> \lambda(D_1) \otimes \lambda(D_2) \\
@V{\widetilde{\mathscr{C}}_{D_1', D_2'} \otimes \widetilde{\mathscr{C}}_{D_1'', D_2''} \circ \mathrm{id} \otimes R \otimes \mathrm{id}}VV @VV{\widetilde{C}_{D_1, D_2}}V \\
\lambda(D_2' \circ D_1') \otimes \lambda(D_2'' \circ D_1'') @>{\Psi_{\mathscr{C}_{\mathscr{T}}(\mathfrak{t}_1, \mathfrak{t}_2)}}>> \lambda(D_2 \circ D_1)
\end{CD}
\tag{2.23}
$$

commutes for all exact Fredholm triples $\mathfrak{t}_1$ and $\mathfrak{t}_2$ in $\mathscr{T}(X_1, X_2; X_1', X_2'; X_1'', X_2'')$ and $\mathscr{T}(X_2, X_3; X_2', X_3'; X_2'', X_3'')$, respectively.

Direct Sums: The isomorphisms

$$\widetilde{\oplus}_{D', D''} \equiv \Psi_{D' \oplus D''} \colon \lambda(D') \otimes \lambda(D'') \longrightarrow \lambda(D)$$

with $D' \in \mathscr{F}(X', Y')$ and $D'' \in \mathscr{F}(X'', Y'')$ provide liftings of (2.7) and (2.8) to determinant line bundles, i.e. the diagram

$$
\begin{CD}
\lambda(D') \otimes \lambda(D'') @>{\widetilde{\oplus}_{D', D''}}>> \lambda(D' \oplus D'') \\
@V{R}VV @VV{\widetilde{\mathscr{I}}_{R_{X', X''}, R_{Y', Y''}; D' \oplus D''}}V \\
\lambda(D'') \otimes \lambda(D') @>{\widetilde{\oplus}_{D'', D'}}>> \lambda(D'' \oplus D')
\end{CD}
\tag{2.24}
$$

commutes for all $D' \in \mathscr{F}(X', Y')$ and $D'' \in \mathscr{F}(X'', Y'')$ and the diagram

$$
\begin{CD}
\lambda(D') \otimes \lambda(D'') \otimes \lambda(D''') @>{\mathrm{id} \otimes \widetilde{\oplus}_{D'', D'''}}>> \lambda(D') \otimes \lambda(D'' \oplus D''') \\
@V{\widetilde{\oplus}_{D', D''} \otimes \mathrm{id}}VV @VV{\widetilde{\oplus}_{D' \oplus D'', D'''}}V \\
\lambda(D' \oplus D'') \otimes \lambda(D''') @>{\widetilde{\oplus}_{D' \oplus D'', D'''}}>> \lambda(D' \oplus D'' \oplus D''')
\end{CD}
\tag{2.25}
$$

commutes for all $D' \in \mathscr{F}(X', Y')$, $D'' \in \mathscr{F}(X'', Y'')$, and $D''' \in \mathscr{F}(X''', Y''')$.

THEOREM 1. *There exist a collection of topologies on the line bundles*

$$\det_{X,Y} \longrightarrow \mathscr{F}(X, Y)$$

corresponding to pairs (X, Y) *of Banach spaces and a collection of line-bundle isomorphisms* (2.20) *which satisfy the Naturality I, II, III, Normalization I, II, Compositions, and Direct Sums properties above.*

Some of the properties listed above are implied by other properties:

- Naturality I follows from the continuity of Ψ and Naturality III;

- Naturality II follows from Normalization II and the last Compositions property, each applied twice;
- Naturality III follows from Normalization II and the two algebraic Compositions properties (via Naturality II);
- the continuity of Ψ follows from Normalization I, II and the two algebraic Compositions properties (see proofs of Lemma 5.1 and Corollary 5.4);
- the system of topologies on the line bundles $\det_{X,Y}$ is determined by Normalization I, II and the two algebraic Compositions properties (see proof of Proposition 5.3).

By the proof of Corollary 4.13, the next property is also implied by the Normalization II property and the two algebraic Compositions properties; in Section 3.2, we deduce the two algebraic Compositions properties from the Exact Squares property.

Exact Squares: For every commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_{\mathrm{TL}} & \xrightarrow{\mathrm{i_T}} & D_{\mathrm{TM}} & \xrightarrow{\mathrm{j_T}} & D_{\mathrm{TR}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{i_L}} & & \downarrow{\scriptstyle \mathrm{i_M}} & & \downarrow{\scriptstyle \mathrm{i_R}} & & \\
0 & \longrightarrow & D_{\mathrm{CL}} & \xrightarrow{\mathrm{i_C}} & D_{\mathrm{CM}} & \xrightarrow{\mathrm{j_C}} & D_{\mathrm{CR}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{j_L}} & & \downarrow{\scriptstyle \mathrm{i_M}} & & \downarrow{\scriptstyle \mathrm{j_R}} & & \\
0 & \longrightarrow & D_{\mathrm{BL}} & \xrightarrow{\mathrm{i_B}} & D_{\mathrm{BM}} & \xrightarrow{\mathrm{j_B}} & D_{\mathrm{BR}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{2.26}
$$

of exact rows and columns of Fredholm operators, the diagram

$$
\begin{array}{ccc}
\lambda(D_{\mathrm{TL}})\otimes\lambda(D_{\mathrm{BL}})\otimes\lambda(D_{\mathrm{TR}})\otimes\lambda(D_{\mathrm{BR}}) & \xrightarrow{\Psi_{\mathrm{T}}\otimes\Psi_{\mathrm{B}}\circ\mathrm{id}\otimes R\otimes\mathrm{id}} & \lambda(D_{\mathrm{TM}})\otimes\lambda(D_{\mathrm{BM}}) \\
\downarrow{\scriptstyle \Psi_{\mathrm{L}}\otimes\Psi_{\mathrm{R}}} & & \downarrow{\scriptstyle \Psi_{\mathrm{M}}} \\
\lambda(D_{\mathrm{CL}})\otimes\lambda(D_{\mathrm{CR}}) & \xrightarrow{\qquad\Psi_{\mathrm{C}}\qquad} & \lambda(D_{\mathrm{CM}})
\end{array}
\tag{2.27}
$$

of graded lines, where $\Psi_\star$ are the isomorphisms (2.20) corresponding to the top, center, and bottom rows and left, middle, and right columns of the diagram (2.26), commutes.

It is thus consistent with [11, Theorem 1] that the Normalization I, II properties and the two algebraic Compositions properties completely determine a compatible system of topologies on determinant line bundles. The two Direct Sums properties follow from the Exact Squares property applied to the two diagrams in Figure 1 and the Naturality II property applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D'' & \longrightarrow & D' \oplus D'' & \longrightarrow & D' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D'' & \longrightarrow & D'' \oplus D' & \longrightarrow & D' & \longrightarrow & 0 \end{array}$$

By the proof of Lemma 5.5, the Normalization I property can be replaced by a dual version. Let

$$\mathscr{F}'(X, Y) \equiv \{D \in \mathscr{F}(X, Y) : \kappa(D) = 0\}$$

be the space of injective Fredholm operators. For each $D_0 \in \mathscr{F}'(X, Y)$, right inverse $S\colon \mathfrak{c}(D_0) \to Y$ for

$$q_{D_0}\colon Y \longrightarrow \mathfrak{c}(D_0), \quad y \longrightarrow y + \operatorname{Im} D_0,$$

and $D \in \mathscr{F}(X, Y)$ sufficiently close to D_0, the homomorphism

$$q_D \circ S\colon \mathfrak{c}(D_0) \longrightarrow \mathfrak{c}(D)$$

is an isomorphism and thus induces an isomorphism

$$\widetilde{\mathscr{I}}_{D_0,S;D}\colon \lambda(D_0) \longrightarrow \lambda(D), \quad 1 \otimes \alpha \longrightarrow 1 \otimes (\alpha \circ \lambda(q_D \circ S)^{-1}). \tag{2.28}$$

Putting these isomorphisms together, we obtain a bundle map

$$\widetilde{\mathscr{I}}_{D_0,S} : U_{D_0,S} \times \lambda(D_0) \longrightarrow \det_{X,Y}|_{U_{D_0,S}} \tag{2.29}$$

covering the identity on an open neighborhood $U_{D_0,S}$ of D_0 in $\mathscr{F}'(X, Y)$.

Normalization I′: The map $\widetilde{\mathscr{I}}_{D_0,S}$ is continuous for every $D_0 \in \mathscr{F}'(X, Y)$, right inverse $S\colon \mathfrak{c}(D_0) \to Y$ of q_{D_0}, and sufficiently small open neighborhood $U_{D_0,S}$ of D_0 in $\mathscr{F}'(X, Y)$.

The determinant line bundle is also compatible with dualizations of Fredholm operators. For each Banach vector space X, let X^* denote the dual Banach vector space, i.e. the space of bounded linear functionals $X \to \mathbb{R}$. For each $D \in \mathscr{F}(X, Y)$, let $D^* \in \mathscr{F}(Y^*, X^*)$ denote the dual operator, i.e.

$$\{D^*\beta\}(x) = \beta(Dx) \quad \forall\, \beta \in Y^*, x \in X.$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & D' & \longrightarrow & D' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & D' \oplus D'' & \longrightarrow & D' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & D'' & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D' & \longrightarrow & D' & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D' \oplus D'' & \longrightarrow & D' \oplus D'' \oplus D''' & \longrightarrow & D''' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & D'' \oplus D''' & \longrightarrow & D''' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

FIGURE 1. Exact squares of Fredholm operators corresponding to the two Direct Sums properties

The map

$$\mathscr{D}\colon \mathscr{F}(X,Y) \longrightarrow \mathscr{F}(Y^*,X^*), \quad D \longrightarrow D^*,$$

is then continuous. For each $D \in \mathscr{F}(X,Y)$, the homomorphisms

(2.30)

$$\mathscr{D}_D\colon \kappa(D) \longrightarrow \mathfrak{c}(D^*)^*, \;\; \{\mathscr{D}_D(x)\}(\alpha + \operatorname{Im} D^*) = \alpha(x) \;\; \forall\, x \in \kappa(D), \alpha \in X^*,$$

$$\mathscr{D}_D\colon \mathfrak{c}(D)^* \longrightarrow \kappa(D^*), \;\; \{\mathscr{D}_D(\beta)\}(y) = \beta(y + \operatorname{Im} D) \quad \forall\, \beta \in \mathfrak{c}(D)^*, y \in Y,$$

are isomorphisms. For each finite-dimensional vector space V, we define

$$\mathscr{P}\colon \lambda(V^*) \longrightarrow \lambda^*(V), \tag{2.31}$$
$$\{\mathscr{P}(\alpha_1 \wedge \ldots \wedge \alpha_n)\}(v_1 \wedge \ldots \wedge v_n) = (-1)^{\binom{n}{2}} \det(\alpha_i(v_j))_{i,j=1,\ldots,n}$$

and denote the inverse of $\mathscr{P}$ also by $\mathscr{P}$. For each exact triple $\mathfrak{t}$ of Fredholm operators as in (2.11), we define the dual triple $\mathfrak{t}^*$ to be given by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y''^* & \xrightarrow{\mathrm{j}_Y^*} & Y^* & \xrightarrow{\mathrm{i}_Y^*} & Y'^* & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle D''^*} & & \Big\downarrow{\scriptstyle D^*} & & \Big\downarrow{\scriptstyle D'^*} & & \\ 0 & \longrightarrow & X''^* & \xrightarrow{\mathrm{j}_X^*} & X^* & \xrightarrow{\mathrm{i}_X^*} & X'^* & \longrightarrow & 0 \end{array} \tag{2.32}$$

This defines an embedding

$$\mathscr{D}_{\mathscr{T}}\colon \mathscr{T}(X,Y;X',Y';X'',Y'') \longrightarrow \mathscr{T}(Y^*,X^*;Y''^*,X''^*;Y'^*,X'^*) \tag{2.33}$$
$$\text{s.t.} \quad \pi_{\mathrm{C}} \circ \mathscr{D}_{\mathscr{T}} = \mathscr{D} \circ \pi_{\mathrm{C}}, \quad \pi_{\mathrm{L}} \circ \mathscr{D}_{\mathscr{T}} = \mathscr{D} \circ \pi_{\mathrm{R}}, \quad \pi_{\mathrm{R}} \circ \mathscr{D}_{\mathscr{T}} = \mathscr{D} \circ \pi_{\mathrm{L}}.$$

The advantage of the isomorphism (2.31) over the isomorphism induced by the first pairing in (3.10) is that the former fits better with short exact sequences; see the last statement of Lemma 4.2.

Dualizations: There exists a collection of (continuous) line bundle isomorphisms

$$\widetilde{\mathscr{D}}\colon \det{}_{X,Y} \longrightarrow \mathscr{D}^* \det{}_{Y^*,X^*} \tag{2.34}$$

over $\mathscr{F}(X,Y)$ parametrized by pairs (X,Y) of Banach vector spaces with the following properties.

Normalization III: For every homomorphism $\delta\colon L \to \{0\}$ from a line,

$$\widetilde{\mathscr{D}}_\delta(x \otimes 1^*) = 1 \otimes \mathscr{P}(\lambda(\mathscr{D}_\delta)x) \quad \forall\, x \in \lambda(L) = L.$$

Dual Exact Triples: The isomorphisms (2.20) and (2.34) provide a lifting of (2.33) to determinant line bundles, i.e. the diagram

$$\begin{array}{ccc} \lambda(D') \otimes \lambda(D'') & \xrightarrow{\quad \Psi_{\mathfrak{t}} \quad} & \lambda(D) \\ \Big\downarrow{\scriptstyle \widetilde{\mathscr{D}}_{D''} \otimes \widetilde{\mathscr{D}}_{D'} \circ R} & & \Big\downarrow{\scriptstyle \widetilde{\mathscr{D}}_D} \\ \lambda(D''^*) \otimes \lambda(D'^*) & \xrightarrow{\quad \Psi_{\mathfrak{t}^*} \quad} & \lambda(D^*) \end{array} \tag{2.35}$$

commutes for every $\mathfrak{t} \in \mathscr{T}(X,Y;X',Y';X'',Y'')$ as in (2.10).

By Corollary 5.7 and Section 3.4, each determinant line bundle system as in Theorem 1 determines a unique system of isomorphisms $\widetilde{\mathscr{D}}$ satisfying the above two properties. Furthermore, there is a somewhat smaller family of line bundle systems that satisfy a stronger version of the Normalization III property:

Normalization III**:* For each $D \in \mathscr{F}^*(X, Y)$, $\widetilde{\mathscr{D}}_D$ is the canonical isomorphism induced by the first equation (2.30) and the pairing (2.31):

$$\lambda(D) \longrightarrow \lambda(D^*), \quad x \otimes 1^* \longrightarrow 1 \otimes \mathscr{P}(\lambda(\mathscr{D}_D)x). \tag{2.36}$$

By Lemma 5.6, the isomorphisms (2.36) give rise to a continuous bundle map over $\mathscr{F}^*(X, Y)$ for any system of topologies on determinant line bundles as in Theorem 1. In the proof of Corollary 5.7, we use this to show that the continuity of (2.34) is implied by the Dual Exact Triples property. However, the latter is compatible with the Normalization III* property only for some determinant line bundle systems, including the one specified by the isomorphisms $\Psi_{\mathfrak{t}}$ of (4.10).

The dualization isomorphisms $\widetilde{\mathscr{D}}_D$ given by (4.13) and the identity isomorphisms $A_{i,1}$ in (1.2) seem rather natural. However, by Theorem 2, the number of systems of topologies on determinant line bundles compatible with these choices is still infinite.

Combining the Dual Exact Triples property with the Naturality II property applied to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D''^* & \xrightarrow{\mathfrak{j}^*} & (D' \oplus D'')^* & \xrightarrow{\mathfrak{i}^*} & D'^* & \xrightarrow{\text{id}} & 0 \\
 & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle (T_X, T_Y)} & & \downarrow & & \\
0 & \longrightarrow & D''^* & \xrightarrow{\mathfrak{i}} & D''^* \oplus D'^* & \xrightarrow{\mathfrak{j}} & D'^* & \longrightarrow & 0,
\end{array} \tag{2.37}$$

$$T_X(\beta) = (\beta|_{Y''}, \beta|_{Y'}), \quad T_Y(\alpha) = (\alpha|_{X''}, \alpha|_{X'}),$$

where $\mathfrak{i} = (\mathfrak{i}_X, \mathfrak{i}_Y)$ and $\mathfrak{j} = (\mathfrak{j}_X, \mathfrak{j}_Y)$ are as in (2.17), we find that the diagram

$$\begin{array}{ccc}
\lambda(D') \otimes \lambda(D'') & \xrightarrow{\widetilde{\oplus}_{D',D''}} & \lambda(D' \oplus D'') \\
\downarrow{\scriptstyle \widetilde{\mathscr{D}}_{D''} \otimes \widetilde{\mathscr{D}}_{D'} \circ R} & & \downarrow{\scriptstyle \widetilde{\mathscr{I}}_{T_X, T_Y; (D' \oplus D'')^*} \circ \widetilde{\mathscr{D}}_{D' \oplus D''}} \\
\lambda(D''^*) \otimes \lambda(D'^*) & \xrightarrow{\widetilde{\oplus}_{D''^*, D'^*}} & \lambda(D''^* \oplus D'^*)
\end{array}$$

commutes, i.e. the dualization and direct sum isomorphisms, $\widetilde{\mathscr{D}}$ and $\widetilde{\oplus}$, on the determinant lines are compatible. Combining the Dual Exact Triples property with the Naturality II property applied to (2.37) with $(D', D'') = (D_2 \circ$

D_1, id_{X_2}), we find that the diagram

$$\begin{array}{ccc} \lambda(D_1)\otimes\lambda(D_2) & \xrightarrow{\widetilde{\mathscr{C}}_{D_1,D_2}} & \lambda(D_2\circ D_1) \\ \Big\downarrow{\scriptstyle \widetilde{\mathscr{D}}_{D_2}\otimes\widetilde{\mathscr{D}}_{D_1}\circ R} & & \Big\downarrow{\scriptstyle \widetilde{\mathscr{D}}_{D_2\circ D_1}} \\ \lambda(D_2^*)\otimes\lambda(D_1^*) & \xrightarrow{\widetilde{\mathscr{C}}_{D_2^*,D_1^*}} & \lambda(D_2\circ D_1) \end{array}$$

commutes, i.e. the dualization and composition isomorphisms, $\widetilde{\mathscr{D}}$ and $\widetilde{\mathscr{C}}$, on the determinant lines are compatible.

Section 4.2 provides explicit formulas for the above isomorphisms Ψ_t, $\widetilde{\mathscr{C}}_{D_1,D_2}$, $\widetilde{\oplus}_{D',D''}$, and $\widetilde{\mathscr{D}}_D$; see (4.10), (4.22), (4.12), and (4.13), respectively. Such formulas may be useful in some applications.

3. Conceptual considerations and comparison of conventions

3.1. Topologizing determinant line bundles

For any Banach vector spaces X and Y, the overlap maps between the trivializations $\widetilde{\mathscr{I}}_{D,T}$ of $\det_{X,Y}$ in (2.6) are continuous. Thus, the trivializations $\widetilde{\mathscr{I}}_{D,T}$ topologize $\det_{X,Y}|_{\mathscr{F}^*(X,Y)}$ as a line bundle over $\mathscr{F}^*(X,Y)$, as required by the Normalization I property on page 207. By Lemma 5.1, the resulting topology is compatible with the Normalization II property on page 212.

For any Banach vector space X and $N \in \mathsf{Z}^{\geq 0}$, let $\iota_{X;N}\colon X \to X\oplus\mathsf{R}^N$ be the natural inclusion. If Y is another Banach vector space, $D\in\mathscr{F}(X,Y)$, and $\Theta\colon\mathsf{R}^N\to Y$ is any homomorphism, define

$$\iota_\Theta\colon \mathscr{F}(X,Y)\longrightarrow\mathscr{F}(X\oplus\mathsf{R}^N,Y)$$

by

$$\iota_\Theta(D)=D_\Theta,\quad D_\Theta(x,u)=Dx+\Theta(u);$$

the map ι_Θ is an embedding. The exact triple

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\iota_{X;N}} & X\oplus\mathsf{R}^N & \xrightarrow{\pi_2} & \mathsf{R}^N & \xrightarrow{\mathrm{id}} & 0 \\ & & \Big\downarrow{\scriptstyle D} & & \Big\downarrow{\scriptstyle D_\Theta} & & \Big\downarrow & & \\ 0 & \longrightarrow & Y & \xrightarrow{\mathrm{id}_Y} & Y & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{3.1}$$

and (2.20) give rise to the isomorphism

$$\hat{\mathscr{I}}_{\Theta;D}\colon\lambda(D)\longrightarrow\lambda(D_\Theta),\quad \hat{\mathscr{I}}_{\Theta;D}(\sigma)=\Psi_t(\sigma\otimes\Omega_N\otimes 1^*), \tag{3.2}$$

where Ω_N is the standard volume tensor on R^N, i.e.

$$\Omega_N=e_1\wedge\ldots\wedge e_N$$

if $e_1, \dots, e_N$ is the standard basis for R^N. By the continuity requirement on the family of isomorphisms Ψ_t and the previous paragraph, the isomorphisms $\hat{\mathscr{I}}_{\Theta;D}$ topologize $\det_{X,Y}$ over the open subset

$$U_{X;\Theta} \equiv \{D \in \mathscr{F}(X, Y) : \mathfrak{c}(D_\Theta) = 0\}.$$

Since these open subsets cover $\mathscr{F}(X, Y)$ as Θ ranges over all homomorphisms $\mathsf{R}^N \to Y$ and N ranges over all nonnegative integers, the isomorphisms $\hat{\mathscr{I}}_{\Theta;D}$ completely specify the topology on $\det_{X,Y}$. However, the overlap map

$$\hat{\mathscr{I}}_{\Theta_2;D} \circ \hat{\mathscr{I}}^{-1}_{\Theta_1;D} \colon \iota^*_{\Theta_1} \det_{X\oplus\mathsf{R}^{N_1},Y} \longrightarrow \iota^*_{\Theta_2} \det_{X\oplus\mathsf{R}^{N_2},Y}$$

must be continuous over $U_{X;\Theta_1} \cap U_{X;\Theta_2}$ for any pair of homomorphisms $\Theta_1 \colon \mathsf{R}^{N_1} \to Y$ and $\Theta_2 \colon \mathsf{R}^{N_2} \to Y$. By Proposition 5.3, this is indeed the case for the isomorphisms Ψ_t given by (4.10); the main ingredient in the proof of this proposition is Proposition 4.8, confirming the first algebraic Compositions property on page 212. By Corollary 5.4, the family of these isomorphisms Ψ_t is continuous with respect to the resulting topology; the main ingredient in the proof of this corollary is Proposition 4.10, confirming the second algebraic Compositions property.

For each homomorphism $\delta : V \to W$ between finite-dimensional vector spaces, there is a natural isomorphism

$$\mathscr{I}_\delta \colon \lambda(\delta) \longrightarrow \lambda(\mathbf{0}) \equiv \lambda(V) \otimes \lambda^*(W). \tag{3.3}$$

As suggested in [15], a suitable collection of these isomorphisms is fundamental to constructing a system of determinant line bundles for Fredholm operators. Unfortunately, [15] makes no mention of what properties of a system of isomorphisms (3.3) are needed for such a construction and gives no explicit formula for these isomorphisms. The discussion in [15] is also limited to Cauchy-Riemann operators on Riemann surfaces. The convention (2.2), which is also used in [12, Section 20.2] and [14, Section 7.4], is compatible with the isomorphisms (3.3) given by

$$x \otimes y^* \longrightarrow (-1)^{(\mathfrak{d}(W)-\mathfrak{d}(\mathfrak{c}(\delta)))\mathfrak{d}(\mathfrak{c}(\delta))} x \wedge_V v \otimes (\lambda(\delta)v \wedge_W y)^*, \tag{3.4}$$
$$\forall\, x \in \lambda(\kappa(\delta)) - 0,\ y \in \lambda(\mathfrak{c}(\delta)) - 0,\ v \in \lambda\left(\frac{V}{\kappa(\delta)}\right) - 0,$$

where $\wedge_V$ and $\wedge_W$ are the isomorphisms of Lemma 4.1. This is precisely the isomorphism of [14, Lemma 7.4.7] and is used directly to topologize determinant line bundles in the proof of [14, Proposition 7.4.8].[1] While the properties

[1] As shown in the proof of Proposition 5.3 in this paper, the restriction to injective homomorphisms Θ in the proof of [14, Proposition 7.4.8] is unnecessary.

of (3.4) necessary for this construction are verified in [14], few of the important properties of the resulting determinant line bundles are checked in [14]. The isomorphism (3.4) appears only indirectly in the construction of this paper; see Remark 4.6.

There are alternative ways of constructing a system of determinant line bundles satisfying the properties in Section 2.

(1) A system of determinant line bundles for bounded complexes of vector bundles and isomorphisms for exact triples of such complexes is constructed in [11, Chapter I]. A system of determinant line bundles for Fredholm operators can then be obtained by associating each Fredholm operator with a two-term complex, deducing the Exact Squares property for Fredholm operators from that for bounded complexes and the two algebraic Compositions properties from the Exact Squares property, and deriving explicit formulas for all isomorphisms. This approach is described in detail in Section 3.2.

(2) One could explicitly specify a collection of isomorphisms $\hat{\mathscr{I}}_{\Theta;D}$ as in (3.2) that are compatible with compositions. This is essentially the approach taken in [13], [14], [16], and [17] to topologize determinant line bundles, without verifying the properties in Section 2. The isomorphisms (3.2) can be used to define Exact Triples isomorphisms (2.20) from the Normalization II property, imposing the commutativity property of Lemma 4.12 by definition, and to derive an explicit formula for these isomorphisms. The Exact Squares property for Fredholm operators can then be obtained from the basic Exact Squares property of Lemma 4.3 as in the proof of Corollary 4.13 and used to confirm the two algebraic Compositions properties.

(3) The commutativity property of Lemma 4.12 could be verified for the isomorphism (4.10) directly, without using Proposition 4.10, and used to obtain the Exact Squares property as in the proof of Corollary 4.13. The two algebraic Compositions properties could then be deduced either from the Exact Squares property or from the corresponding properties for vector spaces by an argument similar to the proof of Corollary 4.13. Unfortunately, the proof of the special case of Proposition 4.10 corresponding to Lemma 4.12 is as elaborate as the proof of Proposition 4.10 itself; the former involves a bit less notation, but exactly the same steps.

In all three approaches, the Dual Exact Triples property can be either checked directly or deduced from more general considerations. The above listed alternatives can be used to replace parts of Section 4 in this paper, but most of Section 5 would still be needed. It appears the overall approach of this paper is more efficient than the three alternatives described above.

The equivalence of the topologies arising from the algebraic approach of [11] and the analytic approach of [15] in many complex-geometric settings is shown in the trilogy [2], [3], [4]; see [2, Theorem 0.1] in particular. Combined with earlier work [7], [8], this trilogy leads to an arithmetic version of the Grothendick-Riemann-Roch Theorem; see [9]. A thorough discussion of the determinant line bundle in Akarelov geometry, which is outside of the scope of this paper, is contained in the books [6], [18].

3.2. Relation with Knudsen-Mumford

The existence of a determinant line bundle system satisfying the properties in Section 2 follows most readily (but still with some work) from the proof of [11, Theorem 1], which constructs determinant line bundles for bounded complexes of vector bundles. Unfortunately, a complete construction of a determinant line bundle based on [11] with a verification of all of the properties listed in Section 2 and with explicit formulas for the relevant isomorphisms does not seem to appear elsewhere; we describe it below.

For each homomorphism $\Theta\colon \mathsf{R}^N \to Y$,

$$\mathscr{K}_\Theta \equiv \big\{(D,x,u)\in U_{X;\Theta}\times X\oplus\mathsf{R}^N : (x,u)\in\kappa(D_\Theta)\big\} \longrightarrow U_{X;\Theta}$$

is a vector bundle. For each $D\in U_{X;\Theta}$, the commutative diagram (3.1) gives rise to an exact sequence

$$0\longrightarrow\kappa(D)\longrightarrow\kappa(D_\Theta)\xrightarrow{\delta_\Theta}\mathsf{R}^N\xrightarrow{\Theta}\mathfrak{c}(D)\longrightarrow 0. \tag{3.5}$$

Thus, each homomorphism $\Theta\colon\mathsf{R}^N\to Y$ determines a two-term graded complex

$$\cdots\longrightarrow 0\longrightarrow\mathscr{K}_\Theta\xrightarrow{\delta_\Theta}U_{X;\Theta}\times\mathsf{R}^N\longrightarrow 0\longrightarrow\cdots \tag{3.6}$$

of vector bundles over $U_{X;\Theta}$, with $\mathscr{K}_\Theta$ placed at the 0-th and 1-st positions, and a Z_2-graded line bundle

$$\mathscr{L}_\Theta\equiv\lambda(\mathscr{K}_\Theta)\otimes\lambda^*(U_{X;\Theta}\times\mathsf{R}^N),$$

the determinant line bundle of the two-term complex (3.6).

For each $D\in U_{X;\Theta}$, let $\Xi\colon\mathfrak{c}(D)\to\mathsf{R}^N$ be a right inverse for the surjective map

$$\mathsf{R}^N\longrightarrow\mathfrak{c}(D),\quad u\longrightarrow\Theta(u)+\operatorname{Im}D. \tag{3.7}$$

The diagram

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & \kappa(D) & \xrightarrow{\;0\;} & \mathfrak{c}(D) & \longrightarrow & 0 & \longrightarrow \cdots \\
& \big\downarrow & & \big\downarrow \mathrm{i}_{D;X} & & \big\downarrow \Xi & & \big\downarrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & \mathscr{K}_\Theta|_D & \xrightarrow{\delta_\Theta} & \{D\}\times \mathsf{R}^N & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

$$\mathrm{i}_{D;X}(x) = (D, x, 0)$$

is then a *quasi-isomorphism of graded complexes* over $\{D\}$, i.e. a homomorphism of graded complexes of vector bundles that induces an isomorphism in homology. By [11, Theorem 1], there is then a canonical isomorphism

$$\hat{\mathscr{I}}'_{\Theta;D} : \lambda(D) \longrightarrow \mathscr{L}_\Theta|_D \approx \lambda(D_\Theta).$$

Since any other right inverse for the homomorphism (3.7) is of the form $\Xi + \delta_\Theta \tilde{\Xi}$ for some homomorphism $\tilde{\Xi}: \mathfrak{c}(D) \to \kappa(D_\Theta)$, $\hat{\mathscr{I}}'_{\Theta;D}$ is independent of the choice of Ξ by [11, Proposition 2]. If $\Theta': \mathsf{R}^{N'} \to Y$ is another homomorphism and $\iota: \mathsf{R}^N \to \mathsf{R}^{N'}$ is a homomorphism such that $\Theta = \Theta' \circ \iota$,

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & \mathscr{K}_\Theta & \xrightarrow{\delta_\Theta} & U_{X;\Theta}\times \mathsf{R}^N & \longrightarrow & 0 & \longrightarrow \cdots \\
& \big\downarrow & & \big\downarrow \mathrm{id}\times\mathrm{id}\times\iota & & \big\downarrow \mathrm{id}\times\iota & & \big\downarrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & \mathscr{K}_{\Theta'} & \xrightarrow{\delta_{\Theta'}} & U_{X;\Theta}\times \mathsf{R}^N & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

is also a quasi-isomorphism of graded complexes. By the proof of [11, Theorem 1], it also induces a canonical isomorphism

$$\mathscr{I}_{\Theta',\Theta} : \mathscr{L}_\Theta \longrightarrow \mathscr{L}_{\Theta'}$$

of line bundles over $U_{X;\Theta}$. By the functoriality of the determinant construction of [11, Theorem 1],

$$\hat{\mathscr{I}}'_{\Theta';D} = \mathscr{I}_{\Theta',\Theta} \circ \hat{\mathscr{I}}'_{\Theta;D} : \lambda(D) \longrightarrow \mathscr{L}_{\Theta'}|_D \approx \lambda(D_{\Theta'}).$$

Since the line bundle maps $\mathscr{I}_{\Theta',\Theta}$ are continuous, the isomorphisms $\hat{\mathscr{I}}'_{\Theta;D}$ topologize $\det_{X,Y}$ over $U_{X;\Theta}$ and endow $\det_{X,Y}$ with a well-defined topology of a line bundle over $\mathscr{F}(X,Y)$, which satisfies the Normalization I and Naturality I properties.

The proof of [11, Theorem 1] produces analogues of the isomorphisms (2.20) for exact triples of graded complexes (3.6) of vector bundles, with isomorphisms of exact triples of Fredholm operators replaced by quasi-isomorphisms of exact triples of bounded complexes. These isomorphisms satisfy analogues of the Normalization II, Naturality II, III, and the Exact Squares properties. By the proof of Corollary 5.4, an exact triple of Fredholm operators gives rise to an exact triple of two-term complexes (over a point). By the analogue of the Naturality II property for two-term complexes, the isomorphisms of [11, Theorem 1] then induce via the isomorphisms $\hat{\mathscr{I}}'_{\Theta;D}$ isomorphisms $\Psi_{\mathfrak{t}}$ for exact triples of Fredholm operators which satisfy the Normalization II and Naturality II, III properties. These isomorphisms depend continuously on $\mathfrak{t}$ by the proofs of Lemma 5.1 and Corollary 5.4. By the proof of Corollary 4.13, an exact square of Fredholm operators as in (2.26) gives rise to an exact square of two-term complexes. By the analogue of the Exact Squares property for two-term complexes and the proof of Corollary 4.13, the induced isomorphisms for exact triples of Fredholm operators satisfy the Exact Squares property for Fredholm operators. The proof of [11, Theorem 1] implies the existence of the bundle maps $\widetilde{\mathscr{D}}_D$ as in (2.34) satisfying the analogue of the Dual Exact Triples property on page 217 for two-term complexes. These bundle maps $\widetilde{\mathscr{D}}_D$ satisfy the analogue of the Normalization III* property on page 218 in the case of the system explicitly constructed in the proof of [11, Theorem 1]; this can be seen from the last paragraph of this section and Section 3.4.

We now show that the two algebraic Compositions properties on page 212 follow from the Exact Squares and Naturality II, III properties, thus fully establishing that [11, Theorem 1] gives rise to a determinant line bundle system satisfying all properties in Section 2. Applying the Exact Square and Naturality III properties to the top and center diagrams in Figure 2 and using the identification

$$\lambda(D_2^\star \circ D_1^\star) \longrightarrow \lambda(D_2^\star \circ D_1^\star) \otimes \lambda(\mathrm{id}_{X_2^\star}), \quad \sigma \longrightarrow \sigma \otimes 1 \otimes 1^*, \tag{3.8}$$

with $\star = {}', {}''$ or blank, we obtain the two commutative squares in the last diagram in Figure 2. The two round arrows are the vertical arrows in (2.23); the two half-disk diagrams commute by the definition of $\widetilde{\mathscr{C}}_{D_1^\star, D_2^\star}$. Thus, the diagram (2.23), which consists of the outermost arrows in the last diagram in Figure 2, commutes.

The derivation of the first algebraic Compositions property is more involved. Applying the Exact Squares property to the top diagram in Figure 3, where the left column, the bottom row, and the center row are the exact triples (2.19) corresponding to the compositions $D_2 \circ D_1$, $D_3 \circ D_2$, and $D_3 \circ (D_2 \circ D_1)$, with the last one augmented by id_{X_2},

$$\mathfrak{i}_X(x_1) = (x_1, D_1x_1, D_2D_1x_1), \quad \mathfrak{j}_Y(x_1, x_2, x_3) = (D_1x_1 - x_2, x_3 - D_2x_2),$$
$$\mathfrak{i}_Y(x_2) = (D_3D_2x_2, x_2, D_2x_2), \quad \mathfrak{j}_Y(x_4, x_2, x_3) = (x_4 - D_3D_2x_2, x_3 - D_2x_2),$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_1' & \longrightarrow & D_1 & \longrightarrow & D_1'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2' \circ D_1' \oplus \mathrm{id}_{X_2'} & \longrightarrow & D_2 \circ D_1 \oplus \mathrm{id}_{X_2} & \longrightarrow & D_2'' \circ D_1'' \oplus \mathrm{id}_{X_2''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2' & \longrightarrow & D_2 & \longrightarrow & D_2'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2' \circ D_1' & \longrightarrow & D_2 \circ D_1 & \longrightarrow & D_2'' \circ D_1'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2' \circ D_1' \oplus \mathrm{id}_{X_2'} & \longrightarrow & D_2 \circ D_1 \oplus \mathrm{id}_{X_2} & \longrightarrow & D_2'' \circ D_1'' \oplus \mathrm{id}_{X_2''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{id}_{X_2'} & \longrightarrow & \mathrm{id}_{X_2} & \longrightarrow & \mathrm{id}_{X_2''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccc}
\lambda(D_1') \otimes \lambda(D_1'') \otimes \lambda(D_2') \otimes \lambda(D_2'') & \xrightarrow{\Psi_{\mathfrak{t}_1} \otimes \Psi_{\mathfrak{t}_2}} & \lambda(D_1) \otimes \lambda(D_2) \\
\downarrow & & \downarrow \\
\lambda(D_2' \circ D_1' \oplus \mathrm{id}_{X_2'}) \otimes \lambda(D_2'' \circ D_1'' \oplus \mathrm{id}_{X_2''}) & \longrightarrow & \lambda(D_2 \circ D_1 \oplus \mathrm{id}_{X_2}) \quad \widetilde{\mathscr{C}}_{D_1,D_2} \\
\uparrow & & \uparrow \\
\lambda(D_2' \circ D_1') \otimes \lambda(D_2'' \circ D_1'') & \xrightarrow{\Psi_{\mathscr{C}_{\mathscr{T}}(\mathfrak{t}_1,\mathfrak{t}_2)}} & \lambda(D_2 \circ D_1)
\end{array}
$$

FIGURE 2. Derivation of the second algebraic Compositions property from the Exact Squares and Naturality III properties

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_1 & \longrightarrow & D_1 & \longrightarrow & \mathbf{0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \scriptstyle{\mathfrak{i}} & & \downarrow & & \\
0 & \longrightarrow & D_2' \circ D_1' \oplus \mathrm{id}_{X_2'} & \longrightarrow & D_3 \circ D_2 \circ D_1 \oplus \mathrm{id}_{X_2} \oplus \mathrm{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \scriptstyle{\mathfrak{j}} & & \downarrow & & \\
0 & \longrightarrow & D_2' & \longrightarrow & D_3 \circ D_2 \oplus \mathrm{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2 \circ D_1 & \longrightarrow & D_3 \circ D_2 \circ D_1 \oplus \mathrm{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \scriptstyle{\mathfrak{i}} & & \downarrow & & \\
0 & \longrightarrow & D_2 \circ D_1 \oplus \mathrm{id}_{X_2} & \longrightarrow & D_3 \circ D_2 \circ D_1 \oplus \mathrm{id}_{X_2} \oplus \mathrm{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \scriptstyle{\mathfrak{j}} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{id}_{X_2} & \longrightarrow & \mathrm{id}_{X_2} & \longrightarrow & \mathbf{0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_3 \circ D_2 \circ D_1 & \longrightarrow & D_3 \circ D_2 \circ D_1 \oplus \mathrm{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \scriptstyle{\mathfrak{i}} & & \downarrow & & \\
0 & \longrightarrow & D_3 \circ D_2 \circ D_1 \oplus \mathrm{id}_{X_2} & \longrightarrow & D_3 \circ D_2 \circ D_1 \oplus \mathrm{id}_{X_2} \oplus \mathrm{id}_{X_3} & \longrightarrow & D_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \scriptstyle{\mathfrak{j}} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{id}_{X_2} & \longrightarrow & \mathrm{id}_{X_2} & \longrightarrow & \mathbf{0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

FIGURE 3. Exact squares of Fredholm operators used in the derivation of the first algebraic Compositions property

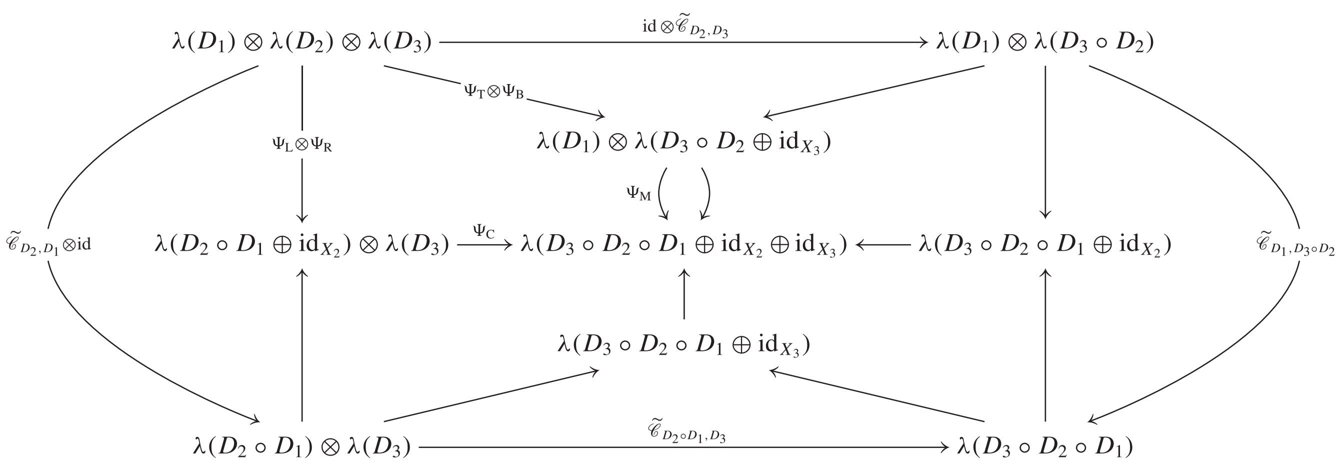


FIGURE 4. Commutative diagram used in the derivation of the first algebraic Compositions property from the Exact Squares and Naturality III properties

and using identifications similar to (3.8), we find that the top left quadrilateral in Figure 4 commutes. The commuting bottom left quadrilateral in Figure 4 is obtained by applying the Exact Squares property to the second diagram in Figure 3; a similar exact square gives the commuting top right quadrilateral. The bottom right quadrilateral arises from the last diagram in Figure 3. The two arrows that run between the same objects in the middle of Figure 4 are related by the isomorphism of exact triples of Fredholm operators,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_1 & \longrightarrow & D_3\circ D_2\circ D_1\oplus \mathrm{id}_{X_2}\oplus \mathrm{id}_{X_3} & \longrightarrow & D_3\circ D_2\oplus \mathrm{id}_{X_3} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle D'} & & \Big\downarrow{\scriptstyle (\phi,\psi)} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & D_1 & \xrightarrow{\mathfrak{i}} & D_3\circ D_2\circ D_1\oplus \mathrm{id}_{X_2}\oplus \mathrm{id}_{X_3} & \xrightarrow{\mathfrak{j}} & D_3\circ D_2\oplus \mathrm{id}_{X_3} & \longrightarrow & 0, \end{array}$$

where the top row of the exact row is the exact triple (2.19) corresponding to the composition $(D_3 \circ D_2) \circ D_1$ augmented by id_{X_3},

$$\phi(x_1, x_2, x_3) = (x_1, x_2, x_3 + D_2x_2), \quad \psi(x_4, x_2, x_3) = (x_4, x_2, x_3 + D_2x_2).$$

Since $\widetilde{\mathscr{I}}_{\phi,\psi} = \mathrm{id}$, these two arrows are in fact the same by the Naturality II property. The two half-disk and two triangular diagrams in Figure 4 commute by the definition of $\widetilde{\mathscr{C}}$. Thus, the diagram (2.22), which consists of the outermost arrows of the diagram in Figure 4, commutes.

The determinant for a complex of vector bundles in [11, p. 31] corresponds to reversing the two factors in (2.2). The isomorphism (3.4) should then be replaced by

$$\begin{aligned} \lambda^*(\mathfrak{c}(\delta)) \otimes \lambda(\kappa(\delta)) &\longrightarrow \lambda^*(W) \otimes \lambda(V), \\ y^* \otimes x &\longrightarrow (-1)^{(\mathfrak{d}(V)-\mathfrak{d}(\kappa(\delta)))\mathfrak{d}(\kappa(\delta))}(\lambda(\delta)v \wedge_W y)^* \otimes x \wedge_V v, \end{aligned} \tag{3.9}$$

with x, y, v as before. This isomorphism differs from the isomorphism (3.4) conjugated by the isomorphisms (2.1) by (-1) to the power of $\mathfrak{d}(\mathrm{Im}\,\delta)$, which equals $N - \mathfrak{d}(\mathfrak{c}(D))$ in the case of (3.5). The dependence on $\mathfrak{d}(\mathfrak{c}(D))$ drops out when taking the overlap maps for the trivializations of the new version of the determinant line bundle, and so the isomorphisms (3.9) still give rise to a well-defined topology on this bundle. The two versions of the determinant line bundle are isomorphic by the maps (2.1) composed with $(-1)^{\mathfrak{c}(D)}$ in the fiber over $D \in \mathscr{F}(X, Y)$; neither of these two maps is continuous, but the composite is continuous. The isomorphism $\Psi_{\mathfrak{t}}$ for exact triples of Fredholm operators described by (4.10) for the topology on $\det_{X,Y}$ specified by (3.4) should then be conjugated by the above isomorphism between the two versions of the determinant line bundle. In particular, this changes the sign exponent in

(4.12) to $(\operatorname{ind} D')\mathfrak{d}(\mathfrak{c}(D''))$, in addition to interchanging the kernel and cokernel factors.

3.3. Other conventions

In [5, Section 3.1], $\lambda(D)$ is defined as the tensor product of $\lambda(\kappa(D))$ and $\lambda(\mathfrak{c}(D)^*)$. In [13, Appendix A.2], $\lambda(D)$ is defined as the tensor product of $\lambda(\kappa(D))$ and $\lambda(\kappa(D^*))$. In light of the second isomorphism in (2.30), these two conventions are essentially identical. They implicitly identify $\lambda^*(\mathfrak{c}(D))$ with $\lambda(\mathfrak{c}(D)^*)$. Such an identification is determined by a pairing of $\lambda(V^*)$ with $\lambda(V)$ for a finite-dimensional vector space V. There are two such standard pairings:

$$\begin{aligned}\alpha_1\wedge\ldots\wedge\alpha_n\otimes v_1\wedge\ldots\wedge v_n &\longrightarrow \det(\alpha_i(v_j))_{i,j=1,\ldots,n} \qquad \text{and}\\ &\longrightarrow (-1)^{\binom{n}{2}}\det(\alpha_i(v_j))_{i,j=1,\ldots,n}.\end{aligned} \tag{3.10}$$

Along with (3.4), these two pairings topologize the new version of the determinant line bundle in two different ways; the resulting line bundles are isomorphic by the multiplication by (-1) to the power of $\binom{\mathfrak{d}(\mathfrak{c}(D))}{2}$ in the fiber over $D\in\mathscr{F}(X,Y)$. Under the second pairing in (3.10), the isomorphism (3.4) precisely corresponds to the isomorphism [5, (3.1)]. On the other hand, the analogue of (3.4) used in the proof of [13, Theorem A.2.2] corresponds under the first pairing in (3.10) to (3.4) without the sign; see [13, Exercise A.2.3]. In the case of (3.5), the exponent of this sign is $(N-\mathfrak{c}(\mathfrak{d}))\mathfrak{c}(D)$, which changes the overlap maps between the trivializations of the determinant line bundle by (-1) to the power of $(N'-N)\mathfrak{d}(\mathfrak{c}(D))$. The overlap maps in the proof of [13, Theorem A.2.2] thus need not be continuous if $N-N'$ is odd and so do not topologize the determinant line bundles.

In [17, Section (11a)], $\lambda(D)$ is defined as the tensor product of $\lambda(\mathfrak{c}(D)^*)$ and $\lambda(\kappa(D))$. In [16, Section 1.2], $\lambda(D)$ is defined as the tensor product of $\lambda(\kappa(D^*))$ and $\lambda(\kappa(D))$. In light of the second isomorphism in (2.30), these conventions are essentially identical. Under the second pairing in (3.10), the isomorphism (3.9) becomes [17, (11.3)]. Under the same pairing, the isomorphism (3.9) corresponds to the isomorphism of [16, Theorem 2.1] multiplied by (-1) to the power of

$$\big(\mathfrak{d}(W)-\mathfrak{d}(\mathfrak{c}(\delta))\big)(\operatorname{ind}\delta)+\mathfrak{d}(\kappa(\delta))\mathfrak{d}(\mathfrak{c}(\delta))\cong\mathfrak{d}(W)(\operatorname{ind}\delta)+\mathfrak{c}(\delta)\mod 2.$$

In the case of (3.5), the sign exponent reduces to $N(\operatorname{ind} D)+\mathfrak{d}(\mathfrak{c}(D))$. The dependence on $\mathfrak{d}(\mathfrak{c}(D))$ drops out when taking the overlap maps for the trivializations of this version of the determinant line bundle, and so the isomorphism of [16, Theorem 2.1] gives rise to a well-defined topology on this bundle. It is isomorphic to the determinant line bundle of [17, Section (11a)] by the multiplication by $(-1)^{\mathfrak{c}(D)}$ in the fiber over $D \in \mathscr{F}(X,Y)$. The interchange of factors in $\lambda(D)$ accounts for the change of the sign exponent in the direct sum formulas, [16, (3)] and [17, (11.2)], from (4.12), as explained at the end of the last paragraph in Section 3.2.

In [15, Section 1] and [10, Appendix D.2], $\lambda(D)$ is defined to be either

$$\lambda(\kappa(D)^*) \otimes \lambda(\mathfrak{c}(D)) \quad \text{or} \quad \lambda^*(\kappa(D)) \otimes \lambda(\mathfrak{c}(D));$$

the notation is somewhat ambiguous, but looks more like the former; the latter is used in [1, Section (f)]. The usage in [15] is more consistent with the latter convention; the usage in [10] is sometimes more consistent with the latter and sometimes more consistent with the former.[2] While $\lambda(\kappa(D)^*)$ and $\lambda^*(\kappa(D))$ are canonically isomorphic, there are at least two choices of such canonical isomorphisms, the two provided by the pairings (3.10). The "construction" of the determinant line bundle in [15] consists of mentioning that each homomorphism $\delta\colon V \to W$ between finite-dimensional vector spaces gives rise to a natural isomorphism

$$\begin{aligned} \lambda(\kappa(D)^*) \otimes \lambda(\mathfrak{c}(D)) &\longrightarrow \lambda(V^*) \otimes \lambda(W) \\ \text{or} \quad \lambda^*(\kappa(D)) \otimes \lambda(\mathfrak{c}(D)) &\longrightarrow \lambda^*(V) \otimes \lambda(W), \end{aligned}$$

but no indication is given what it is. In the proof of [10, Proposition D.2.2], this isomorphism is described as a composition of other isomorphisms, but some of them are not specified.[3] The construction in [10, Appendix D.2] is fundamentally based on [10, Proposition D.2.6], though its proof appears to be incomplete; see Remark 4.9 for details. However, the statement of this proposition is the basis for the construction of the determinant line bundle in this paper and a close cousin of this proposition, Proposition 4.10, is used to verify the continuity of the bundle map (2.20) for families of exact triples of Fredholm operators. The construction in [1] is limited to Hilbert spaces and still omits some details. Neither [1, Section (f)], [10, Appendix D], nor [15] confirms most of the properties of the determinant line bundle stated in Section 2.

As noted in [15, Section 2], the section of $\det_{X,Y}$ in the definitions of [15,

[2] For example, the last equality in the last displayed expression in the proof of [10, Proposition D.2.2] uses the latter definition, while [10, (D.2.9)] uses the former.

[3] In addition, $(\det F)^{-1}$ should be $\det F$ at the end of the statement of this proposition and $\det H_2$ should be $(\det H_2)^*$ in the second-to-last displayed equation in the proof; the first change is necessary for the section (3.11) to be continuous in the finite-dimensional case.

Section 1] and [10, Appendix D.2] given by

$$\sigma(D) = \begin{cases} 1^* \otimes 1, & \text{if } D \text{ is isomorphism;} \\ 0, & \text{otherwise;} \end{cases} \tag{3.11}$$

is continuous; there is no such section if $\det_{X,Y}$ is defined as in (2.2), [5], [14], [16], or [17]. The definition of $\det_{X,Y}$ in [15, Section 1] and [10, Appendix D.2] thus comes with a natural normalization for the topology, but it does not restrict the topology of $\det_{X,Y}$ any further than the properties in Section 2; see Section 3.4. The alternative definitions seem more natural from the geometric viewpoint, as typically the spaces $\kappa(D)$ describe tangent spaces of some, ideally smooth, moduli spaces, and so it seems desirable not to dualize them. The alternative definitions also lead to a somewhat nicer appearance of formulas describing key properties of the determinant line bundle system; for example, [10, Proposition D.2.2] reverses the order of the factors in the isomorphism of Lemma 4.1.

3.4. Classification of determinant line bundles

There are infinitely many systems of determinant line bundles that satisfy all properties in Section 2. Theorem 2, stated and proved in this section, describes all of them.

For each exact triple $\mathfrak{t}$ of Fredholm operators, we denote by $\Psi_{\mathfrak{t}}$ the isomorphism (4.10). Suppose $\{\Psi'_{\mathfrak{t}}\}$ is another collection of isomorphisms for exact triples of Fredholm operators satisfying all properties in Section 2.

Let $\mathfrak{t}$ be an exact triple as in (2.11) and

$$\Theta'\colon \mathsf{R}^{N'} \longrightarrow Y' \quad \text{and} \quad \tilde\Theta''\colon \mathsf{R}^{N''} \longrightarrow Y$$

be homomorphisms such that $D' \in U_{X';\Theta'}$ and $D'' \in U_{X'';\jmath_Y\circ\tilde\Theta''}$. Let $N = N' + N''$, $\imath\colon \mathsf{R}^{N'} \to \mathsf{R}^N$ be the inclusion as $\mathsf{R}^{N'} \times 0^{N''}$, and $\jmath\colon \mathsf{R}^{N'} \to \mathsf{R}^{N''}$ be the projection onto the last N'' coordinates. We define

$$\begin{aligned} &\Theta\colon \mathsf{R}^N \to X, \quad \Theta(x', x'') = \imath_Y(\Theta'(x')) + \tilde\Theta''(x'') \quad \forall\, (x', x'') \in \mathsf{R}^{N'} \oplus \mathsf{R}^{N''}, \\ &\Theta''\colon \mathsf{R}^{N''} \to X'', \quad \Theta''(x'') = \jmath_Y(\tilde\Theta''(x'')) \quad \forall\, x'' \in \mathsf{R}^{N''}. \end{aligned}$$

Thus, the first diagram in Figure 5, where the right column is the exact triple

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{R}^{N'} & \xrightarrow{\imath} & \mathsf{R}^N & \xrightarrow{\jmath} & \mathsf{R}^{N''} & \longrightarrow & 0 \\ & & \big\downarrow j_{N'} & & \big\downarrow j_N & & \big\downarrow j_{N''} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

is an exact square of Fredholm operators. By the Normalization II and Exact Squares properties, the collection $\{\Psi'_{\mathfrak{t}}\}$ is thus determined by the isomorphisms

$$\hat{\mathscr{I}}'_{\Theta;D}\colon \lambda(D) \longrightarrow \lambda(D_\Theta), \quad \hat{\mathscr{I}}'_{\Theta;D}(\sigma) = \Psi'_{\mathfrak{t}}(\sigma \otimes \Omega_N \otimes 1^*),$$

corresponding to the exact triples (3.1) with $D \in U_{X;\Theta}$.

Given $D \in \mathscr{F}(X, Y)$, let $\dot{X} \subset X$ be a linear subspace such that the operator

$$\dot{D}\colon \dot{X} \longrightarrow \operatorname{Im} D, \quad x \longrightarrow Dx,$$

is an isomorphism and $\Theta_D\colon \mathsf{R}^{N_D} \to Y$ be a homomorphism inducing an isomorphism to $\mathfrak{c}(D)$ when composed with the projection $Y \to \mathfrak{c}(D)$. There is an exact square of Fredholm operators as in the second diagram in Figure 5, where the right column is the exact triple

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \kappa(D) & \longrightarrow & \kappa(D) \oplus \mathsf{R}^{N_D} & \longrightarrow & \mathsf{R}^{N_D} & \longrightarrow & 0 \\ & & \downarrow \mathbf{0} & & \downarrow \mathbf{0}_{\Theta_D} & & \downarrow j_{N_D} & & \\ 0 & \longrightarrow & \mathfrak{c}(D) & \longrightarrow & \mathfrak{c}(D) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

By the Naturality II, III and Exact Squares properties, the collection $\{\Psi'_{\mathfrak{t}}\}$ is thus determined by the isomorphisms $\Psi'_{i,c}$ corresponding to the exact triples

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{R}^{i+c} & \longrightarrow & \mathsf{R}^{i+2c} & \longrightarrow & \mathsf{R}^{c} & \longrightarrow & 0 \\ & & \downarrow \mathbf{0} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathsf{R}^{c} & \longrightarrow & \mathsf{R}^{c} & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{3.12}$$

where the middle arrow is the projection onto the last c coordinates.

Let $A_{i,c} \in \mathsf{R}^*$ be such that

$$\Psi'_{i,c} = A_{i,c}\Psi_{i,c}, \tag{3.13}$$

where $\Psi_{i,c}$ is the isomorphism (4.10) for the exact triple (3.12). In particular,

$$\Psi'_{i,c}(\Omega_{i+c} \otimes \Omega_c^* \otimes \Omega_c \otimes 1) = (-1)^c A_{i,c}\, \Omega_{i+c} \otimes 1. \tag{3.14}$$

For each homomorphism $\Theta\colon \mathsf{R}^N \to Y$ and $D \in U_{X;D}$, there is an exact square of Fredholm operators as in the last diagram in Figure 5. By the Naturality III, Normalization II, and Exact Squares properties and (3.13),

$$\hat{\mathscr{I}}'_{\Theta;D} = A_{\operatorname{ind} D, \mathfrak{d}(\mathfrak{c}(D))}\hat{\mathscr{I}}_{\Theta;D}, \qquad A_{i,c} \in \mathsf{R}^+,\ i \in \mathsf{Z},\ c \in \mathsf{Z}^{\geq 0},\ c \geq -i,\ A_{i,0} = 1\ \forall\, i. \tag{3.15}$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D' & \longrightarrow & D'_{\Theta'} & \longrightarrow & j_{N'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & D_{\Theta} & \longrightarrow & j_{N} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'' & \longrightarrow & D''_{\Theta''} & \longrightarrow & j_{N''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dot{D} & \longrightarrow & D & \longrightarrow & \mathbf{0}_{\kappa(D),\mathfrak{c}(D)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dot{D} & \longrightarrow & D_{\Theta_D} & \longrightarrow & (\mathbf{0}_{\kappa(D),\mathfrak{c}(D)})_{\Theta_D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{0} & \longrightarrow & j_{N_D} & \longrightarrow & j_{N_D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & D_{\Theta_D} & \longrightarrow & j_{N_D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & D_{\Theta} & \longrightarrow & j_{N} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{0} & \longrightarrow & j_{N-N_D} & \longrightarrow & j_{N-N_D} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

FIGURE 5. Exact squares of Fredholm operators specifying a determinant line bundle system

The overlap maps between these isomorphisms are the same as before and in particular are continuous. The isomorphisms (3.15) are compatible with the isomorphisms

$$\mathscr{I}'_\delta = \frac{A_{\mathfrak{d}(V)-\mathfrak{d}(W),\mathfrak{d}(\mathfrak{c}(\delta))}}{A_{\mathfrak{d}(V)-\mathfrak{d}(W),\mathfrak{d}(W)}}\mathscr{I}_\delta\colon \lambda(\delta) \longrightarrow \lambda(\mathbf{0}), \tag{3.16}$$

whenever $\delta\colon V \to W$ is a homomorphism between finite-dimensional vector spaces. The isomorphisms

$$\mathscr{I}_D\colon \lambda(D) \longrightarrow \lambda(D), \quad \sigma \longrightarrow A^{-1}_{\mathrm{ind}\,D,\mathfrak{d}(\mathfrak{c}(D))}\sigma,$$

give rise to continuous isomorphisms between the determinant line bundles in the original and new topologies. The suitable exact triples and dualization isomorphisms are given by

$$\begin{aligned} \Psi'_{\mathfrak{t}} &= \mathscr{I}_D \circ \Psi_{\mathfrak{t}} \circ \mathscr{I}^{-1}_{D'} \otimes \mathscr{I}^{-1}_{D''} = \frac{A_{\mathrm{ind}\,D',\mathfrak{d}(\mathfrak{c}(D'))}A_{\mathrm{ind}\,D'',\mathfrak{d}(\mathfrak{c}(D''))}}{A_{\mathrm{ind}\,D,\mathfrak{d}(\mathfrak{c}(D))}}\Psi_{\mathfrak{t}}, \\ \widetilde{\mathscr{D}}'_D &= A^{\mathrm{ind}\,D}_{-1,1}\,\mathscr{I}_{D^*} \circ \widetilde{\mathscr{D}}_D \circ \mathscr{I}^{-1}_D = A^{\mathrm{ind}\,D}_{-1,1}\,\frac{A_{\mathrm{ind}\,D,\mathfrak{d}(\mathfrak{c}(D))}}{A_{-\,\mathrm{ind}\,D,\mathfrak{d}(\kappa(D))}}\widetilde{\mathscr{D}}_D, \end{aligned} \tag{3.17}$$

if $\mathfrak{t}$ is as in (2.11). The extra factors of $A_{-1,1}$ in the second equation above are needed to achieve the Normalization III property on page 217, while preserving the Dual Exact Triples property. In the case of the exact triple (3.1), $\Psi'_{\mathfrak{t}} = \hat{\mathscr{I}}_{\Theta;D}$, as the case should be. The new determinant line bundle system also satisfies the Normalization III* property if and only if $A_{-k,k} = A^k_{-1,1}$ for every $k \in \mathsf{Z}^+$.

The above argument also implies that the Normalization III and Dual Exact Triples properties on page 212 determine the dualization isomorphisms $\widetilde{\mathscr{D}}_D$ completely. Putting everything together, we obtain a complete description of systems of determinant line bundles.

THEOREM 2. *The map specified by* (3.14) *sends each system of determinant line bundles satisfying the properties in Section 2, other than Normalization III***, to the functions*

$$\{(i,c) : i \in \mathsf{Z},\ c \in \mathsf{Z}^+,\ c \geq -i\} \longrightarrow \mathsf{R}^+, \quad (i,c) \longrightarrow A_{i,c},$$

and is a bijection with the set of all such functions. The determinant line bundle systems that also satisfy the Normalization III* *property correspond to the subset of the above functions satisfying* $A_{-k,k} = A^k_{-1,1}$ *for all* $k \in \mathsf{Z}^+$. *In particular, the compatible systems of topologies on determinant line bundles are in one-to-one correspondence with the admissible systems of isomorphisms*

$\mathscr{I}_\delta$ *as in* (3.3)*,* (3.4)*, and* (3.16) *and with admissible systems of isomorphisms* $\Psi_{\mathfrak{t}}$ *as in* (2.20)*.*

By Theorem 2 and the preceding discussion, the section S of $\det^*_{X,Y}$ given by

$$S_D(\sigma) = \begin{cases} c, & \text{if } \sigma = c\,1 \otimes 1^*; \\ 0, & \text{if } D \text{ is not an isomorphism;} \end{cases}$$

is continuous. This is the analogue of the section (3.11) for the convention (2.2).

REMARK 3.1. According to [17, Remark 11.1], there are two possible sign conventions for the determinant line bundle and the sign convention in [17, Section (11a)] is the same as in [11]. As described above, the setup in [17, Section (11a)] corresponds to the setup in [11, Chapter I] via the second pairing in (3.10). The alternatives for [17, (11.2)] and [17, (11.3)] specified in [17, Remark 11.1] for the "other" sign convention do not satisfy the key commutativity requirement on the preceding page in [17]. In order for this requirement to be satisfied, the sign in [17, (11.2)] must be kept precisely the same (contrary to what is explicitly stated in [17, Remark 11.1]); this convention would then correspond to the setup in [11, Chapter I] via the first pairing in (3.10). Furthermore, by Theorem 2, there are infinitely many possible sign conventions, at least several of which seem quite natural. The isomorphisms (3.15) satisfy the two requirements above the diagram on page 150 in [17] provided $A_{0,1} > 0$. These systems of isomorphisms can be narrowed down by replacing the Normalization III property on page 217 with the Normalization III* property ($A_{-k,k} = A^k_{-1,1}$ for all $k \in \mathbb{Z}^+$), by specifying the dualization or direct sum isomorphisms, i.e.

$$A_{-i,i+c} = A^i_{-1,1} A_{i,c} \;\text{ or }\; A_{i,c} = A^c_{0,1} \quad \forall\, i \in \mathbb{Z},\; c \in \mathbb{Z}^{\ge 0},\; c \ge -i,$$

and/or by requiring the isomorphisms $\mathscr{I}_\delta$ to be given by

$$\mathscr{I}_\delta \colon \lambda(\delta) \longrightarrow \lambda(\mathbf{0}), \quad 1 \otimes 1^* \longrightarrow (\det \delta)^{-1}\, v \otimes v^*,$$

whenever $\delta \colon V \to V$ is an isomorphism and $v \in \lambda(V) - 0$ ($A_{0,c} = 1$ for all $c \in \mathbb{Z}^+$). The strongest of these additional conditions, specifying the isomorphisms for direct sums of Fredholm operators, seems to be the least natural requirement to make.

4. Linear algebra

4.1. Finite-dimensional vector spaces

In this subsection, we make a number of purely algebraic observations concerning finite-dimensional vector spaces that lie behind the determinant line construction.

LEMMA 4.1 ([11, Proposition 1(i)]). *Every short exact sequence*

$$0 \longrightarrow V' \xrightarrow{\mathfrak{i}} V \xrightarrow{\mathfrak{j}} V'' \longrightarrow 0 \tag{4.1}$$

induces a natural isomorphism $\wedge_V : \lambda(V') \otimes \lambda(V'') \longrightarrow \lambda(V)$.

PROOF. If $v'_1, \ldots, v'_k$ is a basis for V' and $v_1, \ldots, v_\ell \in V$ are such that the set $\mathfrak{j}(v_1), \ldots, \mathfrak{j}(v_\ell)$ is a basis for V'', $v'_1 \wedge \ldots \wedge v'_k$ and $\mathfrak{j}(v_1) \wedge \ldots \wedge \mathfrak{j}(v_\ell)$ span $\lambda(V')$ and $\lambda(V'')$, respectively. By the exactness of (4.1), $\mathfrak{i}(v'_1), \ldots, \mathfrak{i}(v'_k), v_1, \ldots, v_\ell$ is basis for V and so the map

$$\wedge_V : v'_1 \wedge \ldots \wedge v'_k \otimes \mathfrak{j}(v_1) \wedge \ldots \wedge \mathfrak{j}(v_\ell) \longrightarrow \mathfrak{i}(v'_1) \wedge \ldots \wedge \mathfrak{i}(v'_k) \wedge v_1 \wedge \ldots \wedge v_\ell \tag{4.2}$$

induces an isomorphism $\lambda(V') \otimes \lambda(V'') \to \lambda(V)$. Since the exactness of (4.1), each $v_i \in V$ is determined by $\mathfrak{j}(v_i) \in V''$ up to a linear combination of $\mathfrak{i}(v'_1), \ldots, \mathfrak{i}(v'_k)$, the right-hand side of (4.2) is determined by $v'_1, \ldots, v'_k \in V'$ and $\mathfrak{j}(v_1), \ldots, \mathfrak{j}(v_\ell) \in V''$. Changing the collections $v'_1, \ldots, v'_k \in V'$ and $v_1, \ldots, v_\ell \in V$ by a $k \times k$-matrix A' and an $\ell \times \ell$-matrix A, respectively, changes the wedge products of the first k vectors and the last ℓ vectors by $\det A'$ and $\det A$, respectively, on both sides of (4.2). Thus, the isomorphism induced by (4.2) is independent of the choices of collections $v'_1, \ldots, v'_k \in V'$ and $v_1, \ldots, v_\ell \in V$ as above. It clearly commutes with isomorphisms of short exact sequences.

The next lemma follows immediately from the definitions of $\mathscr{P}$ in (2.31) and of $\wedge_V$ above.

LEMMA 4.2. *For every finite-dimensional vector space V,*

$$\mathscr{P}(v^*) = (\mathscr{P}v)^* \quad \forall\, v \in \lambda(V) - 0. \tag{4.3}$$

For every isomorphism $\delta : V \to W$ between finite-dimensional vector spaces,

$$\lambda(\delta^*)\mathscr{P}((\lambda(\delta)v)^*) = \mathscr{P}(v^*) \quad \forall\, v \in \lambda(V) - 0. \tag{4.4}$$

For every short exact sequence (4.1),

$$\mathscr{P}((\lambda(\mathfrak{i})v' \wedge_V v'')^*) = \mathscr{P}(v''^*) \wedge_{V^*} \lambda(\mathfrak{i}^*)^{-1}\mathscr{P}(v'^*) \qquad \forall\, v' \in \lambda(V') - 0,\ v'' \in \lambda(V/\mathfrak{i}(V')) - 0. \tag{4.5}$$

From (4.2), we immediately find that the isomorphisms $\wedge_V$ of Lemma 4.1 satisfy graded commutativity, as described by the next lemma. Corollary 4.4 below is a special case of this lemma (either $V_{\mathrm{TR}} = 0$ or $V_{\mathrm{BL}} = 0$).

LEMMA 4.3 ([11, Proposition 1(ii)]). *For every commutative diagram*

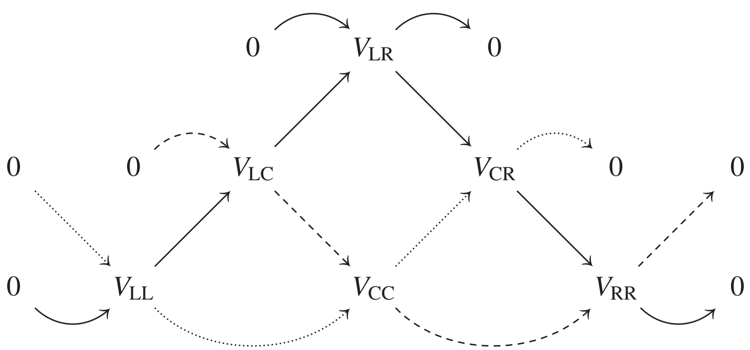
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_{\mathrm{TL}} & \longrightarrow & V_{\mathrm{TM}} & \longrightarrow & V_{\mathrm{TR}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_{\mathrm{CL}} & \longrightarrow & V_{\mathrm{CM}} & \longrightarrow & V_{\mathrm{CR}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_{\mathrm{BL}} & \longrightarrow & V_{\mathrm{BM}} & \longrightarrow & V_{\mathrm{BR}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

of exact rows and columns, the diagram

$$\begin{array}{ccc}
\lambda(V_{\mathrm{TL}}) \otimes \lambda(V_{\mathrm{BL}}) \otimes \lambda(V_{\mathrm{TR}}) \otimes \lambda(V_{\mathrm{BR}}) & \xrightarrow{\wedge_{V_{\mathrm{TM}}} \otimes \wedge_{V_{\mathrm{BM}}} \circ \mathrm{id} \otimes R \otimes \mathrm{id}} & \lambda(V_{\mathrm{TM}}) \otimes \lambda(V_{\mathrm{BM}}) \\
\Big\downarrow {\scriptstyle \wedge_{V_{\mathrm{CL}}} \otimes \wedge_{V_{\mathrm{CR}}}} & & \Big\downarrow {\scriptstyle \wedge_{V_{\mathrm{CM}}}} \\
\lambda(V_{\mathrm{CL}}) \otimes \lambda(V_{\mathrm{CR}}) & \xrightarrow{\qquad \wedge_{V_{\mathrm{CM}}} \qquad} & \lambda(V_{\mathrm{CM}})
\end{array}$$

commutes.

COROLLARY 4.4. *For every commutative diagram*



of 4 exact short sequences, the diagram

$$\begin{CD}
\lambda(V_{\mathrm{LL}})\otimes\lambda(V_{\mathrm{LR}})\otimes\lambda(V_{\mathrm{RR}}) @>{\wedge_{V_{\mathrm{LC}}}\otimes\mathrm{id}}>> \lambda(V_{\mathrm{LC}})\otimes\lambda(V_{\mathrm{RR}})\\
@V{\mathrm{id}\otimes\wedge_{V_{\mathrm{CR}}}}VV @VV{\wedge_{V_{\mathrm{CC}}}}V\\
\lambda(V_{\mathrm{LL}})\otimes\lambda(V_{\mathrm{CR}}) @>{\wedge_{V_{\mathrm{CC}}}}>> \lambda(V_{\mathrm{CC}})
\end{CD}$$

commutes.

4.2. Exact triples of Fredholm operators

We begin this subsection by extending the isomorphism of Lemma 4.1 to exact triples of Fredholm operators. It is immediate from the explicit formula (4.10) for the new isomorphism that it satisfies the Naturality II, III, Normalization II, and Direct Sums properties in Section 2. We verify that it also satisfies the Dual Exact Triples property with $\widetilde{\mathscr{D}}_D$ given by (4.13) and the two algebraic Compositions properties.

We will use the natural pairing of a one-dimensional vector space L with its dual given by

$$L^*\otimes L\longrightarrow \mathsf{R},\quad \alpha\otimes v\longrightarrow \alpha(v).$$

If V is a finite-dimensional vector space and $v\in\lambda(V)$, we denote by

$$\langle v\rangle\equiv \dim V+2\mathsf{Z}\in\mathsf{Z}_2$$

the degree of v as an element of the Z_2-line $\lambda(V)$.

PROPOSITION 4.5 ([10, Proposition D.2.3]). *Every exact triple* $\mathfrak{t}$ *of Fredholm operators as in* (2.11) *induces a natural isomorphism*

$$\Psi_{\mathfrak{t}}\colon \lambda(D')\otimes\lambda(D'')\longrightarrow\lambda(D).$$

PROOF. By the Snake Lemma, (2.11) induces an exact sequence

$$0\longrightarrow\kappa(D')\xrightarrow{\mathfrak{i}_X}\kappa(D)\xrightarrow{\mathfrak{j}_X}\kappa(D'')\xrightarrow{\delta}\mathfrak{c}(D')\xrightarrow{\mathfrak{i}_Y}\mathfrak{c}(D)\xrightarrow{\mathfrak{j}_Y}\mathfrak{c}(D'')\longrightarrow 0. \tag{4.6}$$

By Lemma 4.1, there are then natural isomorphisms

$$\begin{aligned}
&\lambda(\kappa(D))\approx\lambda(\kappa(D'))\otimes\lambda(\mathrm{Im}\,\mathfrak{j}_X),\quad \lambda(\kappa(D''))\approx\lambda(\mathrm{Im}\,\mathfrak{j}_X)\otimes\lambda(\mathrm{Im}\,\delta),\\
&\lambda(\mathfrak{c}(D'))\approx\lambda(\mathrm{Im}\,\delta)\otimes\lambda(\mathrm{Im}\,\mathfrak{i}_Y),\qquad \lambda(\mathfrak{c}(D))\approx\lambda(\mathrm{Im}\,\mathfrak{i}_Y)\otimes\lambda(\mathfrak{c}(D'')).
\end{aligned}\tag{4.7}$$

Putting these isomorphisms together and using the natural evaluation isomorphisms, we obtain

$$\begin{aligned}\lambda(D')\otimes\lambda(D'') &\equiv \lambda(\kappa(D'))\otimes\lambda^*(\mathfrak{c}(D'))\otimes\lambda(\kappa(D''))\otimes\lambda^*(\mathfrak{c}(D''))\\ &\approx \lambda(\kappa(D))\otimes\lambda^*(\operatorname{Im}\mathfrak{j}_X)\otimes\lambda^*(\operatorname{Im}\mathfrak{i}_Y)\otimes\lambda^*(\operatorname{Im}\delta)\\ &\qquad\otimes\lambda(\operatorname{Im}\mathfrak{j}_X)\otimes\lambda(\operatorname{Im}\delta)\otimes\lambda^*(\mathfrak{c}(D))\otimes\lambda(\operatorname{Im}\mathfrak{i}_Y)\\ &\approx \lambda(\kappa(D))\otimes\lambda^*(\mathfrak{c}(D)).\end{aligned}\tag{4.8}$$

This establishes the claim.

For computational purposes, it is essential to specify the isomorphism of Proposition 4.5 explicitly. With the notation as in (2.11) and (4.7), let

$$\epsilon_{\mathfrak{t}} = (\operatorname{ind} D'')\mathfrak{d}(\mathfrak{c}(D')) + \mathfrak{d}(\mathfrak{c}(D))\mathfrak{d}(\operatorname{Im}\delta).\tag{4.9}$$

For $\mathfrak{t}$ corresponding to (2.11), we define

$$\begin{aligned}&\Psi_{\mathfrak{t}}\big(x\otimes(\lambda(\delta)v\wedge_{\mathfrak{c}(D')}w)^*\otimes(\lambda(\mathfrak{j}_X)u\wedge_{\kappa(D'')}v)\otimes(\lambda(\mathfrak{j}_Y)y)^*\big)\\ &\qquad\qquad = (-1)^{\epsilon_{\mathfrak{t}}}(\lambda(\mathfrak{i}_X)x\wedge_{\kappa(D)}u)\otimes(\lambda(\mathfrak{i}_Y)w\wedge_{\mathfrak{c}(D)}y)^*,\end{aligned}\tag{4.10}$$

whenever

$$x\in\lambda(\kappa(D')),\quad u\in\lambda\bigg(\frac{\kappa(D)}{\mathfrak{i}_X(\kappa(D'))}\bigg),\quad v\in\lambda\bigg(\frac{\kappa(D'')}{\mathfrak{j}_X(\kappa(D))}\bigg),$$

$$w\in\lambda\bigg(\frac{\mathfrak{c}(D')}{\delta(\kappa(D''))}\bigg),\quad y\in\lambda\bigg(\frac{\mathfrak{c}(D)}{\mathfrak{i}_Y(\mathfrak{c}(D'))}\bigg),\qquad x,u,v,w,y\neq 0.$$

Thus, $\Psi_{\mathfrak{t}}$ satisfies the Normalization II and Naturality II, III properties.

REMARK 4.6. If $\delta\colon V\to W$ is a homomorphism between finite-dimensional vector spaces, the isomorphism (4.10) applied to the exact sequence

$$0\longrightarrow 0\longrightarrow\kappa(\delta)\longrightarrow V\stackrel{\delta}{\longrightarrow}W\stackrel{q}{\longrightarrow}\mathfrak{c}(\delta)\longrightarrow 0\longrightarrow 0\tag{4.11}$$

induces the isomorphism

$$\Psi_\delta\colon\lambda^*(W)\otimes\lambda(V)\longrightarrow\lambda(\delta),\qquad \Psi_\delta(\beta\otimes x)=\Psi_{\mathfrak{t}_\delta}(1\otimes\beta\otimes x\otimes 1^*),$$

where $\Psi_{\mathfrak{t}_\delta}$ is the isomorphism (4.10) for the exact sequence (4.11). Explicitly,

$$\Psi_\delta\big((\lambda(\delta)v\wedge_W w)^*\otimes(u\wedge_V v)\big)=(-1)^{\mathfrak{d}(V)\mathfrak{d}(W)+(\mathfrak{d}(W)-\mathfrak{d}(\mathfrak{c}(\delta)))\mathfrak{d}(\mathfrak{c}(\delta))}u\otimes w^*,$$

if

$$u\in\lambda(\kappa(\delta))-0,\quad v\in\lambda(V/\kappa(\delta))-0,\quad w\in\lambda(\mathfrak{c}(\delta))-0.$$

Thus,

$$\Psi_{\mathbf{0}} \circ \Psi_{\delta}^{-1} \colon \lambda(\delta) \longrightarrow \lambda(\mathbf{0}) \equiv \lambda(V) \otimes \lambda^*(W),$$
$$u \otimes w^* \longrightarrow (-1)^{(\mathfrak{d}(W)-\mathfrak{d}(\mathfrak{c}(\delta)))\mathfrak{d}(\mathfrak{c}(\delta))}(u \wedge_V v) \otimes (\lambda(\delta)v \wedge_W w)^*,$$

is precisely the isomorphism (3.4).

For any $D' \in \mathscr{F}(X', Y')$ and $D'' \in \mathscr{F}(X'', Y'')$, let

$$\widetilde{\oplus}_{D',D''} \colon \lambda(D') \otimes \lambda(D'') \longrightarrow \lambda(D' \oplus D'')$$

be the isomorphism $\Psi_{\mathfrak{t}}$ in (4.10) corresponding to the diagram (2.17). Thus,

(4.12)
$$\begin{aligned}
&\widetilde{\oplus}_{D',D''}\big((x'_1 \wedge \ldots \wedge x'_{k'}) \otimes (y'_1 \wedge \ldots \wedge y'_{\ell'})^* \otimes (x''_1 \wedge \ldots \wedge x''_{k''}) \otimes (y''_1 \wedge \ldots \wedge y''_{\ell''})^*\big) \\
&\quad = (-1)^{(\operatorname{ind} D'')\mathfrak{d}(\mathfrak{c}(D'))}\big((x'_1, 0) \wedge \ldots \wedge (x'_{k'}, 0) \wedge (0, x''_1) \wedge \ldots \wedge (0, x''_{k''})\big) \\
&\qquad\qquad \otimes \big((y'_1, 0) \wedge \ldots \wedge (y'_{\ell'}, 0) \wedge (0, y''_1) \wedge \ldots \wedge (0, y''_{\ell''})\big)^*,
\end{aligned}$$

whenever

$$x'_1 \wedge \ldots \wedge x'_{k'} \in \lambda(\kappa(D')) - 0, \quad y'_1 \wedge \ldots \wedge x'_{\ell'} \in \lambda(\mathfrak{c}(D')) - 0,$$
$$x''_1 \wedge \ldots \wedge x''_{k''} \in \lambda(\kappa(D'')) - 0, \quad y''_1 \wedge \ldots \wedge y''_{\ell''} \in \lambda(\mathfrak{c}(D'')) - 0.$$

The two Direct Sums properties on page 213 follow immediately from (4.12).

The next proposition shows that the isomorphism

(4.13)
$$\begin{aligned}
&\widetilde{\mathscr{D}}_D \colon \lambda(D) \longrightarrow \lambda(D^*), \\
&\qquad x \otimes \alpha \longrightarrow (-1)^{(\operatorname{ind} D)\mathfrak{d}(\mathfrak{c}(D))}\lambda(\mathscr{D}_D)(\mathscr{P}\alpha) \otimes \mathscr{P}(\lambda(\mathscr{D}_D)x),
\end{aligned}$$

which satisfies the Normalization III* property on page 218, satisfies the Dual Exact Triples property. The extra factor of $(-1)^{\mathfrak{d}(\mathfrak{c}(D))}$ in (4.13) arises for the same reason as in the paragraph containing (3.9). Due to this extra factor, the compositions of $\widetilde{\mathscr{D}}_D$ with $\widetilde{\mathscr{D}}_{D^*}$ are the multiplication by $(-1)^{\operatorname{ind} D}$, not necessarily the identity, whenever the Banach spaces X and Y are reflexive.

PROPOSITION 4.7. *For every exact triple* (2.11) *of Fredholm operators, the diagram* (2.35) *commutes.*

PROOF. With notation as in (2.11) and (2.35), we define

$$\epsilon_{\mathrm{L}} = (\operatorname{ind} D')(\operatorname{ind} D'') + (\operatorname{ind} D')\mathfrak{d}(\mathfrak{c}(D')) + (\operatorname{ind} D'')\mathfrak{d}(\mathfrak{c}(D'')) + \epsilon_{\mathfrak{t}^*},$$
$$\epsilon_{\mathrm{R}} = \epsilon_{\mathfrak{t}} + (\operatorname{ind} D)\mathfrak{d}(\mathfrak{c}(D)).$$

The isomorphisms (2.30) intertwine the analogue of the exact sequence (4.6) for $\mathfrak{t}^*$ and the dual of (4.6):

(4.14)

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & \kappa(D''^*) & \xrightarrow{\mathfrak{j}_Y^*} & \kappa(D^*) & \xrightarrow{\mathfrak{i}_Y^*} & \kappa(D'^*) & \xrightarrow{\delta^*} & \mathfrak{c}(D''^*) & \xrightarrow{\mathfrak{j}_X^*} & \mathfrak{c}(D^*) & \xrightarrow{\mathfrak{i}_X^*} & \mathfrak{c}(D'^*) & \longrightarrow 0 \\
& \uparrow{\scriptstyle \mathscr{D}_{D''}} & & \uparrow{\scriptstyle \mathscr{D}_{D}} & & \uparrow{\scriptstyle \mathscr{D}_{D'}} & & \downarrow{\scriptstyle \mathscr{D}^*_{D''}} & & \downarrow{\scriptstyle \mathscr{D}^*_{D}} & & \downarrow{\scriptstyle \mathscr{D}^*_{D'}} & \\
0 \longrightarrow & \mathfrak{c}(D'')^* & \xrightarrow{\mathfrak{j}_Y^*} & \mathfrak{c}(D)^* & \xrightarrow{\mathfrak{i}_Y^*} & \mathfrak{c}(D')^* & \xrightarrow{\delta^*} & \kappa(D'')^* & \xrightarrow{\mathfrak{j}_X^*} & \kappa(D)^* & \xrightarrow{\mathfrak{i}_X^*} & \kappa(D')^* & \longrightarrow 0
\end{array}$$

In particular,

$$\begin{aligned}\mathfrak{d}(\operatorname{Im}\delta^*) = \mathfrak{d}(\operatorname{Im}\delta) &= \mathfrak{d}(\kappa(D')) + \mathfrak{d}(\kappa(D'')) + \mathfrak{d}(\kappa(D)) \\ &= \mathfrak{d}(\mathfrak{c}(D')) + \mathfrak{d}(\mathfrak{c}(D'')) + \mathfrak{d}(\mathfrak{c}(D))\end{aligned}$$

and so $2|(\epsilon_{\mathrm{L}} - \epsilon_{\mathrm{R}})$.

Let x, u, v, w, y be as in (4.10). By (4.14), we can compute $\Psi_{\mathfrak{t}^*}$ using

$$\check{x} = \lambda(\mathscr{D}_{D''})\mathscr{P}((\lambda(\mathfrak{j}_Y)y)^*) \in \lambda\big(\kappa(D''^*)\big),$$

$$\check{u} = \lambda(\mathscr{D}_{D})\mathscr{P}((\lambda(\mathfrak{i}_Y)w)^*) \in \lambda\left(\frac{\kappa(D^*)}{\mathfrak{j}_Y^*(\kappa(D''^*))}\right),$$

$$\check{v} = \lambda(\mathscr{D}_{D'})\mathscr{P}((\lambda(\delta)v)^*) \in \lambda\left(\frac{\kappa(D'^*)}{\iota_Y^*(\kappa(D^*))}\right),$$

$$\check{w} = \lambda(\mathscr{D}^*_{D''})^{-1}\mathscr{P}((\lambda(\mathfrak{j}_X)u)^*) \in \lambda\left(\frac{\mathfrak{c}(D''^*)}{\delta^*(\kappa(D'^*))}\right),$$

$$\check{y} = \lambda(\mathscr{D}^*_{D})^{-1}\mathscr{P}((\lambda(\mathfrak{i}_X)x)^*) \in \lambda\left(\frac{\mathfrak{c}(D^*)}{\mathfrak{j}_X^*(\mathfrak{c}(D''^*))}\right).$$

By (4.3), (4.4), and the commutativity of the diagram (4.14),

$$\mathscr{P}(\lambda(\mathscr{D}_{D'})x) = (\lambda(\mathfrak{i}_X^*)\check{y})^*, \tag{4.15}$$

$$\begin{aligned}
&\lambda(\mathscr{D}_{D'})\mathscr{P}(w^*) = \lambda(\mathfrak{i}_Y^*)\check{u}, && \lambda(\mathscr{D}_{D'})\lambda(\delta^*)^{-1}\mathscr{P}(v^*) = \check{v},\\
&\lambda(\mathscr{D}_{D})\mathscr{P}(y^*) = \lambda(\mathfrak{j}_Y^*)\check{x}, && \lambda(\mathscr{D}_{D})\lambda(\mathfrak{i}_Y^*)^{-1}\mathscr{P}(w^*) = \check{u},\\
&\lambda(\mathscr{D}_{D''})\lambda(\mathfrak{j}_X)u = \mathscr{P}(\check{w}^*), && \lambda(\mathscr{D}_{D''})v = \lambda(\delta)^{-1}\mathscr{P}(\check{v}^*)\\
&\lambda(\mathscr{D}_{D})\lambda(\mathfrak{i}_X)x = \mathscr{P}(\check{y}^*), && \lambda(\mathscr{D}_{D})u = \lambda(\mathfrak{j}_X)^{-1}\mathscr{P}(\check{w}^*).
\end{aligned}$$

Combining each pair of identities on the last four lines above with (4.5), we obtain

$$\lambda(\mathscr{D}_{D'})\mathscr{P}((\lambda(\delta)v \wedge_{\mathfrak{c}(D')} w)^*) = \lambda(\mathfrak{i}_Y^*)\check{u} \wedge_{\kappa(D'^*)} \check{v}, \tag{4.16}$$

$$\lambda(\mathscr{D}_D)\mathscr{P}((\lambda(\mathfrak{i}_Y)w \wedge_{\mathfrak{c}(D)} y)^*) = \lambda(\mathfrak{j}_Y^*)\check{x} \wedge_{\kappa(D^*)} \check{u}, \tag{4.17}$$

$$\mathscr{P}(\lambda(\mathscr{D}_{D''})(\lambda(\mathfrak{j}_X)u \wedge_{\kappa(D'')} v)) = (\lambda(\delta^*)\check{v} \wedge_{\mathfrak{c}(D''^*)} \check{w})^*, \tag{4.18}$$

$$\mathscr{P}(\lambda(\mathscr{D}_D)(\lambda(\mathfrak{i}_X)x \wedge_{\kappa(D)} u)) = (\lambda(\mathfrak{j}_X^*)\check{w} \wedge_{\mathfrak{c}(D^*)} \check{y})^*, \tag{4.19}$$

respectively. By (4.13), (4.15), (4.16), (4.18), and (4.10), the image of

$$x\otimes(\lambda(\delta)v\wedge_{\mathfrak{c}(D')}w)^*\otimes(\lambda(\mathfrak{j}_X)u\wedge_{\kappa(D'')}v)\otimes(\lambda(\mathfrak{j}_Y)y)^* \in \lambda(D')\otimes\lambda(D'') \tag{4.20}$$

under $\Psi_{\mathfrak{t}^*} \circ \widetilde{\mathscr{D}}_{D''} \otimes \widetilde{\mathscr{D}}_{D'} \circ R$ is

$$(-1)^{\epsilon_{\mathrm{L}}}(\lambda(\mathfrak{j}_Y^*)\check{x} \wedge_{\kappa(D^*)} \check{u}) \otimes (\lambda(\mathfrak{j}_X^*)\check{w} \wedge_{\mathfrak{c}(D^*)} \check{y})^* \in \lambda(D^*).$$

By (4.10), (4.13), (4.17), and (4.19), the image of the element (4.20) under $\widetilde{\mathscr{D}}_D \circ \Psi_{\mathfrak{t}}$ is

$$(-1)^{\epsilon_{\mathrm{R}}}(\lambda(\mathfrak{j}_Y^*)\check{x} \wedge_{\kappa(D^*)} \check{u}) \otimes (\lambda(\mathfrak{j}_X^*)\check{w} \wedge_{\mathfrak{c}(D^*)} \check{y})^* \in \lambda(D^*).$$

Since $2|(\epsilon_{\mathrm{L}} - \epsilon_{\mathrm{R}})$, this establishes the claim.

For any $D_1 \in \mathscr{F}(X_1, X_2)$ and $D_2 \in \mathscr{F}(X_2, X_3)$, let

$$\widetilde{\mathscr{C}}_{D_1,D_2}\colon \lambda(D_1) \otimes \lambda(D_2) \longrightarrow \lambda(D_2 \circ D_1)$$

be the isomorphism $\Psi_{\mathfrak{t}}$ in (4.10) corresponding to the diagram (2.19). The exact sequence (4.6) in this case specializes to

$$0 \longrightarrow \kappa(D_1) \longrightarrow \kappa(D_2 \circ D_1) \xrightarrow{D_1} \kappa(D_2) \xrightarrow{\delta} \mathfrak{c}(D_1)$$
$$\xrightarrow{D_2} \mathfrak{c}(D_2 \circ D_1) \longrightarrow \mathfrak{c}(D_2) \longrightarrow 0,$$

$$\delta(x_2) = -x_2 + \operatorname{Im} D_1. \tag{4.21}$$

Let

$$\epsilon_{D_1,D_2} = (\operatorname{ind} D_2)\mathfrak{d}(\mathfrak{c}(D_1)) + \big(\mathfrak{d}(\mathfrak{c}(D_1)) + \mathfrak{d}(\mathfrak{c}(D_2))\big)\mathfrak{d}(\operatorname{Im}\delta).$$

Then,

$$\begin{aligned}\widetilde{\mathscr{C}}_{D_1,D_2}&\big(x_1 \otimes (v \wedge_{\mathfrak{c}(D_1)} w)^* \otimes (\lambda(D_1)u \wedge_{\kappa(D_2)} v) \otimes y_2^*\big)\\ &= (-1)^{\epsilon_{D_1,D_2}}(x_1 \wedge_{\kappa(D_2\circ D_1)} u) \otimes (\lambda(D_2)w \wedge_{\mathfrak{c}(D_2\circ D_1)} y_2)^*,\end{aligned} \tag{4.22}$$

whenever

$$x_1 \in \lambda(\kappa(D_1)) - 0, \quad y_2 \in \lambda(\mathfrak{c}(D_2)) - 0, \quad u \in \lambda\left(\frac{\kappa(D_2 \circ D_1)}{\kappa(D_1)}\right) - 0,$$
$$v \in \lambda\left(\frac{\kappa(D_2)}{\kappa(D_2) \cap (\operatorname{Im} D_1)}\right) - 0, \quad w \in \lambda\left(\frac{X_2}{\kappa(D_2) + (\operatorname{Im} D_1)}\right) - 0.$$

PROPOSITION 4.8 (Commutativity of (2.22), [10, Proposition D.2.6]). *For any triple of Fredholm operators* $D_1: X_1 \to X_2$, $D_2: X_2 \to X_3$, *and* $D_3: X_3 \to X_4$, *the diagram* (2.22) *commutes.*

PROOF. We denote by $D''D'$ the composition $D'' \circ D'$ of two maps D' and D'' and define

$$\epsilon_{\mathrm{L}} = \epsilon_{D_1,D_2} + \epsilon_{D_2D_1,D_3}, \quad \epsilon_{\mathrm{R}} = \epsilon_{D_2,D_3} + \epsilon_{D_1,D_3D_2}.$$

For $i = 1, 2, 3$, let

$$x_i \in \lambda(\kappa(D_i)) - 0 \quad \text{and} \quad y_i \in \lambda(\mathfrak{c}(D_i)) - 0.$$

For $(i, j) \in \{(1, 2), (2, 3), (1, 23), (12, 3)\}$, let

$$u_{i,j} \in \lambda\left(\frac{\kappa(D_jD_i)}{\kappa(D_i)}\right) - 0,$$
$$v_{i,j} \in \lambda\left(\frac{\kappa(D_j)}{\kappa(D_j) \cap \mathrm{Im}(D_i)}\right) - 0,$$
$$w_{i,j} \in \lambda\left(\frac{X_j}{\kappa(D_j) + \mathrm{Im}(D_i)}\right) - 0,$$

where $D_{12} = D_2D_1$, $D_{23} = D_3D_2$, and $X_{23} = X_2$; see Figure 6. Below we choose these elements in a compatible way.

Applying Lemma 4.1 to the exact sequence (4.21) with D_1 and D_2 replaced by D_i and D_j with (i, j) as above, we obtain

$$\mathfrak{d}(\kappa(D_3)) = \langle u_{12,3}\rangle + \langle v_{12,3}\rangle, \qquad \mathfrak{d}(\mathfrak{c}(D_1)) = \langle v_{1,23}\rangle + \langle w_{1,23}\rangle,$$
$$\mathfrak{d}(\mathfrak{c}(D_jD_i)) = \mathfrak{d}(\mathfrak{c}(D_i)) + \mathfrak{d}(\mathfrak{c}(D_j)) - \langle v_{i,j}\rangle, \quad \mathrm{ind}\, D_jD_i = \mathrm{ind}\, D_i + \mathrm{ind}\, D_j,$$

where $(i, j) = (1, 2), (2, 3)$. From this, we find that

$$\begin{aligned} \epsilon_{\mathrm{L}} &= A + C(\langle v_{1,2}\rangle + \langle v_{12,3}\rangle) + \langle u_{12,3}\rangle\langle v_{1,2}\rangle \mod 2, \\ \epsilon_{\mathrm{R}} &= A + C(\langle v_{2,3}\rangle + \langle v_{1,23}\rangle) + \langle v_{2,3}\rangle\langle w_{1,23}\rangle \mod 2, \end{aligned} \tag{4.23}$$

where

$$A = (\mathrm{ind}\, D_3D_2) \cdot \mathfrak{d}(\mathfrak{c}(D_1)) + (\mathrm{ind}\, D_3) \cdot \mathfrak{d}(\mathfrak{c}(D_2)),$$
$$C = \mathfrak{d}(\mathfrak{c}(D_1)) + \mathfrak{d}(\mathfrak{c}(D_2)) + \mathfrak{d}(\mathfrak{c}(D_3)).$$

In light of the top row in the first diagram in Figure 6, the bottom row in the second diagram, and Lemma 4.1, we can take

$$\begin{aligned} u_{1,23} &= u_{1,2} \wedge_{\frac{\kappa(D_3D_2D_1)}{\kappa(D_1)}} u_{12,3}, \\ w_{12,3} &= \lambda(D_2)w_{1,23} \wedge_{\frac{X_3}{\kappa(D_3)+\mathrm{Im}(D_2D_1)}} w_{2,3}. \end{aligned} \tag{4.24}$$

Along with Corollary 4.4, these equalities insure that

$$\text{(4.25)}\quad \big((x_1 \wedge_{\kappa(D_2D_1)} u_{1,2}) \wedge_{\kappa(D_3D_2D_1)} u_{12,3}\big) \otimes \big(\lambda(D_3)w_{12,3} \wedge_{\mathfrak{c}(D_3D_2D_1)} y_3\big)^* \\ = \big(x_1 \wedge_{\kappa(D_3D_2D_1)} u_{1,23}\big) \otimes \big(\lambda(D_3D_2)w_{1,23} \wedge_{\mathfrak{c}(D_3D_2D_1)} (\lambda(D_3)w_{2,3} \wedge_{\mathfrak{c}(D_3D_2)} y_3)\big)^*$$

in $\lambda(D_3D_2D_1)$. In light of the right column and bottom row in the first diagram in Figure 6, the top row and left column in the second diagram in Figure 6, and Lemma 4.1, we can take

$$\text{(4.26)}\quad \begin{aligned} u_{2,3} &= \lambda(D_1)u_{12,3} \wedge_{\frac{\kappa(D_3D_2)}{\kappa(D_2)}} \mu, & v_{1,23} &= v_{1,2} \wedge_{\frac{\kappa(D_3D_2)}{\kappa(D_3D_2)\cap \mathrm{Im}(D_1)}} \mu, \\ v_{12,3} &= \lambda(D_2)\mu \wedge_{\frac{\kappa(D_3)}{\kappa(D_3)\cap \mathrm{Im}(D_2D_1)}} v_{2,3}, & w_{1,2} &= \mu \wedge_{\frac{X_2}{\kappa(D_2)+\mathrm{Im}(D_1)}} w_{1,23} \end{aligned}$$

for some

$$\mu \in \lambda\left(\frac{\kappa(D_3D_2)}{\kappa(D_2)+\kappa(D_3D_2)\cap \mathrm{Im}(D_1)}\right) - 0.$$

In light of the left column of the first diagram and the right column of the second diagram in Figure 6, (4.26), and Corollary 4.4, we can take

$$\text{(4.27)}\quad \begin{aligned} x_2 &= \lambda(D_1)u_{1,2} \wedge_{\kappa(D_2)} v_{1,2}, \\ y_2 &= v_{2,3} \wedge_{\mathfrak{c}(D_2)} w_{2,3}, \\ y_1 &= v_{1,2} \wedge_{\mathfrak{c}(D_1)} w_{1,2} = v_{1,23} \wedge_{\mathfrak{c}(D_1)} w_{1,23}, \\ x_3 &= \lambda(D_2D_1)u_{12,3} \wedge_{\kappa(D_3)} v_{12,3} = \lambda(D_2)u_{2,3} \wedge_{\kappa(D_3)} v_{2,3}. \end{aligned}$$

Combining the above definitions of x_2 and y_2 with (4.26) and applying Lemma 4.3 to the two diagrams in Figure 6, we find that

$$\text{(4.28)}\quad \begin{aligned} \lambda(D_1)x_2 \wedge_{\kappa(D_3D_2)} u_{2,3} &= (-1)^{\langle u_{12,3}\rangle\langle v_{1,2}\rangle}\lambda(D_1)u_{1,23} \wedge_{\kappa(D_3D_2)} v_{1,23}, \\ \lambda(D_2)w_{1,2} \wedge_{\mathfrak{c}(D_2D_1)} y_2 &= (-1)^{\langle v_{2,3}\rangle\langle w_{1,23}\rangle} v_{12,3} \wedge_{\mathfrak{c}(D_2D_1)} w_{12,3}. \end{aligned}$$

By (4.22), (4.27), and (4.28), the images of

$$x_1 \otimes y_1^* \otimes x_2 \otimes y_2^* \otimes x_3 \otimes y_3^* \in \lambda(D_1) \otimes \lambda(D_2) \otimes \lambda(D_3)$$

under $\widetilde{C}_{D_2\circ D_1, D_3} \circ \widetilde{C}_{D_1,D_2} \otimes \mathrm{id}$ and $\widetilde{C}_{D_1, D_3\circ D_2} \circ \mathrm{id} \otimes \widetilde{C}_{D_2,D_3}$ are

$$(-1)^{\epsilon_{\mathrm{L}}+\langle v_{2,3}\rangle\langle w_{1,23}\rangle}\big((x_1 \wedge_{\kappa(D_2D_1)} u_{1,2}) \wedge_{\kappa(D_3D_2D_1)} u_{12,3}\big) \\ \otimes \big(\lambda(D_3)w_{12,3} \wedge_{\mathfrak{c}(D_3D_2D_1)} y_3\big)^*,$$

$$(-1)^{\epsilon_{\mathrm{R}}+\langle u_{12,3}\rangle\langle v_{1,2}\rangle}\big(x_1 \wedge_{\kappa(D_3D_2D_1)} u_{1,23}\big) \\ \otimes \big(\lambda(D_3D_2)w_{1,23} \wedge_{\mathfrak{c}(D_3D_2D_1)} (\lambda(D_3)w_{2,3} \wedge_{\mathfrak{c}(D_3D_2)} y_3)\big)^*,$$

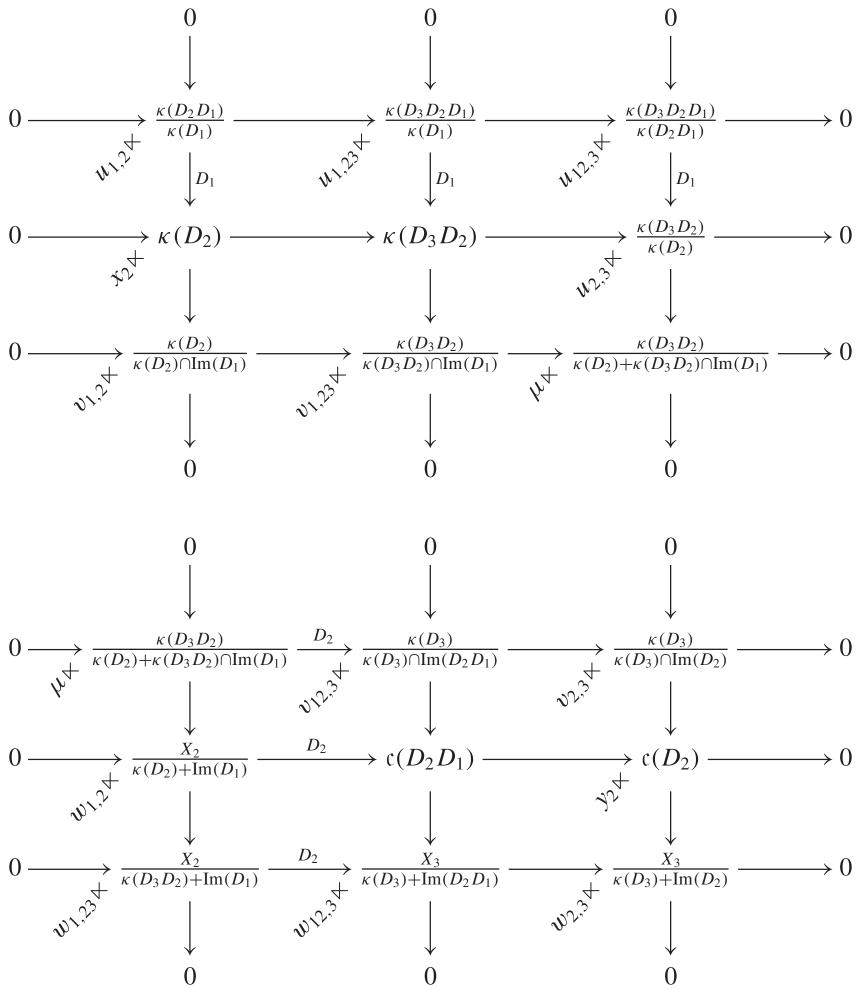


FIGURE 6. Commutative diagrams of exact sequences used in the proof of Proposition 4.8

respectively. By (4.23), the second and third identities in (4.26), and (4.25), these two elements of $\lambda(D_3D_2D_1)$ are the same, which establishes the claim.

REMARK 4.9. The proof of this crucial proposition in [10, Appendix D.2] does not appear to establish anything. Up to notational differences, it describes an expression for

$$\{\widetilde{C}_{D_2\circ D_1,D_3}\circ\widetilde{C}_{D_1,D_2}\otimes\mathrm{id}\}(x_1\otimes y_1^*\otimes x_2\otimes y_2^*\otimes x_3\otimes y_3^*)\in\lambda(D_3\circ D_2\circ D_1)$$

without any signs and simply claims that

$$\{\widetilde{C}_{D_1,D_3\circ D_2}\circ \mathrm{id}\otimes\widetilde{C}_{D_2,D_3}\}(x_1\otimes y_1^*\otimes x_2\otimes y_2^*\otimes x_3\otimes y_3^*)\in\lambda(D_3\circ D_2\circ D_1)$$

is given by the same expression, without providing an explicit formula for $\widetilde{C}_{D_1,D_2}$, using the statement of Lemma 4.3, or indicating the significance of the grading of the lines $\lambda(V)$. As illustrated by the proof of Proposition 4.8 above, the two expressions require auxiliary terms from different vectors spaces and it takes significant care to show that it is possible to choose them compatibly. Furthermore, there are two typos at the end of the proof of the closely related [10, Corollary D.2.4] with two subscripts that should be different being the same and resulting in the order of two factors switched between the statements of [10, Proposition D.2.3] and [10, Corollary D.2.4].

PROPOSITION 4.10 (Commutativity of (2.23)). *For any pairs* $(\mathfrak{t}_1,\mathfrak{t}_2)$ *of exact triples of Fredholm operators as in* (2.14)*, the diagram* (2.23) *commutes.*

PROOF. We continue with the notation described in the first sentence of the proof of Proposition 4.8 and define

$$\begin{aligned}
\mathfrak{t}_{12} &= \mathscr{C}_{\mathscr{T}}(\mathfrak{t}_1,\mathfrak{t}_2),\\
\epsilon_{\mathrm{R}} &= \epsilon_{\mathfrak{t}_1}+\epsilon_{\mathfrak{t}_2}+\epsilon_{D_1,D_2},\\
\epsilon_{\mathrm{L}} &= (\mathrm{ind}\,D_1'')(\mathrm{ind}\,D_2')+\epsilon_{D_1',D_2'}+\epsilon_{D_1'',D_2''}+\epsilon_{\mathfrak{t}_{12}}.
\end{aligned}$$

For $k=1,2$ and $\star={}',{}''$, let

$$x_k^\star\in\lambda(\kappa(D_k^\star))-0,\qquad y_k'\in\lambda(\mathfrak{c}(D_k'))-0,\qquad y_k''\in\lambda\biggl(\frac{X_{k+1}}{\mathrm{Im}(\mathfrak{i}_{k+1})+\mathrm{Im}(D_k)}\biggr)-0.$$

With $\star$ denoting $'$, $''$ or a blank, let

$$\begin{aligned}
u^\star &\in\lambda\biggl(\frac{\kappa(D_2^\star D_1^\star)}{\kappa(D_1^\star)}\biggr)-0,\\
v^\star &\in\lambda\biggl(\frac{\kappa(D_2^\star)}{\kappa(D_2^\star)\cap\mathrm{Im}(D_1^\star)}\biggr)-0,\\
w^\star &\in\lambda\biggl(\frac{X_2^\star}{\kappa(D_2^\star)+\mathrm{Im}(D_1^\star)}\biggr)-0;
\end{aligned}$$

see Figures 7 and 8. For $k=1,2,12$, let

$$\delta_k\colon\kappa(D_k'')\longrightarrow\mathfrak{c}(D_k'),$$

where $D'_{12} = D'_2 D'_1$ and $D''_{12} = D''_2 D''_1$, be the connecting homomorphisms in the sequences (4.6) corresponding to $\mathfrak{t}_1$, $\mathfrak{t}_2$, and $\mathfrak{t}_{12}$, respectively, and

$$u_k \in \lambda\left(\frac{\kappa(D_k)}{\kappa(D_k) \cap \mathrm{Im}(\mathfrak{i}_k)}\right) - 0,$$
$$v_k \in \lambda\left(\frac{\kappa(D''_k)}{\mathfrak{j}_k(\kappa(D_k))}\right) - 0,$$
$$w_k \in \lambda\left(\frac{X'_{k+1}}{\mathfrak{i}_{k+1}^{-1}(\mathrm{Im}(D_k))}\right) - 0,$$

with $\mathfrak{i}_{12} = \mathfrak{i}_1$, $\mathfrak{j}_{12} = \mathfrak{j}_1$, and $12 + 1 \equiv 3$; see Figures 7 and 8. Define

$$\widetilde{w}'' \in \lambda\left(\frac{X_2}{\mathfrak{j}_2^{-1}(\kappa(D''_2)) + \mathrm{Im}(D_1)}\right) - 0 \quad \text{by} \quad w'' = \lambda(\mathfrak{j}_2)\widetilde{w}''.$$

Below we choose these elements in a compatible way.

In order to describe the two relevant signs, we define

$$\begin{aligned}
A &= (\mathrm{ind}\, D''_2)(\mathfrak{c}'_1 + \mathfrak{c}''_1 + \mathfrak{c}'_2) + (\mathrm{ind}\, D''_1 + \mathrm{ind}\, D'_2)\mathfrak{c}'_1 + \mathfrak{c}''_1 \kappa'_2,\\
C &= \mathfrak{c}'_1 + \mathfrak{c}''_1 + \mathfrak{c}'_2 + \mathfrak{c}''_2,\\
A_{\mathrm{L}} &= \kappa''_1 \kappa'_2 + (\kappa''_1 + \kappa''_2)\langle v' \rangle + (\mathfrak{c}'_1 + \mathfrak{c}'_2)\langle v'' \rangle + (\langle v' \rangle + \langle v'' \rangle)\langle v_{12} \rangle,\\
A_{\mathrm{R}} &= \mathfrak{c}''_1 \mathfrak{c}'_2 + (\kappa'_2 + \kappa''_2)\langle v_1 \rangle + (\mathfrak{c}'_1 + \mathfrak{c}''_1)\langle v_2 \rangle + (\langle v_1 \rangle + \langle v_2 \rangle)\langle v \rangle,
\end{aligned}$$

where $\kappa^\star_i = \mathfrak{d}(\kappa(D^\star_i))$ and $\mathfrak{c}^\star_i = \mathfrak{d}(\mathfrak{c}(D^\star_i))$ with $i = 1, 2$ and $\star = {}', {}''$. Applying Lemma 4.1 to the exact sequences (4.6) with $D^\star$ replaced by $D^\star_k$, for $\star = {}', {}''$ and blank and $k = 1, 2, 12$, and (4.21) with D_k replaced by $D^\star_k$, for $\star = {}', {}''$ and blank and $k = 1, 2$, we obtain

$$\begin{aligned}
\mathrm{ind}\, D_2 &= \mathrm{ind}\, D'_2 + \mathrm{ind}\, D''_2,\\
\mathrm{ind}\, D''_2 D''_1 &= \mathrm{ind}\, D''_1 + \mathrm{ind}\, D''_2,\\
\mathfrak{d}(\mathfrak{c}(D_k)) &= \mathfrak{d}(\mathfrak{c}(D'_k)) + \mathfrak{d}(\mathfrak{c}(D''_k)) - \langle v_k \rangle,\\
\mathfrak{d}(\mathfrak{c}(D^\star_2 D^\star_1)) &= \mathfrak{d}(D^\star_1) + \mathfrak{d}(\mathfrak{c}(D^\star_2)) - \langle v^\star \rangle.
\end{aligned} \tag{4.29}$$

From this, we find that

$$\begin{aligned}
\epsilon_{\mathrm{L}} &= A + C(\langle v' \rangle + \langle v'' \rangle + \langle v_{12} \rangle) + A_{\mathrm{R}} + \langle v_{12} \rangle,\\
\epsilon_{\mathrm{R}} &= A + C(\langle v_1 \rangle + \langle v_2 \rangle + \langle v \rangle) + A_{\mathrm{L}} + \langle v_1 \rangle + \langle v_2 \rangle,
\end{aligned} \tag{4.30}$$

modulo 2. By the identities in the second column in (4.29),

$$\langle v \rangle + \langle v_1 \rangle + \langle v_2 \rangle = C - \mathfrak{d}(\mathfrak{c}(D_2 D_1)) = \langle v_{12} \rangle + \langle v' \rangle + \langle v'' \rangle. \tag{4.31}$$

From the exact sequences (4.6) and (4.21), we also find

$$\kappa_i'' = \langle u_i\rangle + \langle v_i\rangle,\quad \mathfrak{c}_i' = \langle v_i\rangle + \langle w_i\rangle,\quad \kappa_2^\star = \langle u^\star\rangle + \langle v^\star\rangle,\quad \mathfrak{c}_1^\star = \langle v^\star\rangle + \langle w^\star\rangle,$$

where $i = 1, 2$ and $\star = {}', {}''$. From this, we find that

$$\begin{aligned}(4.32)\quad \langle u'\rangle\langle u_1\rangle + \langle w''\rangle\langle w_2\rangle + \langle w'\rangle\langle u_2\rangle + \langle u''\rangle\langle w_1\rangle + \langle v\rangle\langle v_{12}\rangle \\ \cong A_{\mathrm{L}} + A_{\mathrm{R}} + (\langle v_1\rangle + \langle v_2\rangle + \langle v_{12}\rangle)(\langle v'\rangle + \langle v''\rangle + \langle v\rangle)\end{aligned}$$

modulo 2.

In light of the bottom row and right column in the first diagram in Figure 7, the top row and left column in the second diagram in Figure 7, and Lemma 4.1, we can take

$$(4.33)\quad \begin{aligned} u &= \lambda(\mathfrak{i}_1)u' \wedge_{\frac{\kappa(D_2D_1)}{\kappa(D_1)}} \mu, & u_{12} &= u_1 \wedge_{\frac{\kappa(D_2D_1)}{\kappa(D_2D_1)\cap\mathrm{Im}(\mathfrak{i}_1)}} \mu,\\ w &= \eta \wedge_{\frac{X_2}{\kappa(D_2)+\mathrm{Im}(D_1)}} \widetilde{w}'', & w_{12} &= \lambda(\mathfrak{i}_3^{-1}\circ D_2)\eta \wedge_{\frac{X_3'}{\mathfrak{i}_3^{-1}(\mathrm{Im}(D_2D_1))}} w_2\end{aligned}$$

for some

$$\mu \in \lambda\left(\frac{\kappa(D_2D_1)}{\kappa(D_1)+\kappa(D_2D_1)\cap\mathrm{Im}(\mathfrak{i}_1)}\right) - 0,$$

$$\eta \in \lambda\left(\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+\mathfrak{j}_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)}\right) - 0.$$

Along with Lemma 4.3 applied to the two diagrams in Figure 7, these equalities insure that

$$\begin{aligned}(4.34)\quad &((\lambda(\mathfrak{i}_1)x_1'\wedge_{\kappa(D_1)}u_1)\wedge_{\kappa(D_2D_1)}u)\otimes(\lambda(D_2)w\wedge_{\mathfrak{c}(D_2D_1)}(\lambda(\mathfrak{i}_3)w_2\wedge_{\mathfrak{c}(D_2)}y_2''))^*\\ &= (-1)^{\langle u'\rangle\langle u_1\rangle+\langle\widetilde{w}''\rangle\langle w_2\rangle}(\lambda(\mathfrak{i}_1)(x_1' \wedge_{\kappa(D_2'D_1')} u') \wedge_{\kappa(D_2D_1)} u_{12})\\ &\qquad\otimes (\lambda(\mathfrak{i}_3)w_{12} \wedge_{\mathfrak{c}(D_2D_1)} (\lambda(D_2)\widetilde{w}'' \wedge_{\frac{X_3}{\mathrm{Im}(\mathfrak{i}_3)+\mathrm{Im}(D_2D_1)}} y_2''))^*,\end{aligned}$$

in $\lambda(D_2D_1)$.

We next make use of the three commutative diagrams in Figure 8. They can be viewed as the three coordinate planes in Z^3, with all three diagrams sharing the center and any pair sharing an axis. We choose $v', w', u_2, v_2, v_1, u'', \mu$ arbitrarily, then find y_1', w_1, v_{12}, v so that

$$(4.35)\quad \begin{aligned} v' \wedge_{\mathfrak{c}(D_1)} w' &= y_1' = \lambda(\delta_1)v_1 \wedge_{\mathfrak{c}(D_1)} w_1,\\ v_1 \wedge_{\frac{\kappa(D_2''D_1'')}{\mathfrak{j}_1(\kappa(D_1))}} u'' &= (-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(\mathfrak{j}_1)\mu \wedge_{\frac{\kappa(D_2''D_1'')}{\mathfrak{j}_1(\kappa(D_2))}} v_{12},\\ \lambda(\mathfrak{i}_2)v' \wedge_{\frac{\kappa(D_2)}{\kappa(D_2)\cap\mathrm{Im}(\mathfrak{i}_2D_1)}} u_2 &= (-1)^{\langle w'\rangle\langle u_2\rangle}\lambda(D_1)\mu \wedge_{\frac{\kappa(D_2)}{\kappa(D_2)\cap\mathrm{Im}(\mathfrak{i}_2D_1)}} v,\end{aligned}$$

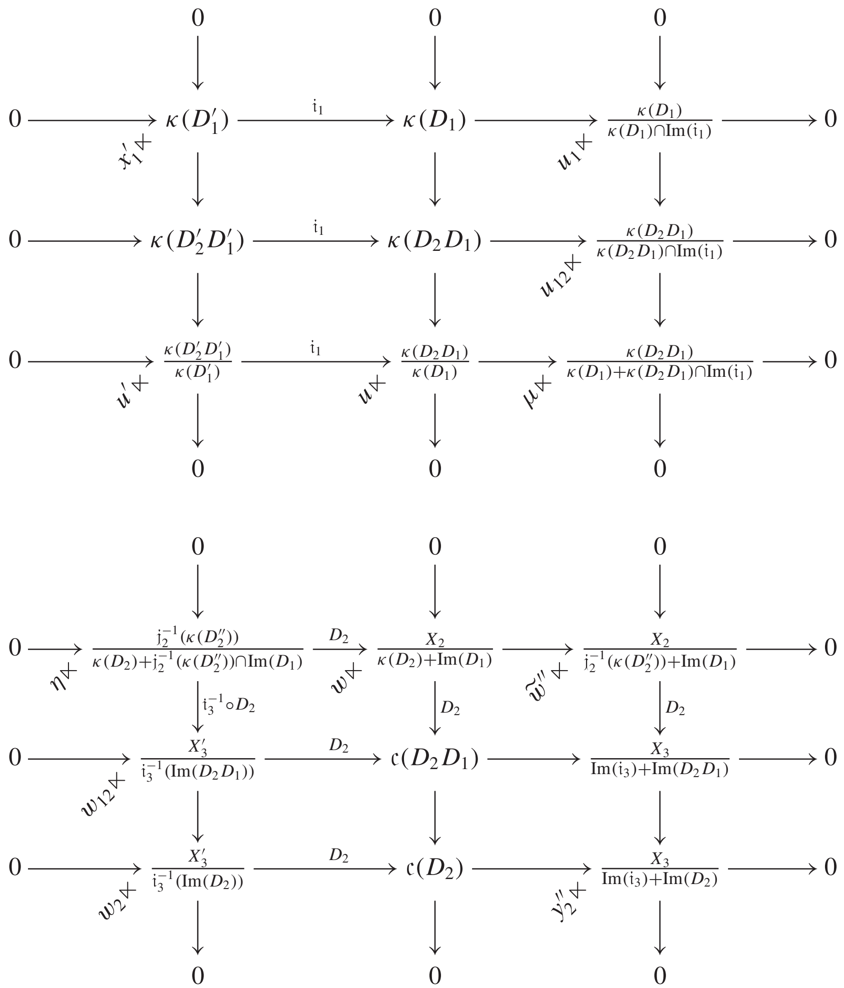


FIGURE 7. Commutative diagrams of exact sequences used in the proof of Proposition 4.10

and finally take η, x_2'', v'' so that

$$\begin{aligned}\lambda(\mathrm{i}_2)w' \wedge_{\frac{\mathrm{j}_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+\mathrm{Im}(\mathrm{i}_2 D_1')}} v_2 &= (-1)^{\langle v\rangle\langle v_{12}\rangle}\lambda(D_1\circ \mathrm{j}_1^{-1})v_{12}\wedge_{\frac{\mathrm{j}_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+\mathrm{Im}(\mathrm{i}_2 D_1')}} \eta,\\ \lambda(\mathrm{j}_2)u_2 \wedge_{\kappa(D_2'')} v_2 = x_2'' &= \lambda(D_1'')u''\wedge_{\kappa(D_2'')} v''.\end{aligned} \tag{4.36}$$

By Lemma 4.3 applied to the three commutative squares in Figure 8, (4.35),

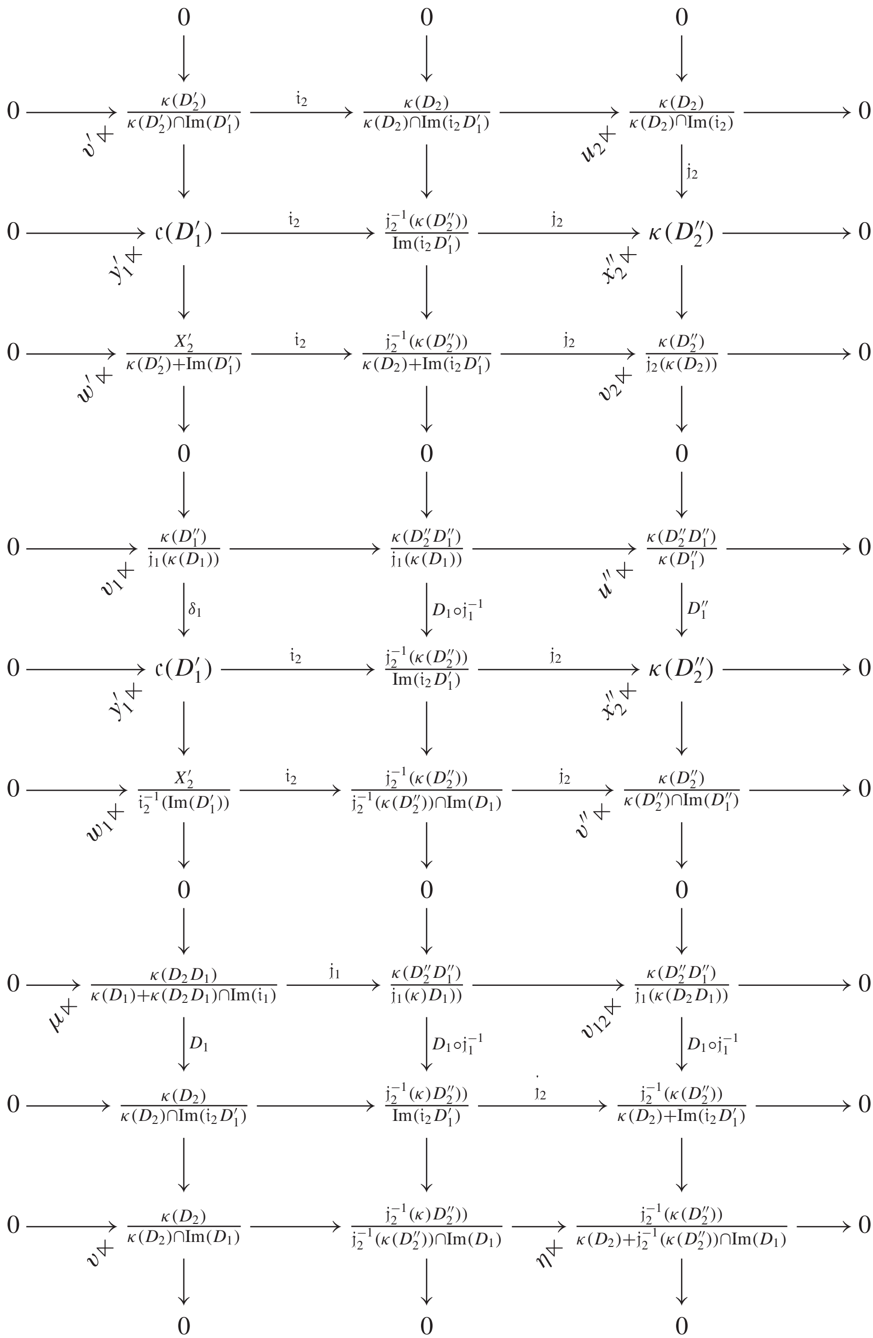


FIGURE 8. Commutative diagrams of exact sequences used in the proof of Proposition 4.10

and (4.36),

$$
\begin{aligned}
&\lambda(D_1\circ\mathfrak{j}_1^{-1})\Big(v_1\wedge_{\frac{\kappa(D_2''D_1'')}{\mathfrak{j}_1(\kappa(D_1))}}u''\Big)\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathrm{Im}(\mathfrak{i}_2D_1')}}\Big(\lambda(\mathfrak{i}_2)w_1\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathfrak{j}_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)}}v''\Big)\\
&\quad=(-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(\mathfrak{i}_2)\big(\lambda(\delta_1)v_1\wedge_{\mathfrak{c}(D_1)}w_1\big)\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathrm{Im}(\mathfrak{i}_2D_1')}}\big(\lambda(D_1'')u''\wedge_{\kappa(D_2'')}v''\big)\\
&\quad=(-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(\mathfrak{i}_2)\big(v'\wedge_{\mathfrak{c}(D_1)}w'\big)\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathrm{Im}(\mathfrak{i}_2D_1')}}\big(\lambda(\mathfrak{j}_2)u_2\wedge_{\kappa(D_2'')}v_2\big)\\
&\quad=(-1)^{\langle u''\rangle\langle w_1\rangle+\langle w'\rangle\langle u_2\rangle}\Big(\lambda(\mathfrak{i}_2)v'\wedge_{\frac{\kappa(D_2)}{\kappa(D_2)\cap\mathrm{Im}(\mathfrak{i}_2D_1)}}u_2\Big)\\
&\qquad\qquad\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathrm{Im}(\mathfrak{i}_2D_1')}}\Big(\lambda(\mathfrak{i}_2)w'\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+\mathrm{Im}(\mathfrak{i}_2D_1')}}v_2\Big)\\
&\quad=(-1)^{\langle u''\rangle\langle w_1\rangle+\langle v\rangle\langle v_{12}\rangle}\Big(\lambda(D_1)\mu\wedge_{\frac{\kappa(D_2)}{\kappa(D_2)\cap\mathrm{Im}(\mathfrak{i}_2D_1)}}v\Big)\\
&\qquad\qquad\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathrm{Im}(\mathfrak{i}_2D_1')}}\Big(\lambda(D_1\circ\mathfrak{j}_1^{-1})v_{12}\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\kappa(D_2)+\mathrm{Im}(\mathfrak{i}_2D_1')}}\eta\Big)\\
&\quad=(-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(D_1\circ\mathfrak{j}_1^{-1})\Big(\lambda(\mathfrak{j}_1)\mu\wedge_{\frac{\kappa(D_2''D_1'')}{\mathfrak{j}_1(\kappa(D_2))}}v_{12}\Big)\\
&\qquad\qquad\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathrm{Im}(\mathfrak{i}_2D_1')}}\Big(v\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathfrak{j}_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)}}\eta\Big).
\end{aligned}
$$

Along with the second equation in (4.35), this gives

$$
\lambda(\mathfrak{i}_2)w_1\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathfrak{j}_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)}}v''=\Big(v\wedge_{\frac{\mathfrak{j}_2^{-1}(\kappa(D_2''))}{\mathfrak{j}_2^{-1}(\kappa(D_2''))\cap\mathrm{Im}(D_1)}}\eta\Big). \tag{4.37}
$$

In addition to the choices of y_1' and x_2'' specified in (4.35) and (4.36), we take

$$
\begin{aligned}
x_1''&=\lambda(\mathfrak{j}_1)u_1\wedge_{\kappa(D_1'')}v_1, &\quad \lambda(\mathfrak{j}_2)y_1''&=v''\wedge_{\mathfrak{c}(D_2')}w'',\\
x_2'&=\lambda(D_1')u'\wedge_{\kappa(D_2')}v', & y_2'&=\lambda(\delta_2)v_2\wedge_{\mathfrak{c}(D_2')}w_2.
\end{aligned} \tag{4.38}
$$

By (4.33), the last two equations in (4.35), the first equation in (4.36), (4.37), and Corollary 4.4,

$$\begin{aligned}
\lambda(\mathfrak{i}_2)w_1 \wedge_{\mathfrak{c}(D_1)} y_1'' &= v \wedge_{\mathfrak{c}(D_1)} w,\\
\lambda(\mathfrak{i}_2)x_2' \wedge_{\lambda(D_2)} u_2 &= (-1)^{\langle w'\rangle\langle u_2\rangle}\lambda(D_1)u \wedge_{\kappa(D_2)} v,\\
\lambda(D_2')w' \wedge_{\mathfrak{c}(D_2'D_1')} y_2' &= (-1)^{\langle v\rangle\langle v_{12}\rangle}\lambda(\delta_{12})v_{12} \wedge_{\mathfrak{c}(D_2'D_1')} w_{12},\\
x_1'' \wedge_{\kappa(D_2''D_1'')} u'' &= (-1)^{\langle u''\rangle\langle w_1\rangle}\lambda(\mathfrak{j}_1)u_{12} \wedge_{\kappa(D_2''D_1'')} v_{12};
\end{aligned} \tag{4.39}$$

the third identity above also uses

$$\lambda(D_2') = \lambda(\mathfrak{i}_3^{-1}\circ D_2)\circ\lambda(\mathfrak{i}_2), \quad \lambda(\delta_2) = \lambda(\mathfrak{i}_3^{-1}\circ D_2)\circ\lambda(\mathfrak{j}_2)^{-1},$$
$$\lambda(\delta_{12}) = \lambda(\mathfrak{i}_3^{-1}\circ D_2)\circ\lambda(D_1\circ\mathfrak{j}_1^{-1}).$$

By (4.10), the second equality in the first identity in (4.35), the first equality in the last identity in (4.36), the first and last equations in (4.38), (4.22), and the first two equations in (4.39), the image of

$$x_1' \otimes y_1'^* \otimes x_1'' \otimes (\lambda(\mathfrak{j}_2)y_1'')^* \otimes x_2' \otimes y_2'^* \otimes x_2'' \otimes (\lambda(\mathfrak{j}_3)y_2'')^*$$
$$\in \lambda(D_1') \otimes \lambda(D_1'') \otimes \lambda(D_2') \otimes \lambda(D_2'')$$

under $\widetilde{C}_{D_1,D_2} \circ \Psi_{\mathfrak{t}_1} \otimes \Psi_{\mathfrak{t}_2}$ is

$$(-1)^{\epsilon_{\mathrm{R}}+\langle w'\rangle\langle u_2\rangle}((\lambda(\mathfrak{i}_1)x_1' \wedge_{\kappa(D_1)} u_1) \wedge_{\kappa(D_2D_1)} u)$$
$$\otimes (\lambda(D_2)w \wedge_{\mathfrak{c}(D_2D_1)} (\lambda(\mathfrak{i}_3)w_2 \wedge_{\mathfrak{c}(D_2)} y_2''))^*.$$

By (4.22), the first equality in the first identity in (4.35), the second equality in the last identity in (4.36), the second and third equations in (4.38), (4.10), and the last two equations in (4.39), the image of this element under the isomorphism $\Psi_{\mathfrak{t}_{12}} \circ \widetilde{\mathscr{C}}_{D_1',D_2'} \otimes \widetilde{\mathscr{C}}_{D_1'',D_2''} \circ \mathrm{id} \otimes R \otimes \mathrm{id}$ is

$$(-1)^{\epsilon_{\mathrm{L}}+\langle v\rangle\langle v_{12}\rangle+\langle u''\rangle\langle w_1\rangle}(\lambda(\mathfrak{i}_1)(x_1' \wedge_{\kappa(D_2'D_1')} u') \wedge_{\kappa(D_2D_1)} u_{12})$$
$$\otimes \Big(\lambda(\mathfrak{i}_3)w_{12} \wedge_{\mathfrak{c}(D_2D_1)} (\lambda(D_2)\widetilde{w}'' \wedge_{\frac{X_3}{\mathrm{Im}(\mathfrak{i}_3)+\mathrm{Im}(D_2D_1)}} y_2'')\Big)^*.$$

By (4.34) and (4.30)-(4.32), these two elements of $\lambda(D_2D_1)$ are the same, which establishes the claim.

4.3. Stabilizations of Fredholm operators

We now describe stabilizations of Fredholm operators which are used to topologize determinant line bundles in the next section. In this subsection, we use them to deduce the Exact Squares property on page 214 from Lemma 4.3.

For any Banach vector space X, $N \in \mathsf{Z}^{\geq 0}$, and homomorphism $\Theta\colon \mathsf{R}^N \to Y$, let

$$\iota_{X;N}\colon X \longrightarrow X \oplus \mathsf{R}^N, \quad D_\Theta\colon X \oplus \mathsf{R}^N \longrightarrow Y, \text{ and } \hat{\mathscr{I}}_{\Theta;D}\colon \lambda(D) \longrightarrow \lambda(D_\Theta)$$

be as in Section 3. Since $D = D_\Theta \circ \iota_{X;N}$ and the projection $\pi_2\colon X \oplus \mathsf{R}^N \to \mathsf{R}^N$ identifies $\mathfrak{c}(\iota_{X;N})$ with R^N, (4.22) gives rise to the isomorphism
(4.40)

$$\mathscr{I}_{\Theta;D}\colon \lambda(D_\Theta) \longrightarrow \lambda(D), \quad \mathscr{I}_{\Theta;D}(\sigma) = \tilde{\mathscr{C}}_{\iota_{X;N},D_\Theta}(1 \otimes (\Omega_N^* \circ \lambda(\pi_2)) \otimes \sigma),$$

where Ω_N is the standard volume tensor on R^N as before.

LEMMA 4.11. *Let X and Y be Banach vector spaces. For any homomorphism $\Theta\colon \mathsf{R}^N \to Y$ and $D \in \mathscr{F}(X, Y)$,*

$$\mathscr{I}_{\Theta;D} \circ \hat{\mathscr{I}}_{\Theta;D} = (-1)^{(\operatorname{ind} D)N}\, \mathrm{id}_{\lambda(D)}\,.$$

PROOF. Let

$$\delta\colon \kappa(D_\Theta) \longrightarrow \mathfrak{c}(\iota_{X;N}) \overset{\pi_2}{\approx} \mathsf{R}^N \quad \text{and} \quad \hat{\delta}\colon \mathsf{R}^N \longrightarrow \mathfrak{c}(D)$$

be the connecting homomorphisms in the exact sequences (4.21) and (4.6) corresponding to the composition $D = D_\Theta \circ \iota_{X;N}$ and the exact triple (3.1), respectively. From the exact sequences

$$0 \longrightarrow \operatorname{Im}\delta \longrightarrow \mathfrak{c}(\iota_{X;N}) \longrightarrow \mathfrak{c}(D) \longrightarrow \mathfrak{c}(D_\Theta) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im}\hat{\delta} \longrightarrow \mathfrak{c}(D) \longrightarrow \mathfrak{c}(D_\Theta) \longrightarrow 0,$$

we find that

$$\mathfrak{d}(\mathfrak{c}(D_\Theta)) = \mathfrak{d}(\mathfrak{c}(D)) - N + \mathfrak{d}(\operatorname{Im}\delta) = \mathfrak{d}(\mathfrak{c}(D)) - \mathfrak{d}(\operatorname{Im}\hat{\delta}),$$

(4.41)
$$\epsilon \cong \hat{\epsilon} + (\operatorname{ind} D)N \mod 2,$$

where ϵ and $\hat{\epsilon}$ are the sign exponents in the equations (4.22) and (4.10) corresponding to the composition $D = D_\Theta \circ \iota_{X;N}$ and the exact triple (3.1), respectively.

In order to describe the maps $\mathscr{I}_{\Theta;D}$ and $\hat{\mathscr{I}}_{\Theta;D}$, we choose

$$x \in \lambda(\kappa(D)) - 0, \qquad y \in \lambda\left(\frac{Y}{\operatorname{Im} D + \operatorname{Im}\Theta}\right) - 0,$$

$$v \in \lambda\left(\frac{\kappa(D_\Theta)}{\iota_{X;N}(\kappa(D))}\right), \quad w \in \lambda\left(\frac{X \oplus \mathsf{R}^N}{\kappa(D_\Theta) + X \oplus 0}\right)$$

such that

$$\lambda(\pi_2)\Big(v \wedge_{\frac{X\oplus \mathsf{R}^N}{X\oplus 0}} w\Big) = \Omega_N.$$

From (4.22) and (4.10), we find that

(4.42)
$$\begin{aligned}\mathscr{I}_{\Theta;D}\big((\lambda(\iota_{X;N})x \wedge_{\kappa(D_\Theta)} v)\otimes y^*\big) &= (-1)^{\epsilon} x \otimes \big(\lambda(D_\Theta)w \wedge_{\mathfrak{c}(D)} y\big)^*,\\ \hat{\mathscr{I}}_{\Theta;D}\big(x\otimes(\lambda(D_\Theta)w\wedge_{\mathfrak{c}(D)} y)^*\big) &= (-1)^{\hat{\epsilon}}\big(\lambda(\iota_{X;N})x\wedge_{\kappa(D_\Theta)} v\big)\otimes y^*.\end{aligned}$$

Combining this with (4.41), we obtain the claim.

For any exact triple $\mathfrak{s}$ of vector-space homomorphisms of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{R}^{N'} & \xrightarrow{\ \mathfrak{i}\ } & \mathsf{R}^{N} & \xrightarrow{\ \mathfrak{j}\ } & \mathsf{R}^{N''} & \longrightarrow & 0\\ & & \big\downarrow{\scriptstyle \Theta'} & & \big\downarrow{\scriptstyle \Theta} & & \big\downarrow{\scriptstyle \Theta''} & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\ \mathfrak{i}_Y\ } & Y & \xrightarrow{\ \mathfrak{j}_Y\ } & Y'' & \longrightarrow & 0 \end{array} \tag{4.43}$$

we define $A_{\mathfrak{s}} \in \mathsf{R}^*$ by

$$\lambda(\mathfrak{i})\Omega_{N'} \wedge_{\mathsf{R}^N} \Omega_{N''} = A_{\mathfrak{s}}\,\Omega_N; \tag{4.44}$$

this number of course depends only on the first row in (4.43). If $\mathfrak{t}$ is an exact triple of Fredholm operators as in (2.11), we denote by $\mathfrak{t}_{\mathfrak{s}}$ the exact triple

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X'\oplus\mathsf{R}^{N'} & \xrightarrow{\ \mathfrak{i}_X\oplus\mathfrak{i}\ } & X\oplus\mathsf{R}^{N} & \xrightarrow{\ \mathfrak{j}_X\oplus\mathfrak{j}\ } & X''\oplus\mathsf{R}^{N''} & \longrightarrow & 0\\ & & \big\downarrow{\scriptstyle D'_{\Theta'}} & & \big\downarrow{\scriptstyle D_{\Theta}} & & \big\downarrow{\scriptstyle D''_{\Theta''}} & & \\ 0 & \longrightarrow & Y' & \xrightarrow{\ \mathfrak{i}_Y\ } & Y & \xrightarrow{\ \mathfrak{j}_Y\ } & Y' & \longrightarrow & 0 \end{array}$$

of Fredholm operators.

LEMMA 4.12. *For every exact triple* $\mathfrak{t}$ *of Fredholm operators as in* (2.11) *and for every exact triple* $\mathfrak{s}$ *of homomorphisms as in* (4.43)*, the diagram*

$$\begin{array}{ccc} \lambda(D'_{\Theta'})\otimes\lambda(D''_{\Theta''}) & \xrightarrow{\qquad \Psi_{\mathfrak{t}_{\mathfrak{s}}} \qquad} & \lambda(D_\Theta)\\ \big\downarrow{\scriptstyle \mathscr{I}_{\Theta';D'}\otimes\mathscr{I}_{\Theta'';D''}} & & \big\downarrow{\scriptstyle \Theta''}\\ \lambda(D')\otimes\lambda(D'') & \xrightarrow{\ (-1)^{(\operatorname{ind} D')N''} A_{\mathfrak{s}}\Psi_{\mathfrak{t}}\ } & \lambda(D) \end{array}$$

commutes.

PROOF. By our assumptions, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \xrightarrow{\ \mathfrak{i}_X\ } & X & \xrightarrow{\ \mathfrak{j}_X\ } & X'' & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \iota_{X';N'}} & & \Big\downarrow{\scriptstyle \iota_{X;N}} & & \Big\downarrow{\scriptstyle \iota_{X'';N''}} & & \\
0 & \longrightarrow & X' \oplus \mathsf{R}^{N'} & \xrightarrow{\ \mathfrak{i}_X \oplus \mathfrak{i}\ } & X \oplus \mathsf{R}^{N} & \xrightarrow{\ \mathfrak{j}_X \oplus \mathfrak{j}\ } & X'' \oplus \mathsf{R}^{N''} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle D'_{\Theta'}} & & \Big\downarrow{\scriptstyle D_{\Theta}} & & \Big\downarrow{\scriptstyle D''_{\Theta''}} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\ \mathfrak{i}_Y\ } & Y & \xrightarrow{\ \mathfrak{j}_Y\ } & Y' & \longrightarrow & 0
\end{array}$$

commutes. By (4.10) applied to the exact triple $\mathfrak{t}_{\mathrm{T}}$ in the top half of this diagram and (4.44),

$$\Psi_{\mathfrak{t}_{\mathrm{T}}}\big(1\otimes(\Omega^*_{N'}\circ\lambda(\pi_2'))\otimes 1\otimes(\Omega^*_{N''}\circ\lambda(\pi_2''))\big) = A_{\mathfrak{s}}^{-1}(-1)^{N'N''}\, 1\otimes(\Omega^*_{N}\circ\lambda(\pi_2)),$$

where $\pi_2^\star\colon X^\star \oplus \mathsf{R}^{N^\star} \to \mathsf{R}^{N^\star}$ is the projection on the second component and $\star = {}', {}''$ or blank. Thus, the claim follows from Proposition 4.10 applied to the above diagram, along with (4.40).

COROLLARY 4.13 (Exact Squares). *For every commutative diagram* (2.26) *of exact rows and columns of Fredholm operators, the corresponding diagram* (2.27) *of graded lines commutes as well.*

PROOF. We augment the domains in (2.26) by a commuting grid of finite-dimensional vector spaces, obtaining a version of the commutative diagram (2.26) with surjective Fredholm operators; the conclusion of this corollary holds for such a diagram by Lemma 4.3. The diagrams (2.27) corresponding to the original and new diagrams (2.26) are related by Lemma 4.12. This gives rise to a cube of commuting diagrams; see Figure 9. We put the new version of (2.27) on the back face and the diagrams arising from Lemma 4.12 on the top, right, bottom, and left faces; this forces signs on each edge of the front face in order to make the last four diagrams commute. The resulting sign distribution on the edges of the front face may be different from the sign distribution (no signs) on the original version of (2.27). However, by Lemma 4.3, the two sign distributions are equivalent at least if the original diagram consists of surjective Fredholm operators. Since the signs involve only the dimensions of the supplementary finite-dimensional vector spaces and the indices of the Fredholm operators (not the dimensions of their kernels or cokernels), it follows that the two signs distributions are equivalent in all cases; this establishes Corollary 4.13.

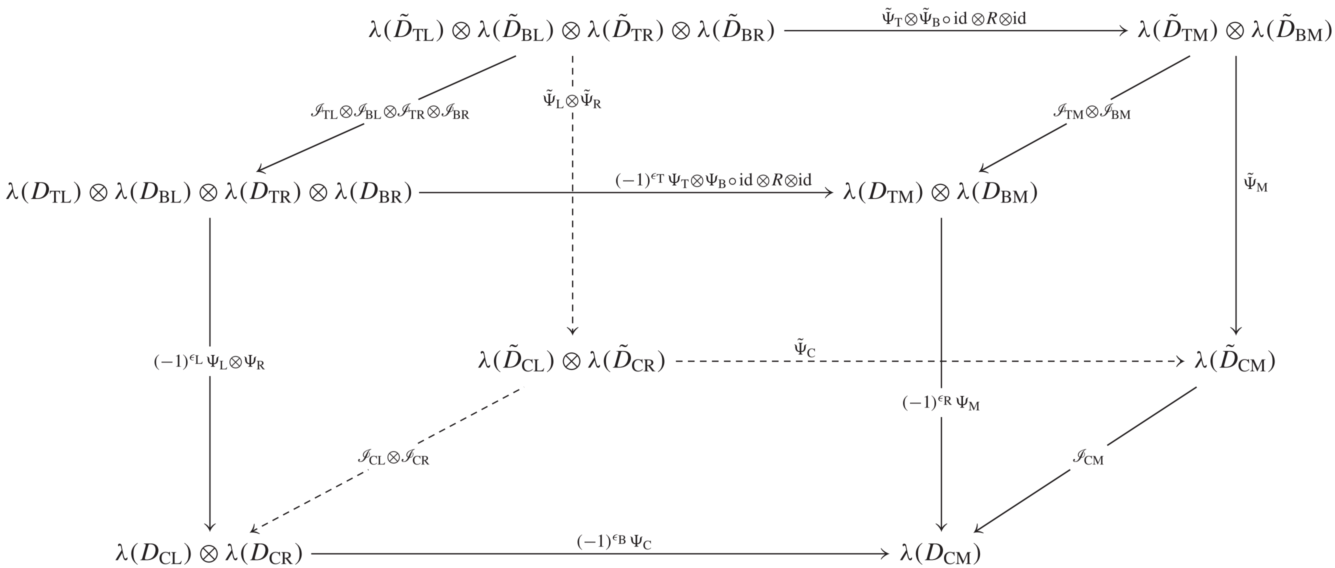


FIGURE 9. The cube of commutative diagrams used in the proof of Corollary 4.13, where $\tilde{D}_{\star *} = (D_{\star *})_{\Theta_{\star *}}$, $\mathscr{I}_{\star *} = \mathscr{I}_{\Theta_{\star *}; D_{\star *}}$, and $\tilde{\Psi}_{\star}$ are the isomorphisms (2.20) corresponding to the **t**op, **c**enter, and **b**ottom rows and **l**eft, **m**iddle, and **r**ight columns of the regularized version of the diagram (2.26) described in the proof

We denote the range of the operator $D_{\star *}$ by $Y_{\star *}$. Let

$$\Theta_{\mathrm{TL}}\colon \mathsf{R}^{N_{\mathrm{TL}}} \longrightarrow Y_{\mathrm{TL}}, \quad \tilde{\Theta}_{\mathrm{TR}}\colon \mathsf{R}^{N_{\mathrm{TR}}} \longrightarrow Y_{\mathrm{TM}},$$
$$\tilde{\Theta}_{\mathrm{BL}}\colon \mathsf{R}^{N_{\mathrm{BL}}} \longrightarrow Y_{\mathrm{CL}}, \quad \tilde{\Theta}_{\mathrm{BR}}\colon \mathsf{R}^{N_{\mathrm{BR}}} \longrightarrow Y_{\mathrm{CM}},$$

be homomorphisms such that

$$(4.45) \quad \mathfrak{c}((D_{\mathrm{TL}})_{\Theta_{\mathrm{TL}}}), \mathfrak{c}((D_{\mathrm{TR}})_{\mathfrak{j}_{\mathrm{T}}\circ\tilde{\Theta}_{\mathrm{TR}}}), \mathfrak{c}((D_{\mathrm{BL}})_{\mathfrak{j}_{\mathrm{L}}\circ\tilde{\Theta}_{\mathrm{BL}}}), \mathfrak{c}((D_{\mathrm{BR}})_{\mathfrak{j}_{\mathrm{R}}\circ\mathfrak{j}_{\mathrm{C}}\circ\tilde{\Theta}_{\mathrm{BR}}}) = 0.$$

Let

$$N_{\mathrm{TM}} = N_{\mathrm{TL}} + N_{\mathrm{TR}}, \quad N_{\mathrm{CL}} = N_{\mathrm{TL}} + N_{\mathrm{BL}}, \quad N_{\mathrm{CR}} = N_{\mathrm{TR}} + N_{\mathrm{BR}},$$
$$N_{\mathrm{BM}} = N_{\mathrm{BL}} + N_{\mathrm{BR}}, \quad N_{\mathrm{CM}} = N_{\mathrm{CL}} + N_{\mathrm{CR}} = N_{\mathrm{TM}} + N_{\mathrm{BM}}.$$

We define $\Theta_{\star *}\colon \mathsf{R}^{N_{\star *}} \to Y_{\star *}$ for $(\star, *) \in \{\mathrm{T}, \mathrm{C}, \mathrm{B}\} \times \{\mathrm{L}, \mathrm{M}, \mathrm{R}\} - \{(\mathrm{T}, \mathrm{L})\}$ by

$$\begin{aligned}
\Theta_{\mathrm{TR}} &= \mathfrak{j}_{\mathrm{T}} \circ \tilde{\Theta}_{\mathrm{TR}},\\
\Theta_{\mathrm{BL}} &= \mathfrak{j}_{\mathrm{L}} \circ \tilde{\Theta}_{\mathrm{BL}},\\
\Theta_{\mathrm{BR}} &= \mathfrak{j}_{\mathrm{R}} \circ \mathfrak{j}_{\mathrm{C}} \circ \tilde{\Theta}_{\mathrm{BR}} = \mathfrak{j}_{\mathrm{B}} \circ \mathfrak{j}_{\mathrm{M}} \circ \tilde{\Theta}_{\mathrm{BR}},\\
\Theta_{\mathrm{TM}}(x_{\mathrm{TL}}, x_{\mathrm{TR}}) &= \mathfrak{i}_{\mathrm{T}}(\Theta_{\mathrm{TL}}(x_{\mathrm{TL}})) + \tilde{\Theta}_{\mathrm{TR}}(x_{\mathrm{TR}}),\\
\Theta_{\mathrm{CL}}(x_{\mathrm{TL}}, x_{\mathrm{BL}}) &= \mathfrak{i}_{\mathrm{L}}(\Theta_{\mathrm{TL}}(x_{\mathrm{TL}})) + \tilde{\Theta}_{\mathrm{BL}}(x_{\mathrm{BL}}),\\
\Theta_{\mathrm{CR}}(x_{\mathrm{TR}}, x_{\mathrm{BR}}) &= \mathfrak{i}_{\mathrm{R}}(\Theta_{\mathrm{TR}}(x_{\mathrm{TR}})) + \mathfrak{j}_{\mathrm{C}}(\tilde{\Theta}_{\mathrm{BR}}(x_{\mathrm{BR}})),\\
\Theta_{\mathrm{BM}}(x_{\mathrm{BL}}, x_{\mathrm{BR}}) &= \mathfrak{i}_{\mathrm{B}}(\Theta_{\mathrm{BL}}(x_{\mathrm{BL}})) + \mathfrak{j}_{\mathrm{M}}(\tilde{\Theta}_{\mathrm{BR}}(x_{\mathrm{BR}})),\\
\Theta_{\mathrm{CM}}(x_{\mathrm{TL}}, x_{\mathrm{TR}}, x_{\mathrm{BL}}, x_{\mathrm{BR}}) &= \mathfrak{i}_{\mathrm{M}}(\Theta_{\mathrm{TM}}(x_{\mathrm{TL}}, x_{\mathrm{TR}}))\\
&\qquad + \mathfrak{i}_{\mathrm{C}}(\tilde{\Theta}_{\mathrm{BL}}(x_{\mathrm{BL}})) + \tilde{\Theta}_{\mathrm{BR}}(x_{\mathrm{BR}})
\end{aligned}$$

for all $x_{\star *} \in \mathsf{R}^{N_{\star *}}$ with $(\star, *) \in \{\mathrm{T}, \mathrm{B}\} \times \{\mathrm{L}, \mathrm{R}\}$. For any $N' \leq N$, we denote by $\mathfrak{i}\colon \mathsf{R}^{N'} \to \mathsf{R}^{N}$ and $\mathfrak{j}\colon \mathsf{R}^{N} \to \mathsf{R}^{N'}$ the inclusion as $\mathsf{R}^{N'} \oplus 0^{N-N'}$ and the projection onto the last N' coordinates, respectively. We also define

$$\mathfrak{i}'\colon \mathsf{R}^{N_{\mathrm{CL}}} \longrightarrow \mathsf{R}^{N_{\mathrm{CM}}}, \quad \mathfrak{i}'(x_{\mathrm{TL}}, x_{\mathrm{BL}}) = (x_{\mathrm{TL}}, 0, x_{\mathrm{BL}}, 0),$$
$$\mathfrak{j}'\colon \mathsf{R}^{N_{\mathrm{CM}}} \longrightarrow \mathsf{R}^{N_{\mathrm{CR}}}, \quad \mathfrak{i}'(x_{\mathrm{TL}}, x_{\mathrm{TR}}, x_{\mathrm{BL}}, x_{\mathrm{BR}}) = (x_{\mathrm{TR}}, x_{\mathrm{BR}}),$$

for all $x_{\star *} \in \mathsf{R}^{N_{\star *}}$. In particular, the diagram in Figure 10 commutes and its 6 rows and 6 columns are exact.

By the commutativity and exactness properties of the diagram in Figure 10, the diagram (2.26) with $D_{\star *}$ replaced by $\tilde{D}_{\star *} \equiv (D_{\star *})_{\Theta_{\star *}}$, $\mathfrak{i}_{\mathrm{C}}\colon X_{\mathrm{CL}} \to X_{\mathrm{CM}}$ and $\mathfrak{j}_{\mathrm{C}}\colon X_{\mathrm{CM}} \to X_{\mathrm{CR}}$ replaced by

$$\mathfrak{i}_{\mathrm{C}} \oplus \mathfrak{i}'\colon X_{\mathrm{CL}} \oplus \mathsf{R}^{N_{\mathrm{CL}}} \longrightarrow X_{\mathrm{CM}} \oplus \mathsf{R}^{N_{\mathrm{CM}}}$$

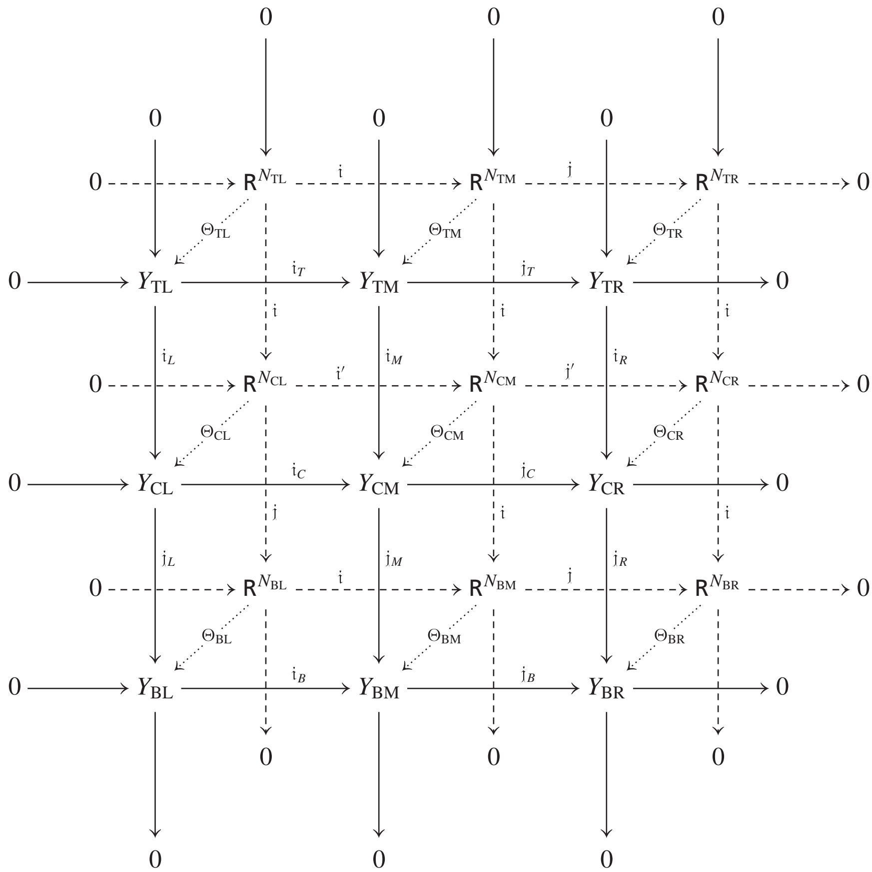


FIGURE 10. The panel of commutative diagrams, with exact rows and columns, used in the proof of Corollary 4.13 to regularize the square grid (2.26)

and

$$\mathfrak{j}_C \oplus \mathfrak{j}' \colon X_{\mathrm{CM}} \oplus \mathsf{R}^{N_{\mathrm{CM}}} \longrightarrow X_{\mathrm{CR}} \oplus \mathsf{R}^{N_{\mathrm{CR}}},$$

respectively, and the remaining homomorphisms $\mathfrak{i}_\star$ and $\mathfrak{j}_\star$ on $X_{*\circ}$ by $\mathfrak{i}_\star \oplus \mathfrak{i}$ and $\mathfrak{j}_\star \oplus \mathfrak{j}$ on $X_{*\circ} \oplus \mathsf{R}^{N_{*\circ}}$, respectively, still commutes and its 3 rows and 3 columns are still exact. Thus, by (4.45), the Normalization II property on page 212, and Lemma 4.3, the diagram on the back face of the cube in Figure 9 commutes. Let $\mathscr{I}_{\star*} = \mathscr{I}_{\Theta_{\star*};D_{\star*}}$ be the isomorphisms defined by (4.40) and

$$\begin{gathered}\epsilon_{\mathrm{T}} = N_{\mathrm{TR}} N_{\mathrm{BL}} + (\operatorname{ind} D_{\mathrm{CL}}) N_{\mathrm{TR}} + (\operatorname{ind} D_{\mathrm{TR}}) N_{\mathrm{BL}} + (\operatorname{ind} D_{\mathrm{BL}}) N_{\mathrm{BR}},\\ \epsilon_{\mathrm{R}} = (\operatorname{ind} D_{\mathrm{TM}}) N_{\mathrm{BM}}, \quad \epsilon_{\mathrm{B}} = N_{\mathrm{TR}} N_{\mathrm{BL}} + (\operatorname{ind} D_{\mathrm{CL}}) N_{\mathrm{CR}},\\ \epsilon_{\mathrm{L}} = (\operatorname{ind} D_{\mathrm{TL}}) N_{\mathrm{BL}} + (\operatorname{ind} D_{\mathrm{TR}}) N_{\mathrm{BR}}.\end{gathered}$$

By the commutativity of the 3 pairs of exact rows and 3 pairs of exact columns in Figure 10 and Lemma 4.12, the diagrams on the top, right, bottom, and left faces in Figure 9 commute. This implies that the diagram on the front face of Figure 9 commutes as well. By Lemma 4.3,

$$\epsilon_{\mathrm{T}} + \epsilon_{\mathrm{R}} + \epsilon_{\mathrm{B}} + \epsilon_{\mathrm{L}} \in 2\mathbb{Z} \tag{4.46}$$

if $\mathfrak{c}(D_{\star *}) = \{0\}$ for $(\star, *) \in \{\mathrm{T}, \mathrm{B}\} \times \{\mathrm{L}, \mathrm{R}\}$. Thus, (4.46) always holds (which can also be checked directly), which establishes Corollary 4.13 in all cases.

5. Topology

It remains to topologize each set $\det_{X,Y}$ as a line bundle over $\mathscr{F}(X, Y)$ so that the Normalization I property in Section 2 holds and the fiberwise homomorphisms (4.10) give rise to continuous maps.

5.1. Continuity of overlap and exact triple maps

For Banach vector spaces X, Y, X', Y', X'', Y'', let

$$\mathscr{T}^*(X, Y; X', Y'; X'', Y'') \subset \mathscr{T}(X, Y; X', Y'; X'', Y'')$$

denote the subspace of short exact sequences as in (2.11) with surjective Fredholm operators D, D', D''.

LEMMA 5.1. *Let X, Y, X', Y', X'', Y'' be Banach vector spaces. The family of maps $\Psi_{\mathfrak{t}}$ given by* (4.10) *induces a continuous bundle map*

$$\Psi\colon \pi_{\mathrm{L}}^* \det_{X',Y'} \otimes \pi_{\mathrm{R}}^* \det_{X'',Y''} \longrightarrow \pi_{\mathrm{C}}^* \det_{X,Y}$$

over $\mathscr{T}^*(X, Y; X', Y'; X'', Y'')$.

PROOF. We abbreviate $\mathscr{T}^*(X, Y; X', Y'; X'', Y'')$ as $\mathscr{T}^*$. Let $\mathfrak{t}_0 \in \mathscr{T}^*$ be as in (2.11), with all seven homomorphisms carrying subscript 0, and $T\colon Y \to X$, $T'\colon Y' \to X'$, and $T''\colon Y'' \to X''$ be right inverses for D_0, D_0', and D_0'', respectively. For each $\mathfrak{t}$ as in (2.11) sufficiently close to $\mathfrak{t}_0$ and $\star = {}', {}''$ or blank, let

$$\Phi_{D_0^\star;\mathfrak{t}}\colon \kappa(D^\star) \longrightarrow \kappa(D_0^\star)$$

be as in (2.5); this map depends on the choice of $T^\star$. We need to show that the map

$$\Psi_{\mathfrak{t}_0;\mathfrak{t}}\colon \lambda(\kappa(D_0')) \otimes \lambda(\kappa(D_0'')) \longrightarrow \lambda(\kappa(D_0)) \tag{5.1}$$

described by

$$\Psi_{\mathfrak{t}}\big(\lambda(\Phi_{D_0';\mathfrak{t}}^{-1})x' \otimes 1^* \otimes \lambda(\Phi_{D_0'';\mathfrak{t}}^{-1})x'' \otimes 1^*\big) = \big(\lambda(\Phi_{D_0;\mathfrak{t}}^{-1})\Psi_{\mathfrak{t}_0;\mathfrak{t}}(x' \otimes x'')\big) \otimes 1^*$$

depends continuously on $\mathfrak{t} \in \mathscr{T}^*$ near $\mathfrak{t}_0$.

Since any homomorphism D' or D sufficiently close to D'_0 or D_0 is surjective, the sign in (4.10) is $+1$. Let $x'_1, \ldots, x'_k$ be a basis for $\kappa(D'_0)$ and $x_1, \ldots, x_\ell \in \kappa(D_0)$ be such that $\mathfrak{j}_{X;0}(x_1), \ldots, \mathfrak{j}_{X;0}(x_\ell)$ is a basis for $\kappa(D''_0)$. If $\mathfrak{t}$ as in (2.11) is sufficiently close to $\mathfrak{t}_0$, $\Phi^{-1}_{D'_0;\mathfrak{t}}(x'_1), \ldots, \Phi^{-1}_{D'_0;\mathfrak{t}}(x'_k)$ is a basis for $\kappa(D')$ and

$$\mathfrak{j}_X(\Phi^{-1}_{D_0;\mathfrak{t}}(x_1)), \ldots, \mathfrak{j}_X(\Phi^{-1}_{D_0;\mathfrak{t}}(x_\ell)) \in X''$$

is a basis for $\kappa(D'')$. In particular,

$$\begin{aligned}
&\mathfrak{j}_X(\Phi^{-1}_{D_0;\mathfrak{t}}(x_1)) \wedge \ldots \wedge \mathfrak{j}_X(\Phi^{-1}_{D_0;\mathfrak{t}}(x_\ell)) \\
&= f(\mathfrak{t})\, \Phi^{-1}_{D''_0;\mathfrak{t}}(\mathfrak{j}_{X;0}(x_1)) \wedge \ldots \wedge \Phi^{-1}_{D''_0;\mathfrak{t}}(\mathfrak{j}_{X;0}(x_\ell)), \\
&\mathfrak{i}_X(\Phi^{-1}_{D'_0;\mathfrak{t}}(x'_1)) \wedge \ldots \wedge \mathfrak{i}_X(\Phi^{-1}_{D'_0;\mathfrak{t}}(x'_k)) \wedge \Phi^{-1}_{D_0;\mathfrak{t}}(x_1) \wedge \ldots \wedge \Phi^{-1}_{D_0;\mathfrak{t}}(x_\ell) \\
&= g(\mathfrak{t})\, \Phi^{-1}_{D_0;\mathfrak{t}}(\mathfrak{i}_{X;0}(x'_1)) \wedge \ldots \wedge \Phi^{-1}_{D_0;\mathfrak{t}}(\mathfrak{i}_{X;0}(x'_k)) \wedge \Phi^{-1}_{D_0;\mathfrak{t}}(x_1) \wedge \ldots \wedge \Phi^{-1}_{D_0;\mathfrak{t}}(x_\ell)
\end{aligned}$$

for some $\mathbb{R}^+$-valued continuous functions f and g. The homomorphism (5.1) is given by

$$\begin{aligned}
\Psi_{\mathfrak{t}_0;\mathfrak{t}}\big((x'_1 \wedge \ldots \wedge x'_k) \otimes (\mathfrak{j}_{X;0}(x_1) \wedge \ldots \wedge \mathfrak{j}_{X;0}(x_\ell))\big) \\
= \frac{g(\mathfrak{t})}{f(\mathfrak{t})}\, \mathfrak{i}_{X;0}(x'_1) \wedge \ldots \wedge \mathfrak{i}_{X;0}(x'_k) \wedge x_1 \wedge \ldots \wedge x_\ell
\end{aligned}$$

and thus is continuous.

COROLLARY 5.2. *Let X and Y be Banach vector spaces. For any homomorphism $\Theta: \mathbb{R}^N \to Y$, the family of isomorphisms $\mathscr{I}_{\Theta;D}$ given by* (4.40) *induces a continuous bundle map*

$$\mathscr{I}_\Theta: \iota^*_\Theta \det_{X\oplus\mathbb{R}^N,Y} \longrightarrow \det_{X,Y}$$

over $\mathscr{F}^*(X,Y)$.

PROOF. By Lemma 4.11, $\mathscr{I}_{\Theta;D}$ is the inverse of the isomorphism $\pm\hat{\mathscr{I}}_{\Theta,D}$ given by (3.2). By Lemma 5.1, the family of isomorphisms $\hat{\mathscr{I}}_{\Theta;D}$ induce a continuous bundle map

$$\hat{\mathscr{I}}_\Theta: \det_{X,Y} \longrightarrow \iota^*_\Theta \det_{X\oplus\mathbb{R}^N,Y}$$

over $\mathscr{F}^*(X,Y)$. This implies the claim.

Let X and Y be as above. The subsets

$$U_{X;\Theta} \equiv \{D \in \mathscr{F}(X,Y) : \mathfrak{c}(D_\Theta) = 0\}$$

form an open cover of $\mathscr{F}(X, Y)$ as Θ ranges over all homomorphisms $\mathsf{R}^N \to Y$ and N ranges over all nonnegative integers. We topologize $\det_{X,Y}|_{U_{X;\Theta}}$ by requiring the bundle isomorphism

$$\iota_\Theta^* \det_{X\oplus\mathsf{R}^N,Y} \longrightarrow \det_{X,Y}, \quad \sigma \longrightarrow \mathscr{I}_{\Theta;D}(\sigma) \quad \forall\, \sigma \in \lambda(D_\Theta),\, D \in \mathscr{F}(X,Y),$$

to be a homeomorphism over $U_{X;\Theta}$ with respect to the topology on the domain induced by the topology on $\det_{X\oplus\mathsf{R}^N,Y}|_{\mathscr{F}^*(X\oplus\mathsf{R}^N,Y)}$ described at the beginning of this section. We next show that this topology is well-defined.

Proposition 5.3 (Continuity of transition maps). *Let X and Y be Banach vector spaces. For any homomorphisms $\Theta_1\colon \mathsf{R}^{N_1} \to Y$ and $\Theta_2\colon \mathsf{R}^{N_2} \to Y$, the bundle map*

$$\mathscr{I}_{\Theta_2;D}^{-1} \circ \mathscr{I}_{\Theta_1;D}\colon \iota_{\Theta_1}^* \det_{X\oplus\mathsf{R}^{N_1},Y} \longrightarrow \iota_{\Theta_2}^* \det_{X\oplus\mathsf{R}^{N_2},Y},$$

is continuous over $U_{X;\Theta_1} \cap U_{X;\Theta_2}$.

Proof. Let $N = N_1 + N_2$,

$$\iota_1, \iota_2\colon \mathsf{R}^{N_1}, \mathsf{R}^{N_2} \longrightarrow \mathsf{R}^{N_1} \oplus \mathsf{R}^{N_2} = \mathsf{R}^N$$

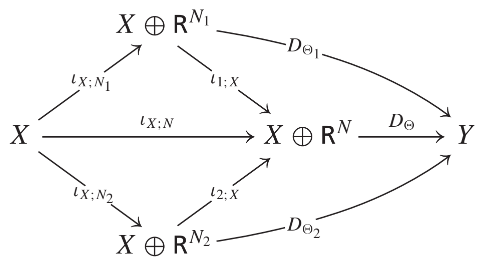
be the canonical embeddings, and

$$\iota_{k;X} = \mathrm{id}_X \oplus \iota_k\colon X \oplus \mathsf{R}^{N_k} \longrightarrow X \oplus \mathsf{R}^N$$

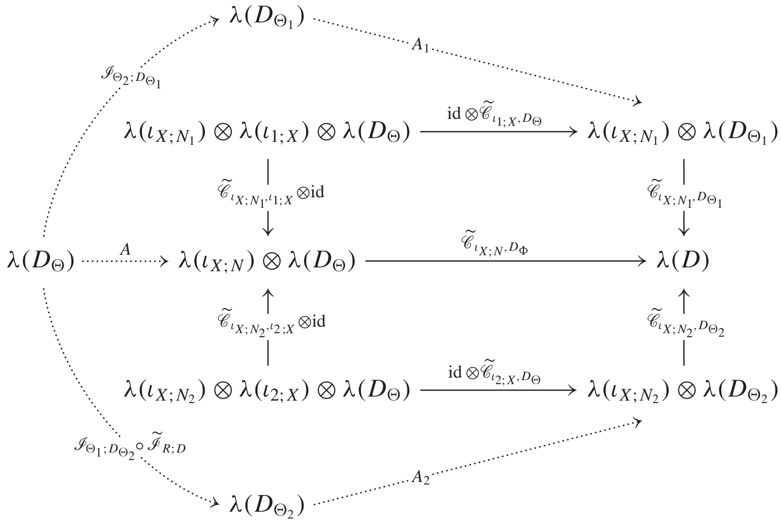
for $k = 1, 2$. Define

$$\Theta\colon \mathsf{R}^N \longrightarrow Y \qquad \text{by} \quad \Theta(u_1, u_2) = \Theta_1(u_1) + \Theta_2(u_2).$$

Thus, the diagram



commutes. By Proposition 4.8, the diagram



also commutes (excluding the dotted arrows). We define the isomorphisms A, A_1, A_2 in this diagram by

$$A(\sigma) = 1\otimes\Omega_N^*\circ\lambda(\pi_2)\otimes\sigma, \quad A_k(\sigma_k) = 1\otimes\Omega_{N_k}^*\circ\lambda(\pi_2)\otimes\sigma_k, \tag{5.2}$$

where $\pi_2\colon \mathfrak{c}(\iota_{X;N_k}) \to \mathsf{R}^{N_k}$ is the isomorphism induced by the projection map $X\oplus\mathsf{R}^{N_k}\to\mathsf{R}^{N_k}$; thus,

$$\mathscr{I}_{\Theta;D} = \widetilde{\mathscr{C}}_{\iota_{X;N},D_\Theta}\circ A, \quad \mathscr{I}_{\Theta_k;D} = \widetilde{\mathscr{C}}_{\iota_{X;N_k},D_{\Theta_k}}\circ A_k, \quad k=1,2.$$

Let $R\colon X\oplus\mathsf{R}^N \to X\oplus\mathsf{R}^N$ be the isomorphism given by

$$R(x,u_1,u_2) = (x,u_2,u_1) \quad \forall\,(x,u_1,u_2)\in X\oplus\mathsf{R}^{N_1}\oplus\mathsf{R}^{N_2}$$

and

$$\begin{aligned}
\widetilde{\mathscr{I}}_{R;D} &\equiv \widetilde{\mathscr{I}}_{R,\mathrm{id}_Y}|_{D_\Theta}\colon \lambda(D_\Theta)\longrightarrow\lambda(D_\Theta\circ R^{-1})\\
&= \lambda(\mathscr{I}_{R,\mathrm{id}_Y}(D_\Theta)),\\
\widetilde{\mathscr{I}}_{R;N_1} &\equiv \widetilde{\mathscr{I}}_{\mathrm{id}_{X\oplus\mathsf{R}^{N_2}},R^{-1}}|_{\iota_{X\oplus\mathsf{R}^{N_2};N_1}}\colon \lambda(\iota_{X\oplus\mathsf{R}^{N_2};N_1})\longrightarrow\lambda(R^{-1}\circ\iota_{X\oplus\mathsf{R}^{N_2};N_1})\\
&= \lambda(\iota_{2;X})
\end{aligned}$$

be the corresponding isomorphisms as in (2.4).

Since $\iota_{1;X} = \iota_{X\oplus\mathsf{R}^{N_1};N_2}$,

$$\begin{aligned}
&\{\mathrm{id}\otimes\widetilde{\mathscr{C}}_{\iota_{1;X;},D_\Theta}\}^{-1}(A_1(\mathscr{I}_{\Theta_2;D_{\Theta_1}}(\sigma)))\\
&\qquad\qquad = 1\otimes(\Omega_{N_1}^*\circ\lambda(\pi_2))\otimes1\otimes(\Omega_{N_2}^*\circ\lambda(\pi_{\mathsf{R};N_2}))\otimes\sigma,
\end{aligned}$$

where $\pi_{\mathsf{R};N_2}\colon \mathfrak{c}(\iota_{1;X}) \to \mathsf{R}^{N_2}$ is the isomorphism induced by the projection map $X \oplus \mathsf{R}^N \to \mathsf{R}^{N_2}$ onto the last N_2 Euclidean coordinates. Since

$$\widetilde{\mathscr{C}}_{\iota_{X;N_1},\iota_{1;X}}\big(1 \otimes (\Omega_{N_1}^* \circ \lambda(\pi_2)) \otimes 1 \otimes (\Omega_{N_2}^* \circ \lambda(\pi_{\mathrm{R};N_2}))\big) \\ = (-1)^{N_1 N_2} 1 \otimes (\Omega_N^* \circ \lambda(\pi_2))$$

by (4.22), we find that

$$\mathscr{I}_{\Theta;D}^{-1} \circ \mathscr{I}_{\Theta_1;D} = (-1)^{N_1 N_2} \mathscr{I}_{\Theta_2;D_{\Theta_1}}^{-1}. \tag{5.3}$$

On the other hand, $\iota_{2;X} = R^{-1} \circ \iota_{X\oplus\mathsf{R}^{N_2};N_1}$. By Proposition 4.8 applied to the composition $D_\Theta \circ R^{-1} \circ \iota_{X\oplus\mathsf{R}^{N_2};N_1}$ and the Naturality III property on page 212, the diagram

$$\begin{array}{ccc} \lambda(\iota_{X\oplus\mathsf{R}^{N_2};N_1}) \otimes \lambda(D_\Theta) & \xrightarrow{\widetilde{\mathscr{I}}_{R;N_1}\otimes\mathrm{id}} & \lambda(\iota_{2;X}) \otimes \lambda(D_\Theta) \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\widetilde{\mathscr{I}}_{R;D}} & & \Big\downarrow{\scriptstyle \widetilde{\mathscr{C}}_{\iota_{2;X},D_\Theta}} \\ \lambda(\iota_{X\oplus\mathsf{R}^{N_2};N_1}) \otimes \lambda(D_\Theta \circ R^{-1}) & \xrightarrow{\widetilde{\mathscr{C}}_{\iota_{X\oplus\mathsf{R}^{N_2};N_1},D_\Theta\circ R^{-1}}} & \lambda(D_{\Theta_2}) \end{array}$$

commutes. Since

$$\widetilde{\mathscr{I}}_{R;N_1}(1 \otimes (\Omega_{N_1}^* \circ \lambda(\pi_{\mathrm{R};N_1}))) = 1 \otimes (\Omega_{N_1}^* \circ \lambda(\pi_{\mathrm{L};N_1})),$$

where $\pi_{\mathrm{L};N_1}\colon \mathfrak{c}(\iota_{2;X}) \to \mathsf{R}^{N_1}$ is the isomorphism induced by the projection map $X \oplus \mathsf{R}^N \to \mathsf{R}^{N_1}$ onto the first N_1 Euclidean coordinates,

$$\{\mathrm{id}\otimes\widetilde{\mathscr{C}}_{\iota_{2;X;},D_\Theta}\}^{-1}(A_2(\mathscr{I}_{\Theta_1;D_{\Theta_2}}(\widetilde{\mathscr{I}}_{R;D}(\sigma)))) \\ = 1 \otimes (\Omega_{N_2}^* \circ \lambda(\pi_2)) \otimes 1 \otimes (\Omega_{N_1}^* \circ \lambda(\pi_{\mathrm{L};N_1})) \otimes \sigma.$$

Since

$$\begin{aligned} &\widetilde{\mathscr{C}}_{\iota_{X;N_2},\iota_{2;X}}\big(1 \otimes (\Omega_{N_2}^* \circ \lambda(\pi_2)) \otimes 1 \otimes (\Omega_{N_1}^* \circ \lambda(\pi_{\mathrm{L};N_1}))\big) \\ &\quad = (-1)^{N_1 N_2} 1 \otimes (\lambda(\pi_{\mathrm{R};N_2}^{-1})\Omega_{N_2} \wedge \lambda(\pi_{\mathrm{L};N_1}^{-1})\Omega_{N_1})^* \\ &\quad = 1 \otimes (\Omega_N^* \circ \lambda(\pi_2)) \end{aligned}$$

by (4.22), we find that

$$\mathscr{I}_{\Theta;D}^{-1} \circ \mathscr{I}_{\Theta_2;D} = \widetilde{\mathscr{I}}_{R;D}^{-1} \circ \mathscr{I}_{\Theta_1;D_{\Theta_2}}^{-1}. \tag{5.4}$$

From (5.2), (5.3), and (5.4), we conclude that

$$\mathscr{I}_{\Theta_2;D}^{-1} \circ \mathscr{I}_{\Theta_1;D} = (-1)^{N_1 N_2} \mathscr{I}_{\Theta_1;D_{\Theta_2}} \circ \widetilde{\mathscr{I}}_{R;D} \circ \mathscr{I}_{\Theta_2;D_{\Theta_1}}^{-1}.$$

The outer maps on the right-hand side above are continuous over $U_{X;\Theta_1} \cap U_{X;\Theta_2}$ by Corollary 5.2, while the middle map is continuous over $U_{X;\Theta_1} \cap U_{X;\Theta_2}$ by Lemma 5.1.

COROLLARY 5.4 (Continuity of (2.20)). *Let X, Y, X', Y', X'', Y'' be Banach vector spaces. The family of maps $\Psi_{\mathfrak{t}}$ given by* (4.10) *induces a continuous bundle map*

$$\Psi\colon \pi_{\mathrm{L}}^* \det_{X',Y'} \otimes \pi_{\mathrm{R}}^* \det_{X'',Y''} \longrightarrow \pi_{\mathrm{C}}^* \det_{X,Y}$$

over $\mathscr{T}(X, Y; X', Y'; X'', Y'')$.

PROOF. We abbreviate $\mathscr{T}(X, Y; X', Y'; X'', Y'')$ as $\mathscr{T}$. Let $\mathfrak{t}_0 \in \mathscr{T}$ be as in (2.11), with all seven homomorphisms carrying subscript 0, and

$$\Theta'\colon \mathsf{R}^{N'} \longrightarrow Y' \quad \text{and} \quad \tilde{\Theta}''\colon \mathsf{R}^{N''} \longrightarrow Y$$

be homomorphisms such that $D_0' \in U_{X';\Theta'}$ and $D_0'' \in U_{X'';\mathfrak{j}_{Y;0}\circ\tilde{\Theta}''}$. Let $N = N' + N''$, $\mathfrak{i}\colon \mathsf{R}^{N'} \to \mathsf{R}^N$ be the inclusion as $\mathsf{R}^{N'} \times 0^{N''}$, and $\mathfrak{j}\colon \mathsf{R}^{N'} \to \mathsf{R}^{N''}$ be the projection onto the last N'' coordinates. For each $\mathfrak{t} \in \mathscr{T}$ as in (2.11), define

$$\Theta_{\mathfrak{t}}\colon \mathsf{R}^N \longrightarrow X, \quad \Theta_{\mathfrak{t}}(x', x'') = \mathfrak{i}_Y(\Theta'(x')) + \tilde{\Theta}''(x'') \;\; \forall\, (x', x'') \in \mathsf{R}^{N'} \oplus \mathsf{R}^{N''},$$

and

$$\Theta_{\mathfrak{t}}''\colon \mathsf{R}^{N''} \longrightarrow X'', \quad \Theta_{\mathfrak{t}}''(x'') = \mathfrak{j}_Y(\tilde{\Theta}''(x'')) \;\; \forall\, x'' \in \mathsf{R}^{N''}.$$

Thus, the diagram $\mathfrak{s}(\mathfrak{t})$ given by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{R}^{N'} & \xrightarrow{\;\mathfrak{i}\;} & \mathsf{R}^{N} & \xrightarrow{\;\mathfrak{j}\;} & \mathsf{R}^{N''} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \Theta'} & & \big\downarrow{\scriptstyle \Theta_{\mathfrak{t}}} & & \big\downarrow{\scriptstyle \Theta_{\mathfrak{t}}''} & & \\
0 & \longrightarrow & Y' & \xrightarrow{\;\mathfrak{i}_Y\;} & Y & \xrightarrow{\;\mathfrak{j}_Y\;} & Y' & \longrightarrow & 0
\end{array}$$

commutes for every exact triple $\mathfrak{t}$ as in (2.11), and we obtain an embedding

$$\mathscr{T} \longrightarrow \mathscr{T}(X \oplus \mathsf{R}^N, Y; X' \oplus \mathsf{R}^{N'}, Y'; X'' \oplus \mathsf{R}^{N''}, Y''), \quad \mathfrak{t} \longrightarrow \mathfrak{t}_{\mathfrak{s}(\mathfrak{t})}.$$

Since $A_{\mathfrak{s}(\mathfrak{t})} = 1$, by Lemma 4.12 the diagram

$$\begin{array}{ccc}
\lambda(D'_{\Theta'}) \otimes \lambda(D''_{\Theta''_{\mathfrak{t}}}) & \xrightarrow{\;\Psi_{\mathfrak{t}_{\mathfrak{s}(\mathfrak{t})}}\;} & \lambda(D_{\Theta_{\mathfrak{t}}}) \\
\Big\downarrow{\scriptstyle \mathscr{I}_{\Theta';D'} \otimes \mathscr{I}_{\Theta''_{\mathfrak{t}};D''}} & & \Big\downarrow{\scriptstyle \mathscr{I}_{\Theta_{\mathfrak{t}};D}} \\
\lambda(D') \otimes \lambda(D'') & \xrightarrow{\;(-1)^{(\operatorname{ind} D')N''}\Psi_{\mathfrak{t}}\;} & \lambda(D)
\end{array}$$

commutes. By the definition of the topologies on the determinant line bundles, the vertical arrows in the above diagram induce continuous line-bundle isomorphisms over the open subset of $\mathscr{T}$ consisting of the exact triples $\mathfrak{t}$ as in (2.11) so that $D' \in U_{X';\Theta'}$ and $D'' \in U_{X'';\Theta''_{\mathfrak{t}}}$. By Lemma 5.1, the top arrow induces a continuous line-bundle isomorphism over the same open subset. Thus, the bottom arrow in this diagram induces a continuous line-bundle isomorphism as well.

5.2. Continuity of dualization isomorphisms

We begin by verifying the Normalization I′ property on page 215; see Lemma 5.5. This allows us to confirm the continuity of (2.36) over $\mathscr{F}^*(X, Y)$; see Lemma 5.6. The continuity of (2.34) over $\mathscr{F}(X, Y)$ then follows from the Dual Exact Triples property on page 217; see the proof of Corollary 5.7. For each $D \in \mathscr{F}(X, Y)$, let

$$q_D: Y \longrightarrow \mathfrak{c}(D), \quad y \longrightarrow y + \operatorname{Im} D,$$

be the projection map as before.

LEMMA 5.5 (Normalization I′). *Let X, Y be Banach vector spaces. For every $D_0 \in \mathscr{F}'(X, Y)$ and right inverse $S: \mathfrak{c}(D_0) \to Y$ for q_{D_0}, there exists a neighborhood $U_{D_0,S}$ of D_0 in $\mathscr{F}'(X, Y)$ so that the bundle isomorphism* (2.29) *is well-defined and continuous.*

PROOF. By the Open Mapping Theorem for Banach vector space,

$$U_{D_0,S} \equiv \{D \in \mathscr{F}'(X, Y) : Y = \operatorname{Im} D \oplus \operatorname{Im} S\}$$

is an open neighborhood of D_0 in $\mathscr{F}'(X, Y)$. Let

$$\pi_X, \pi_S: Y = \operatorname{Im} D_0 \oplus \operatorname{Im} S \longrightarrow \operatorname{Im} D_0, \ \operatorname{Im} S$$

be the projection maps and $D_0^{-1}: \operatorname{Im} D_0 \to X$ be the inverse of the isomorphism

$$D_0: X \longrightarrow \operatorname{Im} D_0, \quad x \longrightarrow D_0 x.$$

For each $D \in U_{D_0,S}$, the map

$$\psi_{D_0;D}: Y \longrightarrow Y, \quad y \longrightarrow D \circ D_0^{-1} \circ \pi_X(y) + \pi_S(y) \ \ \forall\, y \in Y,$$

is an isomorphism so that

$$D = \psi_{D;D_0} \circ D_0 \circ \mathrm{id}_X^{-1} \quad \text{and} \quad \psi_{D;D_0}(y) - \psi_{D;D_0}(S(q_{D_0}(y))) \in \operatorname{Im} D.$$

By the last property,

$$\widetilde{\mathscr{I}}_{\mathrm{id}_X,\psi_{D;D_0}} = \widetilde{\mathscr{I}}_{D_0,S;D}\colon \lambda(D_0) \longrightarrow \lambda(D).$$

Since $\psi_{D_0;D}$ depends continuously on D, the claim follows from the continuity of (2.20) and the Naturality III property.

LEMMA 5.6. *Let X, Y be Banach vector spaces. The family of maps $\widetilde{\mathscr{D}}_D$ given by* (2.36) *induces a continuous bundle map*

$$\widetilde{\mathscr{D}}\colon \det_{X,Y} \longrightarrow \mathscr{D}^* \det_{Y^*,X^*}$$

over $\mathscr{F}^*(X, Y)$.

PROOF. Let $D_0 \in \mathscr{F}^*(X, Y)$, $T\colon Y \to X$ be a right inverse for D_0, and

$$\pi_T\colon X = \kappa(D_0) \oplus \operatorname{Im} T \longrightarrow \kappa(D), \quad x \longrightarrow x - TDx \ \ \forall x \in X,$$

be the projection map. Thus, the homomorphism

$$S\colon \mathfrak{c}(D_0^*) \longrightarrow X^*, \quad \alpha + \operatorname{Im} D_0^* \longrightarrow \alpha|_{\kappa(D_0)} \circ \pi_T,$$

is a right inverse for $q_{D_0^*}$. By the Normalization I property on page 207 and Lemma 5.5, it is sufficient to show that the map

$$\widetilde{\mathscr{I}}^{-1}_{D_0^*,S;D^*} \circ \widetilde{\mathscr{D}}_D \circ \widetilde{\mathscr{I}}_{D_0,T;D-D_0}\colon \lambda(D_0) \longrightarrow \lambda(D) \longrightarrow \lambda(D^*) \longrightarrow \lambda(D_0^*)$$

depends continuously on $D \in U_{D_0,S}$. By (2.5), (2.36), and (2.28), this map is given by

$$x \otimes 1^* \longrightarrow 1 \otimes \mathscr{P}(\lambda(\mathscr{D}_{D_0})x),$$

which establishes the claim.

COROLLARY 5.7 (Continuity of (2.34)). *Let X, Y be Banach vector spaces. The family of maps $\widetilde{\mathscr{D}}_D$ given by* (4.13) *induces a continuous bundle map*

$$\widetilde{\mathscr{D}}\colon \det_{X,Y} \longrightarrow \mathscr{D}^* \det_{Y^*,X^*}$$

over $\mathscr{F}(X, Y)$.

PROOF. Let $D \in \mathscr{F}(X, Y)$ and $\Theta\colon \mathsf{R}^N \to Y$ be a homomorphism so that $D \in U_{X;\Theta}$. By the Dual Exact Triples property for the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ \mathrm{i}\ } & X \oplus \mathsf{R}^N & \xrightarrow{\ \mathrm{j}\ } & \mathsf{R}^N & \longrightarrow & 0 \\
 & & \big\downarrow D & & \big\downarrow D_\Theta & & \big\downarrow j & & \\
0 & \longrightarrow & Y & \xrightarrow{\ \mathrm{id}_Y\ } & Y & \xrightarrow{\ \mathrm{j}_Y\ } & 0 & \longrightarrow & 0
\end{array}$$

the diagram

$$\begin{array}{ccc} \lambda(D)\otimes\lambda(j) & \xrightarrow{\quad\Psi_t\quad} & \lambda(D_\Theta) \\ \Big\downarrow{\scriptstyle \widetilde{\mathscr{D}}_j\otimes\widetilde{\mathscr{D}}_D\circ R} & & \Big\downarrow{\scriptstyle \widetilde{\mathscr{D}}_{D_\Theta}} \\ \lambda(j^*)\otimes\lambda(D^*) & \xrightarrow{\quad\Psi_{t^*}\quad} & \lambda(D_\Theta^*) \end{array}$$

commutes. The horizontal arrows in this diagram induce continuous bundle maps by the continuity of (2.20); the right vertical arrow induces a continuous bundle map over $U_{X;\Theta}$ by Lemma 5.6. The isomorphisms R and $\widetilde{\mathscr{D}}_j$ on the left-hand side of this diagram do not depend on D. Thus, the isomorphisms $\widetilde{\mathscr{D}}_D$ also induce continuous bundle maps over $U_{X;\Theta}$.

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