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EXISTENCE OF CONTINUOUS FUNCTIONS THAT ARE ONE-TO-ONE ALMOST EVERYWHERE

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(Dedicated to the memory of Mary Ellen Rudin)

Abstract

It is shown that given a metric space X and a σ-finite positive regular Borel measure μ on X, there exists a bounded continuous real-valued function on X that is one-to-one on the complement of a set of μ measure zero.

1. Introduction

In [2] the author and Bo Li studied the question of how many functions are needed to generate an algebra dense in various L^p-spaces. In connection with this, they proved [2, Theorem 1.10] that on every smooth manifold-with-boundary there exists a bounded continuous real-valued function that is one-to-one on the complement of a set of measure zero. It was suggested by Lee Stout that this result would generalize to a metric space context. In this paper we show that this is indeed the case. The author would like to thank Stout for sharing his insight. We state the result using the following terminology introduced in [2].

Definition 1.1. We call a map F defined on a measure space X *one-to-one almost everywhere* if there is a subset E of X of measure zero such that the restriction of F to $X \setminus E$ is one-to-one.

Theorem 1.2. *Let X be a metric space and μ be a σ-finite positive regular Borel measure on X. Then there exists a bounded continuous real-valued function on X that is one-to-one almost everywhere.*

The boundedness of the function is not really important; given an unbounded function with the other properties, we can obtain a bounded one by post composing with a homeomorphism of R onto the interval $(-1, 1)$. The point of the theorem is that the function is continuous *everywhere* and one-to-one *almost everywhere*. Note that the metric space X can be of arbitrarily large

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cardinality, but the set of full measure on which the function is one-to-one can have cardinality at most that of the continuum. Note also that the theorem becomes false if the σ-finiteness condition is dropped as is exemplified by the case of counting measure on a discrete space with cardinality greater than that of the continuum.

The result about continuous one-to-one almost everywhere functions in [2] was used there to show that on every Riemannian manifold-with-boundary M of finite volume there exists a bounded continuous real-valued function f such that the set of polynomials in f is dense in $L^p(M)$ for all $1 \leq p < \infty$ [2, Theorem 1.2]. The argument given there can now be repeated using Theorem 1.2 above in place of [2, Theorem 1.10] to establish the following more general result. This result also strengthens [2, Theorem 1.1].

THEOREM 1.3. *Let X be a metric space and μ be a finite positive regular Borel measure on X. Then there exists a bounded continuous real-valued function f on X such that the set of polynomials in f is dense in $L^p(\mu)$ for all $1 \leq p < \infty$.*

2. Proof of Theorem 1.2

We begin with several lemmas. The first of these is probably well-known, and it appears with proof as [2, Lemma 3.1]. Throughout the paper, by “a Cantor set” we mean any space that is homeomorphic to the standard middle thirds Cantor set.

LEMMA 2.1. *If C is a Cantor set and $\mathscr{U}$ is an open cover of C, then C can be written as a finite union $C = C_1 \cup \ldots \cup C_N$ of disjoint Cantor sets $C_1, \ldots, C_N$ each of which lies in some member of $\mathscr{U}$.*

LEMMA 2.2. *Let X be a topological space and μ be a σ-finite positive regular Borel measure on X. Then there exists a countable collection $\{K_n\}$ of disjoint compact sets in X such that $\mu\big(X \setminus \big(\bigcup K_n\big)\big) = 0$.*

PROOF. By hypothesis $X = \bigcup_{n=1}^{\infty} X_n$ with $\mu(X_n) < \infty$ for each n, and without loss of generality the X_n can be taken to be disjoint. For each fixed n, the regularity of μ enables us to inductively choose disjoint compact sets $X_n^1, X_n^2, \ldots$ contained in X_n, such that $\mu\big(X_n \setminus (X_n^1 \cup \ldots \cup X_n^j)\big) < 1/j$ for each $j = 1, 2, \ldots$. Then $\mu\big(X_n \setminus \big(\bigcup_{j=1}^{\infty} X_n^j\big)\big) = 0$. Hence $\{X_n^j\}_{n,j}$ is a countable collection of disjoint compact sets in X such that $\mu\big(X \setminus \big(\bigcup_{n,j} X_n^j\big)\big) = 0$.

LEMMA 2.3. *Let X be a (nonempty) compact metric space without isolated points, and let μ be a positive regular Borel measure on X. Fix $\varepsilon > 0$ and $\delta > 0$. Then for every sufficiently large positive integer r, there exists a collection*

$\{U_1, \ldots, U_r\}$ *of nonempty open sets in* X *with disjoint closures such that*

$$\mu\big(X \setminus (U_1 \cup \ldots \cup U_r)\big) < \varepsilon$$

and

$$\text{diameter}(U_j) < \delta \qquad \textit{for every} \quad j = 1, \ldots, r.$$

PROOF. Since X is a compact metric space, X is totally bounded. Thus X can be covered by finitely many balls $A_1, \ldots, A_s$ of diameters less than δ. Set $E_1 = A_1$ and $E_j = A_j \setminus (A_1 \cup \ldots \cup A_{j-1})$ for each $j = 2, \ldots, s$. Then the E_j are disjoint and $\bigcup_{j=1}^{s} E_j = X$. By the regularity of μ, for each $j = 1, \ldots, s$, we can choose a compact set K_j contained in E_j such that $\mu(E_j \setminus K_j) < \varepsilon/s$. Then the sets $K_1, \ldots, K_s$ are disjoint and have diameters less than δ. Hence we can choose open neighborhoods $U_1, \ldots, U_s$ of $K_1, \ldots, K_s$, respectively, so that the closures of the U_j are disjoint and

$$\text{diameter}(U_j) < \delta \qquad \text{for every} \quad j = 1, \ldots, s.$$

Then also

$$\mu\big(X \setminus (U_1 \cup \ldots \cup U_s)\big) \leq \mu\big(X \setminus (K_1 \cup \ldots \cup K_s)\big) = \sum_{j=1}^{s} \mu(E_j \setminus K_j) < \varepsilon.$$

The above argument establishes that the desired nonempty open sets can be obtained for *some* positive integer $r \leq s$. To show that r can be taken arbitrarily large, it suffices by induction, to show that r can be increased by 1. To this end, suppose that $U_1, \ldots, U_r$ are as in the statement of the lemma. Let $\gamma = \varepsilon - \mu\big(X \setminus (U_1 \cup \ldots \cup U_r)\big) > 0$, and choose a point $p \in U_r$. Because X has no isolated points and μ is regular, there is a nonempty compact set K in $U_r \setminus \{p\}$ such that $\mu\big((U_r \setminus \{p\}) \setminus K\big) < \gamma$. Choose open neighborhoods U_r' and U_{r+1}' of $\{p\}$ and K, respectively, contained in U_r with disjoint closures. Then $U_1, \ldots, U_{r-1}, U_r', U_{r+1}'$ is a collection of $r+1$ nonempty open sets with the required properties.

LEMMA 2.4. *Given a* (*nonempty*) *compact metric space* X *without isolated points, a positive regular Borel measure* μ *on* X, *and* $\varepsilon > 0$, *there exists a Cantor set* C *in* X *such that* $\mu(X \setminus C) < \varepsilon$.

A result close to Lemma 2.4 appears in the paper [1] by Bernard Gelbaum. (The author would like to thank Bo Li for pointing this out.) Lemma 2.4 is more general than the result in [1], since in [1] the measure is required to be nonatomic and there is no such requirement in Lemma 2.4. The author was surprised to find that the proof in [1] is very different from the one given here.

PROOF. By the preceding lemma, there are nonempty open sets $U_1, \dots, U_{r_1}$ (for some r_1) with disjoint closures such that

$$\mu\big(X \setminus (U_1 \cup \ldots \cup U_{r_1})\big) < \varepsilon/2$$

and

$$\text{diameter}(U_{j_1}) < 1 \qquad \text{for every} \quad j_1 = 1, \dots, r_1.$$

Each $\overline{U}_{j_1}$ is a compact set without isolated points, so we can apply the preceding lemma to each $\overline{U}_{j_1}$ to obtain nonempty relatively open subsets V_{j_1,j_2} for $j_1 = 1, \dots, r_1$ and $j_2 = 1, \dots, r_2$ (for some r_2) with disjoint closures such that

(i) $V_{j_1,j_2} \subset \overline{U}_{j_1}$,

(ii) $\mu\big(\overline{U}_{j_1} \setminus (V_{j_1,1} \cup \ldots \cup V_{j_1,r_2})\big) < \dfrac{\varepsilon}{2^2\, r_1}$, and

(iii) $\text{diameter}(V_{j_1,j_2}) < 1/2$.

Setting $U_{j_1,j_2} = V_{j_1,j_2} \cap U_{j_1}$, we obtain nonempty open subsets of X such that

(i′) $U_{j_1,j_2} \subset U_{j_1}$,

(ii′) $\mu\big(U_{j_1} \setminus (U_{j_1,1} \cup \ldots \cup U_{j_1,r_2})\big) < \dfrac{\varepsilon}{2^2\, r_1}$, and

(iii′) $\text{diameter}(U_{j_1,j_2}) < 1/2$.

In general, assume that we have chosen, for each $s = 1, \dots, k$, nonempty open subsets $U_{j_1,\dots,j_s}$ of X for each $j_1 = 1, \dots, r_1; \dots; j_s = 1, \dots, r_s$ (for some $r_1, \dots, r_s$) with disjoint closures such that

(i″) $U_{j_1,\dots,j_s} \subset U_{j_1,\dots,j_{s-1}}$,

(ii″) $\mu\big(U_{j_1,\dots,j_{s-1}} \setminus (U_{j_1,\dots,j_{s-1},1} \cup \ldots \cup U_{j_1,\dots,j_{s-1},r_s})\big) < \dfrac{\varepsilon}{2^s\, r_{s-1}}$, and

(iii″) $\text{diameter}(U_{j_1,\dots,j_s}) < 1/s$.

Each $\overline{U}_{j_1,\dots,j_k}$ is a compact set without isolated points to which we can apply the procedure above to obtain open sets $U_{j_1,\dots,j_{k+1}}$ for each $j_1 = 1, \dots, r_1; \dots; j_{k+1} = 1, \dots, r_{k+1}$ (for some r_{k+1}) with disjoint closures such that conditions (i″)–(iii″) hold with s replaced by $k+1$. Thus by induction the construction can be continued.

Now consider the sets $K_s = \bigcup_{j_1=1}^{r_1} \cdots \bigcup_{j_s=1}^{r_s} \overline{U}_{j_1,\dots,j_s}$. These are nonempty compact sets such that $K_1 \supset K_2 \supset \cdots$, so their intersection $C = \bigcap_{s=1}^{\infty} K_s$ is nonempty. Moreover, one easily verifies that $\mu(X \setminus C) < \varepsilon$. Finally we claim that C is a Cantor set. To verify this, note that for each sequence $(j_1, j_2, \dots) \in \prod_{k=1}^{\infty}\{1, \dots, r_k\}$ we have

$$\overline{U}_{j_1} \supset \overline{U}_{j_1,j_2} \supset \overline{U}_{j_1,j_2,j_3} \supset \cdots,$$

so the intersection of these sets is nonempty, and because the diameters of these sets go to zero, the intersection consists of a single point. Thus there is a well-defined map

$$F : \prod_{k=1}^{\infty}\{1, \ldots, r_k\} \to C$$

sending the sequence $(j_1, j_2, \ldots)$ to the point in the intersection. One easily verifies that F is a bijection by using that, for each fixed s, the sets $\overline{U}_{j_1,\ldots,j_s}$ (as $j_1, \ldots, j_s$ vary) are disjoint. One easily verifies that F is continuous using that the diameters of the sets $\overline{U}_{j_1,\ldots,j_s}$ go to zero as $s \to \infty$. Hence, by compactness, F is a homeomorphism. Thus since $\prod_{k=1}^{\infty}\{1, \ldots, r_k\}$ is a Cantor set, so is C.

LEMMA 2.5. *Given a (nonempty) compact metric space X without isolated points and a positive regular Borel measure μ on X, there exists an at most countable collection $\{C_n\}$ of disjoint Cantor sets in X such that $\mu\big(X \setminus \big(\bigcup C_n\big)\big) = 0$.*

PROOF. We construct the sets C_n inductively. By the preceding lemma, there exists a Cantor set C_1 in X such that $\mu(X \setminus C_1) < 1$. In general, assume that disjoint Cantor sets $C_1, \ldots, C_k$ have been chosen such that $\mu\big(X \setminus (C_1 \cup \ldots \cup C_k)\big) < 1/2^k$. If in fact $\mu\big(X \setminus (C_1 \cup \ldots \cup C_k)\big) = 0$, then we are done. Otherwise, by the regularity of μ, there is an open neighborhood $U \subsetneq X$ of $C_1 \cup \ldots \cup C_k$ such that $\mu\big(U \setminus (C_1 \cup \ldots \cup C_k)\big) < 1/2^{k+2}$. Now choose an open neighborhood V of $C_1 \cup \ldots \cup C_k$ such that $\overline{V} \subset U$. Let $Y = \overline{X \setminus \overline{V}}$. Then Y is a nonempty compact set disjoint from $C_1 \cup \ldots \cup C_k$ and $X = U \cup Y$. Because Y is the closure of the open set $X \setminus \overline{V}$, we see that Y has no isolated points. Therefore, the preceding lemma gives that there is a Cantor set C_{k+1} in Y such that $\mu(Y \setminus C_{k+1}) < 1/2^{k+2}$. Since $C_{k+1} \subset Y$, we know that C_{k+1} is disjoint from the sets $C_1, \ldots, C_k$. Since $X = U \cup Y$ we have

$$\begin{aligned}\mu\big(X \setminus (C_1 \cup \ldots \cup C_{k+1})\big) &\le \mu\big(U \setminus (C_1 \cup \ldots \cup C_k)\big) + \mu(Y \setminus C_{k+1}) \\ &< 1/2^{k+2} + 1/2^{k+2} = 1/2^{k+1}.\end{aligned}$$

Thus by induction we obtain a sequence of disjoint Cantor sets $C_1, C_2, \ldots$, such that $\mu\big(X \setminus (C_1 \cup \ldots \cup C_j)\big) < 1/2^j$ for every j. Hence $\mu(X \setminus \bigcup_{n=1}^{\infty} C_n) = 0$.

LEMMA 2.6. *Given a metric space X and a σ-finite positive regular Borel measure μ on X, there exist an at most countable collection $\{C_n\}$ of disjoint Cantor sets in X and an at most countable set S in X disjoint from each C_n such that $\mu\big(X \setminus \big(\big(\bigcup C_n\big) \cup S\big)\big) = 0$.*

PROOF. By Lemma 2.2 there exists a countable collection $\{K_n\}$ of disjoint compact sets in X such that $\mu\big(X \setminus \big(\bigcup K_n\big)\big) = 0$. By the Cantor-Bendixson theorem [3, Theorem 2A.1], each of the compact sets K_n is a disjoint union of a perfect set P_n and an at most countable set S_n. By Lemma 2.5 each nonempty perfect set P_n contains an at most countable collection $\{K_n^j\}_j$ of disjoint Cantor sets such that $\mu\big(P_n \setminus \big(\bigcup_j K_n^j\big)\big) = 0$. Now $\{K_n^j\}_{n,j}$ is an at most countable collection of disjoint Cantor sets, the set $S = \bigcup S_n$ is at most countable and disjoint from each K_n^j, and $\mu\big(X \setminus \big(\big(\bigcup_{n,j} K_n^j\big) \cup S\big)\big) = 0$.

With these preliminaries, we can now prove Theorem 1.2 by essentially repeating the proof of [2, Theorem 1.10]. Minor changes are required on account of the (possible) presence of the at most countable set S in Lemma 2.6. The proof will be carried out as if the collection $\{C_n\}$ and the set S in Lemma 2.6 are both countably infinite. If either is actually finite, then in the inductive procedure below one simply ceases to carry out the part of the construction that no longer makes sense once the collection $\{C_n\}$, or the set S, has been exhausted. If both the collection $\{C_n\}$ and the set S are finite, then the procedure terminates, but in that case the result is rather trivial, so the construction below is not really needed then.

Proof of Theorem 1.2. By Lemma 2.6 there exist in X disjoint sets S and $C_1, C_2, \dots$ such that S is at most countable, each C_j is a Cantor set, and $\mu\big(X \setminus \big(\big(\bigcup C_j\big) \cup S\big)\big) = 0$. Let the points of S be denoted by $x_1, x_2, \dots$. We will construct a sequence $(f_n)_{n=1}^{\infty}$ of continuous functions from X into $[0, 1]$ such that for each n

(i) f_n is one-to-one on $C_1 \cup \dots \cup C_n \cup \{x_1, \dots, x_n\}$,

(ii) f_{n+1} agrees with f_n on $C_1 \cup \dots \cup C_n \cup \{x_1, \dots, x_n\}$, and

(iii) $\|f_{n+1} - f_n\|_\infty \le 1/2^n$.

Suppose for the moment that such a sequence of functions has been constructed. Then on account of condition (iii), the sequence (f_n) converges uniformly to a continuous limit function f. Due to condition (ii), f_m agrees with f_n on $C_1 \cup \dots \cup C_n \cup \{x_1, \dots, x_n\}$ for all $m \ge n$, and hence the limit function f also agrees with f_n on $C_1 \cup \dots \cup C_n \cup \{x_1, \dots, x_n\}$. Now given distinct points a and b in $\big(\bigcup_{j=1}^{\infty} C_j\big) \cup S$, choose N such that both a and b lie in $C_1 \cup \dots \cup C_N \cup \{x_1, \dots, x_N\}$. Then $f(a) = f_N(a) \ne f_N(b) = f(b)$. Hence f is one-to-one on $\big(\bigcup C_j\big) \cup S$. Thus it suffices to construct a sequence of functions satisfying conditions (i)–(iii).

We will construct the sequence of functions f_n by induction. For the purpose of carrying out the induction we will also require the additional condition that for each n

(iv) $\{\, f_n(C_1), \dots, f_n(C_n)\,\}$ is a collection of disjoint Cantor sets in $[0, 1]$.

We begin by defining f_1. Choose a Cantor set $\widetilde{C}_1$ in [0, 1] and a point y_1 in $[0, 1]\setminus\widetilde{C}_1$. Choose a homeomorphism g_1 of C_1 onto $\widetilde{C}_1$. By the Tietze extension theorem, there is an extension of g_1 to a continuous function of X into [0, 1] that maps x_1 to y_1. Let f_1 be the extension.

Now to carry out the induction, assume that functions $f_1, \ldots, f_k$ have been defined so that conditions (i)–(iv) hold for those values of n for which they are meaningful. We wish to define f_{k+1}. By the continuity of f_k, there is an open cover $\mathscr{U}$ of C_{k+1} such that for each member U of $\mathscr{U}$ we have that $f_k(U)$ is contained in an interval of length $1/2^k$. By Lemma 2.1 we can write C_{k+1} as a finite union $C_{k+1} = C_{k+1}^1 \cup \ldots \cup C_{k+1}^N$ of disjoint Cantor sets $C_{k+1}^1, \ldots, C_{k+1}^N$ each of which is contained in some member of $\mathscr{U}$. Then for each $j = 1, \ldots, N$, the set $f_k(C_{k+1}^j)$ is contained in an interval $I_{k+1}^j \subset [0, 1]$ of length $1/2^k$. Since $f_k(C_1), \ldots, f_k(C_k)$ are disjoint Cantor sets, their union is also a Cantor set and in particular has empty interior in [0, 1]. Consequently, we can choose disjoint Cantor sets $\widetilde{C}_{k+1}^1, \ldots, \widetilde{C}_{k+1}^N$ with $\widetilde{C}_{k+1}^j$ contained in $I_{k+1}^j \setminus \big(f_k(C_1) \cup \ldots \cup f_k(C_k) \cup \{f_k(x_1), \ldots, f_k(x_k)\}\big)$ for each j, and we can choose a point y_{k+1} in $[0, 1] \setminus \big(f_k(C_1) \cup \ldots \cup f_k(C_k) \cup \{f_k(x_1), \ldots, f_k(x_k)\} \cup \widetilde{C}_{k+1}^1 \cup \ldots \cup \widetilde{C}_{k+1}^N\big)$ with $|f_k(x_{k+1}) - y_{k+1}| < 1/2^k$. Choose a homeomorphism g_{k+1}^j of C_{k+1}^j onto $\widetilde{C}_{k+1}^j$ for each j, and then define g_{k+1} on $C_1 \cup \ldots \cup C_{k+1} \cup \{x_1, \ldots, x_{k+1}\}$ by

$$g_{k+1}(x) = \begin{cases} f_k(x) & \text{if } x \in C_1 \cup \ldots \cup C_k \cup \{x_1, \ldots, x_k\} \\ g_{k+1}^j(x) & \text{if } x \in C_{k+1}^j \qquad (j = 1, \ldots, N) \\ y_{k+1} & \text{if } x = x_{k+1} \end{cases}$$

Then g_{k+1} is a homeomorphism of $C_1 \cup \ldots \cup C_{k+1} \cup \{x_1, \ldots, x_{k+1}\}$ onto $f(C_1) \cup \ldots \cup f(C_k) \cup \widetilde{C}_{k+1}^1 \cup \ldots \cup \widetilde{C}_{k+1}^N \cup \{y_1, \ldots, y_{k+1}\}$ taking C_{k+1} onto $\widetilde{C}_{k+1}^1 \cup \ldots \cup \widetilde{C}_{k+1}^N$. Note that

$$\sup\big\{|f_k(x) - g_{k+1}(x)| : x \in C_1 \cup \ldots \cup C_{k+1} \cup \{x_1, \ldots, x_{k+1}\}\big\} \leq 1/2^k$$

since for each j both $f_k(C_{k+1}^j)$ and $g_{k+1}(C_{k+1}^j)$ are contained in the interval I_{k+1}^j of length $1/2^k$ and $|f_k(x_{k+1}) - y_{k+1}| < 1/2^k$. By the Tietze extension theorem, there is a continuous function h_{k+1} on X that agrees with $f_k - g_{k+1}$ on $C_1 \cup \ldots \cup C_{k+1} \cup \{x_1, \ldots, x_{k+1}\}$ and satisfies

$$\|h_{k+1}\|_\infty \leq 1/2^k.$$

Define a function f_{k+1} on X by

$$f_{k+1}(x) = \begin{cases} f_k(x) - h_{k+1}(x) & \text{if } 0 \leq f_k(x) - h_{k+1}(x) \leq 1 \\ 0 & \text{if } f_k(x) - h_{k+1}(x) \leq 0 \\ 1 & \text{if } f_k(x) - h_{k+1}(x) \geq 1 \end{cases}$$

Then f_{k+1} is a continuous functions from X into $[0, 1]$ such that $f_{k+1} = g_{k+1}$ on $C_1 \cup \ldots \cup C_{k+1} \cup \{x_1, \ldots, x_{k+1}\}$ and $\|f_{k+1} - f_k\|_\infty \leq 1/2^k$. It follows that $f_1, \ldots, f_{k+1}$ satisfy the required conditions (i)–(iv) for those values of n for which the conditions are meaningful. Therefore, by induction we obtain the desired sequence (f_n), and the proof is complete.

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