MATH. SCAND. 118 (2016), 285–290

A CHARACTERIZATION OF TOTALLY REAL CARLEMAN SETS AND AN APPLICATION TO PRODUCTS OF STRATIFIED TOTALLY REAL SETS

BENEDIKT S. MAGNUSSON and ERLEND FORNÆSS WOLD

Abstract

We give a characterization of stratified totally real sets that admit Carleman approximation by entire functions. As an application we show that the product of two stratified totally real Carleman sets is a Carleman set.

1. Introduction

In one complex variable, the so called Carleman sets are well understood: A closed subset X of C is a Carleman set if and only if (i) it has no interior, (ii) X is polynomially convex, and (iii) $\mathsf{C} \setminus X$ is locally connected at infinity (Keldych and Lavrentieff [1]). In the complex plane property (iii) is equivalent to what we call having bounded E-hulls, but in C^n there is no topological characterization. In C^n it is clear that (i)–(iii) is not sufficient for any type of approximation, as is shown by considering an affine complex line. In particular (i) must be substituted by some other "lack of complex structure". In this article we prove the analogue of the one dimensional result for totally real sets.

THEOREM 1.1. *Let $M \subset \mathsf{C}^n$ be a closed stratified totally real set. Then M is a Carleman set if and only if M is polynomially convex and has bounded E-hulls.*

(For the definition of Carleman approximation and bounded E-hulls, see Section 2, and for the definition of a stratified totally real set, see Section 3.)

It is already known [2] that in the totally real setting, the property of bounded E-hulls implies Carleman approximation. The remaining result is therefore the following, which does not rely on the notion of being totally real:

THEOREM 1.2. *If $M \subset \mathsf{C}^n$ is a closed set which admits Carleman approximation by entire functions, then M has bounded E-hulls.*

Received 25 October 2013.

Theorem 1.2 generalizes the main theorem of [2] where the implication was shown under the assumption that M is totally real and admits $\mathscr{C}^1$-Carleman approximation. As an application of this theorem we prove the following partial answer to a question raised by E. L. Stout (private communication):

THEOREM 1.3. *Let $M_j \subset \mathbb{C}^{n_j}$ be stratified totally real sets for $j = 1, 2$ which admit Carleman approximation by entire functions. Then $M_1 \times M_2 \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ admits Carleman approximation by entire functions.*

(For the definition of a stratified totally real set see Section 3.)

The precise question is more general: *if M_j are Carleman sets in $\mathbb{C}^{n_j}$ for $j = 1, 2$, is $M_1 \times M_2$ Carleman*?

A natural generalization of one of the main results of [2] which we will use in the proof of Theorem 1.3 is the following:

THEOREM 1.4. *Let $M \subset \mathbb{C}^n$ be a stratified totally real set which has bounded E-hulls, and let $K \subset \mathbb{C}^n$ be a compact set such that $K \cup M$ is polynomially convex. Then any $f \in \mathscr{C}(K \cup M) \cap \mathscr{O}(K)$ is approximable in the Whitney $\mathscr{C}^0$-topology by entire functions.*

As a corollary to Theorem 1.1 we also obtain the following:

COROLLARY 1.5. *Let $M \subset \mathbb{C}^n$ be a totally real manifold of class $\mathscr{C}^k$ and assume that M admits Carleman approximation by entire functions. Then M admits $\mathscr{C}^k$-Carleman approximation by entire functions.*

PROOF. By [2] this follows from the fact that M has bounded E-hulls.

For more on the topic of Carleman approximation, see *e.g.* the monograph [5].

2. Proof of Theorem 1.2

DEFINITION 2.1. Let $M \subset \mathbb{C}^n$ be a closed set. We say that M is a *Carleman set*, or that *M admits Carleman approximation by entire functions*, if $\mathscr{O}(\mathbb{C}^n)$ is dense in $\mathscr{C}(M)$ in the Whitney $\mathscr{C}^0$-topology.

DEFINITION 2.2. Let $X \subset \mathbb{C}^n$ be a closed subset. Given a compact normal exhaustion X_j of X we define the *polynomial hull* of X, denoted by $\widehat{X}$, by $\widehat{X} := \cup_j \widehat{X}_j$ (this is independent of the exhaustion). We also set $h(X) := \widehat{X} \setminus X$. If $h(X)$ is empty we say that X is polynomially convex. Note that $\widehat{X}$ is closed and polynomially convex.

DEFINITION 2.3. We say that a closed set $M \subset \mathbb{C}^n$ has *bounded E-hulls* if for any compact set $K \subset \mathbb{C}^n$ the set $h(K \cup M)$ is bounded.

We give two lemmas preparing for the proof of Theorem 1.2. The first one is a simple well known result à la Mittag-Leffler and Weierstrass, which we state for the lack of a suitable reference.

LEMMA 2.4. *Let* $E = \{x_j\}_{j\in\mathbb{N}}$ *be a discrete sequence in* $\mathbb{C}^n$. *Then for any sequence* $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ *there exists an entire function* $f \in \mathcal{O}(\mathbb{C}^n)$ *with* $f(x_j) = a_j$ *for all* $j \in \mathbb{N}$, *and there exist holomorphic functions* $f_1, \dots, f_n$ *such that* $E = Z(f_1, \dots, f_n)$.

PROOF. By Theorem 3.7 in [3] there exists an injective holomorphic map $F = (\tilde{f}_1, \dots, \tilde{f}_n) : \mathbb{C}^n \to \mathbb{C}^n$ such that $F(x_j) = j \cdot \mathbf{e}_1$. For the first claim we let $g \in \mathcal{O}(\mathbb{C})$ be an entire function with $g(j) = a_j$ for all $j \in \mathbb{N}$ and set $f = g \circ \tilde{f}_1$. For the second claim let $g \in \mathcal{O}(\mathbb{C})$ be an entire function whose zero set is precisely $\{j\}_{j\in\mathbb{N}}$. Now set $f_1 = g \circ \tilde{f}_1$ and $f_k = \tilde{f}_k$ for $k = 2, \dots, n$.

LEMMA 2.5. *Let* $M \subset \mathbb{C}^n$ *be a Carleman set and let* $E = \{x_j\}_{j\in\mathbb{N}}$ *be a discrete set of points with* $E \subset \mathbb{C}^n \setminus M$. *Then* $M \cup E$ *is a Carleman set, with interpolation on* E.

PROOF. Assume that $q : M \cup E \to \mathbb{C}$ and $\varepsilon : M \cup E \to \mathbb{R}_+$ are continuous functions. By Lemma 2.4 there exist functions $f_1, \dots, f_n \in \mathcal{O}(\mathbb{C}^n)$ such that $f_j(x_k) = 0$ for all $x_k \in E$ and $j = 1, \dots, n$, and such that $Z(f_1, \dots, f_n) \cap M = \emptyset$. So there exist continuous functions $g_j \in \mathscr{C}(M)$ such that

$$g_1 \cdot f_1 + \cdots + g_n \cdot f_n = 1$$

on M. Since M admits Carleman approximation we may approximate the g_j's by entire functions $\tilde{g}_j \in \mathcal{O}(\mathbb{C}^n)$ such that the function

$$\varphi = \tilde{g}_1 \cdot f_1 + \cdots + \tilde{g}_n \cdot f_n$$

satisfies $\varphi(x) \neq 0$ for all $x \in M$. Obviously $\varphi(z) = 0$ for all $z \in E$.

By the Mittag-Leffler Theorem there exists an entire function $h \in \mathcal{O}(\mathbb{C}^n)$ such that $h(z) = q(z)$ for all $z \in E$. Let $\psi \in \mathscr{C}(M)$ be the function $\psi(x) := \frac{h(x)-q(x)}{\varphi(x)}$. Since M admits Carleman approximation we may approximate ψ by en entire function $\sigma \in \mathcal{O}(\mathbb{C}^n)$, and if the approximation is good enough, the function $h(z) - \varphi(z) \cdot \sigma(z)$ is ε-close to q on $M \cup E$.

PROOF OF THEOREM 1.2. Aiming for a contradiction we assume that M does not have bounded exhaustion hulls, *i.e.*, there exists a compact set K such that $h(K \cup M)$ is not bounded. This implies that there is a discrete sequence of points $E = \{x_j\} \subset h(K \cup M)$. By Lemma 2.5 there exists an entire function $q \in \mathcal{O}(\mathbb{C}^n)$ such that $|q(z)| < \frac{1}{2}$, for $z \in M$ and $q(x_j) = j$ for $x_j \in E$. Define $C = \|q\|_K$. For $j > C$ we then have that $|q(x_j)| > \sup_{z\in K\cup M}\{|q(z)|\}$ which contradicts the assumption that $E \subset h(K \cup M)$.

3. Proof of Theorem 1.4

DEFINITION 3.1. Let $M \subset \mathbb{C}^n$ be a closed set. We say that M is a *stratified totally real set* if M is the increasing union $M_0 \subset M_1 \subset \cdots \subset M_N = M$ of closed sets, such that $M_j \setminus M_{j-1}$ is a totally real set (a set which is locally contained in a totally real manifold) for $j = 1, \ldots, N$, and with M_0 totally real.

The proof of Theorem 1.4 is an inductive construction depending on the following lemma [4, Theorem 4.5]

LEMMA 3.2. *Let $K \subset \mathbb{C}^n$ be a compact set, let $M \subset \mathbb{C}^n$ be a compact stratified totally real set, and assume that $K \cup M$ is polynomially convex. Then any function $f \in \mathscr{C}(K \cup M) \cap \mathscr{O}(K)$ is uniformly approximable by entire functions.*

PROOF. Set $X_j := K \cup M_j$ for $j = 0, \ldots, N$. Then X_0 is polynomially convex (see the proof of Theorem 4.5 in [4]) and so it follows from [2] that $f|_{X_0}$ is uniformly approximable by entire functions. The result is now immediate from Theorem 4.5 in [4].

PROOF OF THEOREM 1.4. Choose a normal exhaustion K_j of $\mathbb{C}^n$ such that $K_j \cup M$ is polynomially convex for each j. Assume that we are given $f \in \mathscr{C}(K \cup M) \cap \mathscr{O}(K)$, $\varepsilon \in \mathscr{C}(K \cup M)$ with $\varepsilon(x) > 0$ for all $x \in K \cup M$. Set $K_0 = K_1 = K$, $f_0 = f_1 = f$. We will construct an approximation of f by induction on j, and we assume that, for $k = 1, \ldots, j$, we have constructed $f_k \in \mathscr{C}(K_k \cup M) \cap \mathscr{O}(K_k)$ such that

$$(1_k) \qquad |f_k(x) - f(x)| < \varepsilon(x)/2 \qquad \text{for all} \quad x \in M,$$

and

$$(2_k) \qquad \|f_k - f_{k-1}\|_{K_{k-1}} < (1/2)^k.$$

Choose $\chi_j \in \mathscr{C}_0^\infty(K_{j+2}^\circ)$ with $\chi_j \equiv 1$ near K_{j+1}. For any $0 < \delta_j < (1/2)^{j+1}$ it follows from Lemma 3.2 that there exists an entire function g_j such that $|g_j(x) - f_j(x)| < \delta_j$ for all $x \in K_j \cup (M \cap K_{j+2})$.

We define $f_{j+1} := \chi_j \cdot g_j + (1 - \chi_j)(f_j)$ on M, and $f_{j+1} := g_j$ near K_{k+1}. It is clear that we get

$$(2_{j+1}) \qquad \|f_{j+1} - f_j\|_{K_j} < \delta_j < (1/2)^{j+1},$$

and on $M \setminus K_j$ we have that

$$f_{j+1} - f = (f_j - f) + \chi_j \cdot (g_j - f_j),$$

so if δ_j is sufficiently small we also get that

$$(1_{j+1}) \qquad |f_{j+1}(x) - f(x)| < \varepsilon(x)/2 \qquad \text{for all} \quad x \in M.$$

It follows from (2_k) that f_k converges to an entire function $\tilde{f}$, and it follows from (1_k) that $|\tilde{f}(x) - f(x)| < \varepsilon(x)$ for all $x \in K \cup M$.

4. Proof of Theorem 1.3

Note first that $M_1 \times M_2$ is a stratified totally real set which is polynomially convex, so by Theorem 1.4 it suffices to show that $M_1 \times M_2$ has bounded E-hulls. Let $K \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ be compact. Since both M_1 and M_2 are Carleman sets, they have bounded exhaustion hulls by Theorem 1.2. Choose compact sets $\tilde{K}_j$ in $\mathbb{C}^{n_j}$ for $j = 1, 2$, with $K \subset \tilde{K}_1 \times \tilde{K}_2$. Set $K_j := \tilde{K}_j \cup \overline{h(\tilde{K}_j \cup M_j)}$ which are now compact sets.

We claim that $(K_1 \times K_2) \cup (M_1 \times M_2)$ is polynomially convex, from which it follows that $h(K \cup (M_1 \times M_2)) \subset K_1 \times K_2$. Let $(z_0, w_0) \in (\mathbb{C}^{n_1} \times \mathbb{C}^{n_2}) \setminus [(K_1 \times K_2) \cup (M_1 \times M_2)]$. We consider several cases.

(i) $z_0 \notin K_1 \cup M_1$. Here, $K_1 \cup M_1$ is polynomially convex and we simply use a function in the variable z_0 only.

(ii) $w_0 \notin K_2 \cup M_2$. Analogous to (i).

(iii) $z_0 \in M_1 \cap K_1$. Then $w_0 \notin K_2 \cup M_2$, so we are in case (ii).

(iv) $z_0 \in M_1 \setminus K_1$. Then $w_0 \in K_2 \setminus M_2$, unless we are in case (ii). By Lemma 2.5 there exists $f \in \mathcal{O}(\mathbb{C}^{n_2})$ such that $f(w_0) = 1$ and $|f(w)| < 1/2$ for all $w \in M_2$. Set $N = \|f\|_{K_2}$. By Theorem 1.4 there exists $g \in \mathcal{O}(\mathbb{C}^{n_1})$ such that $\|g\|_{K_1} < 1/(2N)$, $|g(z)| < 3/2$ for all $z \in M_1$ and $g(z_0) = 1$. Set $h(z, w) = f(w) \cdot g(z)$. Then $h(z_0, w_0) = 1$. For $(z, w) \in K_1 \times K_2$ we have $|h(z, w)| = |f(w)||g(z)| \leq N \cdot 1/(2N) = 1/2$. If $(z, w) \in M_1 \times M_2$ then $|h(z, w)| = |f(w)||g(z)| \leq 1/2 \cdot 3/2 = 3/4$.

(v) $z_0 \in K_1 \setminus M_1$. Then $w_0 \in M_2 \setminus K_2$ unless we are in case (ii), but this is the same as (iv) with the roles of z_0 and w_0 switched.

REFERENCES

1. Keldych, M., and Lavrentieff, M., *Sur un problème de M. Carleman*, C. R. (Doklady) Acad. Sci. URSS (N. S.) 23 (1939), 746–748.

2. Manne, P. E., Wold, E. F., and Øvrelid, N., *Holomorphic convexity and Carleman approximation by entire functions on Stein manifolds*, Math. Ann. 351 (2011), no. 3, 571–585.

3. Rosay, J.-P., and Rudin, W., *Holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^n$*, Trans. Amer. Math. Soc. 310 (1988), no. 1, 47–86.

4. Samuelsson, H., and Wold, E. F., *Uniform algebras and approximation on manifolds*, Invent. Math. 188 (2012), no. 3, 505–523.
5. Stout, E. L., *Polynomial convexity*, Progress in Mathematics, vol. 261, Birkhäuser Boston, Inc., Boston, MA, 2007.

SCIENCE INSTITUTE
UNIVERSITY OF ICELAND
DUNHAGA 5
107 REYKJAVIK
ICELAND
E-mail: bsm@hi.is

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSLO
P.O. BOX 1053 BLINDERN
NO-0316 OSLO
NORWAY
E-mail: erlendfw@math.uio.no