ON PLANE CURVES WITH DOUBLE
AND TRIPLE POINTS

NANCY ABDALLAH*

Abstract

We describe in simple geometric terms the Hodge filtration on the cohomology $H^*(U)$ of the complement $U = \mathsf{P}^2 \setminus C$ of a plane curve C with ordinary double and triple points. Relations to Milnor algebra, syzygies of the Jacobian ideal and pole order filtration on $H^2(U)$ are given.

1. Introduction

Let $S = \oplus_r S_r = \mathsf{C}[x, y, z]$ be the graded ring of polynomials with complex coefficients, where S_r is the vector space of homogeneous polynomials of S of degree r. For a homogeneous polynomial f of degree N, define the Jacobian ideal of f to be the ideal J_f generated in S by the partial derivatives f_x, f_y, f_z of f with respect to x, y and z. The graded *Milnor algebra* of f is given by

$$M(f) = \oplus_r M(f)_r = S/J_f.$$

The study of such Milnor algebras is related to the singularities of the projective curve $C \subset \mathsf{P}^2$ defined by $f = 0$, see [2], as well as to the mixed Hodge theory of the curve C and that of its complement $U = \mathsf{P}^2 \setminus C$, see the foundational article by Griffiths [14] and also [6], [10], [11] treating singular hypersurfaces in P^n. For other relations with algebraic geometry see [17]. A key question is to relate the Hodge filtration F to the pole order filtration P (whose definition is given in [7, ch. 6]). on the cohomology group $H^2(U)$.

The Milnor algebra can be seen (up to a twist in grading) as the top cohomology $H^3(K^*(f))$ of the Koszul complex of f_x, f_y and f_z defined in section 3.

The aim of this paper is to generalize the results given by A. Dimca and G. Sticlaru in [13] on nodal curves to curves whose singularities are nodes (alias ordinary double points A_1) and ordinary triple points D_4. In the case of

nodal curves, by the work of Deligne [5], one has the equality $F = P$ between the two filtrations on $H^2(U)$. In the case at hand, this equality no longer holds, and this explains why the results are more involved to state and to prove.

In section 2, we recall some basic facts on the mixed Hodge theory and extend one of the main results of A. Dimca and G. Sticlaru in [12] from the nodal to the double and triple points, see Theorem 2.1.

In section 3, we give a brief introduction about syzygies of the Jacobian ideal of f and the relation between Koszul complex cohomology and Hodge theory.

In the final section we state and prove the main result, which is an estimation of the dimensions of certain homogeneous components of the cohomology groups $H^m(K^*(f))$ in terms of simple geometrical invariants, like the number of double and triple points and the genera of the irreducible components of C, see Theorem 4.1.

As a consequence, we find examples of curves with only ordinary double and triple points for which $P^2 H^2(U) \neq F^2 H^2(U)$. The previous known examples involved curves with non-ordinary multiple points, see Examples 3.2, 3.3 and 3.4 in [13]. We also discuss in Example 4.5 the two distinct realization of the Pappus configuration 9_3 and point out another subtle difference between nodal curves and curves with ordinary double and triple points.

2. Hodge theory of plane curve complements

Let X be an algebraic variety. Define $H_c^*(X, \mathsf{C})$ to be the cohomology groups with compact support of X. If X is smooth, then $H_c^m(X, \mathsf{C})$ is dual to $H^{2n-m}(X, \mathsf{C})$, where $n = \dim_\mathsf{C} X$, (see Theorem 6.23 in [16, p. 155]). Recall that the mixed Hodge numbers $h^{p,q}(H^s(X))$ are defined by

$$h^{p,q}(H^s(X)) = \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^s(X, \mathsf{C}),$$

and we have

(2.1) $$h^{p,q}(H^m(X, \mathsf{C})) = h^{n-q,n-p}(H_c^{2n-m}(X, \mathsf{C}))$$

for every $p, q \leq n$. Consider now $U = \mathsf{P}^2 \setminus C$, a smooth affine variety. By Deligne [5], $H^*(U, \mathsf{C})$ has a mixed Hodge structure. In particular, for the Hodge filtration F on $H^2(U)$ one has

(2.2) $$H^2(U) = F^0 = F^1 \supset F^2 \supset F^3 = 0,$$

where the second equality is proven in [7, p. 185].

The dimensions of the associated graded groups $\mathrm{Gr}_F^m H^2(U)$ with respect to the Hodge filtration are described in the following theorem, which is our first main result.

THEOREM 2.1. *Let $C \subset \mathbb{P}^2$ be a curve of degree N. Suppose that C has n nodes and t ordinary triple points as singularities and set $U = \mathbb{P}^2 \setminus C$. Let $C = \bigcup_{j=1,r} C_j$ be the decomposition of C as a union of irreducible components, $g_j = g(\tilde{C}_j)$ be the genus of the normalization $\tilde{C}_j$ of C_j. Then one has*

$$\dim \mathrm{Gr}_F^1 H^2(U, \mathbb{C}) = \sum_{j=1}^r g_j$$

and
$$\dim \mathrm{Gr}_F^2 H^2(U, \mathbb{C}) = \frac{(N-1)(N-2)}{2} - t.$$

PROOF. For any quasi-projective variety X, the Hodge-Deligne polynomial is given by

$$P(X)(u, v) = \sum_{p,q} E^{p,q}(X) u^p v^q$$

where $E^{p,q}(X) = \sum_s (-1)^s \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^s(X, \mathbb{C})$. It is known that this polynomial is additive with respect to constructible partitions, i.e. $P(X) = P(X \setminus Y) + P(Y)$ for a closed subvariety Y of X.

Suppose now that the curve $C_j : f_j = 0$ has degree N_j, and has n_j nodes and t_j triple points. We have,

$$\begin{aligned}
P(C_j) &= P(C_j \setminus (C_j)_{\mathrm{sing}}) + P((C_j)_{\mathrm{sing}}) \\
&= P(\tilde{C}_j \setminus \{(2n_j + 3t_j) \text{ points}\}) + n_j + t_j \\
&= P(\tilde{C}_j) - P(\{(2n_j + 3t_j) \text{ points}\}) + n_j + t_j \\
&= uv - g_j u - g_j v + 1 - n_j - 2t_j.
\end{aligned}$$

Indeed, it is known that for a smooth curve C, the genus $g(C)$ is exactly the Hodge number $h^{1,0}(X) = h^{0,1}(X)$. Now,

$$\begin{aligned}
P(C) &= P(C_1 \cup \ldots \cup C_r) \\
&= \sum_{j=1}^r P(C_j) - \sum_{1 \le i < j \le r} P(C_i \cap C_j) + \sum_{1 \le i < j < k \le r} P(C_i \cap C_j \cap C_k) \\
&= ruv - \left(\sum_{j=1}^r g_j\right) u - \left(\sum_{j=1}^r g_j\right) v + r - \sum_{j=1}^r (n_j + 2t_j) \\
&\quad - \left(\sum_{1 \le i < j \le r} N_i N_j - s\right) + t'.
\end{aligned}$$

where s (respectively t') denotes the total number of triple points which are intersection of only two (respectively three) curves. The term $\sum_{1 \le i < j \le r} N_i N_j - s$

is a result of Bézout's Theorem. Indeed, the total number of intersection points counted with multiplicities is $\sum_{1 \le i < j \le r} N_i N_j$, and it is easy to see that the intersection multiplicity of each of the s triple points is 2, i.e., each one is counted twice. Now since $P(\mathsf{P}^2) = u^2 v^2 + uv + 1$, then, using the additivity again, we get

$$
\begin{aligned}
P(U) &= P(\mathsf{P}^2) - P(C) \\
&= u^2 v^2 - (r-1)uv + \left(\sum_{j=1}^{r} g_j\right) u + \left(\sum_{j=1}^{r} g_j\right) v + \sum_{1 \le i < j \le r} N_i N_j \\
&\quad + \sum_{j=1}^{r} (n_j + 2t_j) - r - s - t' + 1.
\end{aligned}
$$

Since U is smooth, then the group $H_c^4(U, \mathsf{C})$ is dual to the group $H^0(U, \mathsf{C})$, which is 1-dimensional of type $(0,0)$. It follows by equation (2.1), that $H_c^4(U,\mathsf{C})$ is 1-dimensional of type $(2, 2)$ and its contribution to $P(U)$ is exactly the term $u^2 v^2$.

The group $H_c^3(U, \mathsf{C})$ is dual to the group $H^1(U, \mathsf{C})$, which is $(r-1)$-dimensional of type $(1, 1)$. Indeed, by Theorem (C.24) in [7], the only non-zero weights on $H^1(U, \mathsf{C})$ are $m = 1$ and $m = 2$. But W_1 is zero since $W_1 = j^*(H^1(\mathsf{P}^2)) = 0$ where $j: U \to \mathsf{P}^2$ is the compactification of U and we apply again Theorem (C.24) in [7]. It follows that the contribution of $H_c^3(U, \mathsf{C})$ to $P(U)$ is exactly the term $-(r-1)uv$.

The remaining terms come from the group $H_c^2(U, \mathsf{C})$, which is dual to the group $H^2(U, \mathsf{C})$. By Theorem (C.24) in [7], the only non-zero weights on $H^2(U, \mathsf{Q})$ are $m = 2, 3$ and 4. On the other hand, $W_2 = 0$, since $W_2 = j^* H^2(\mathsf{P}^2)$. But $H^2(\mathsf{P}^2) = \alpha \mathsf{Q}$, where $\alpha = c_1(L)$, c_1 denotes the Chern class of $L = \mathcal{O}(1)$. Therefore, by the naturality property of Chern classes (see [1]), $N\alpha = j^*(c_1(L^{\otimes N})) = c_1(j^*(L^{\otimes N})) = c_1(L^{\otimes N}|U)$ which is equal to zero since it has a nowhere vanishing section given by f, the defining equation of C. This implies that $H^2(U, \mathsf{C})$ has only classes of type $(2, 1)$, $(1, 2)$, and $(2, 2)$. Therefore the dimension $\dim \mathrm{Gr}_F^1 H^2(U, \mathsf{C})$ is the number of independent classes of type $(1, 2)$, which correspond to classes of type $(1, 0)$ in $H_c^2(U)$, according to equation (2.1), and hence to the terms in u in $P(U)$. This gives the first equality.

Now the dimension $\dim \mathrm{Gr}_F^2 H^2(U, \mathsf{C})$ is the number of independent classes of type $(2, 1)$ or $(2, 2)$, which correspond respectively to the terms in v or the constant terms in the polynomial $P(U)$. This yields

$$
\dim \mathrm{Gr}_F^2 H^2(U, \mathsf{C}) = \sum_{j=1}^{r} (g_j + n_j + 3t_j - 1) + \sum_{1 \le i < j \le r} N_i N_j + 1 - \left(\sum_{j=1}^{r} t_j + s + t'\right).
$$

Recall the formula

$$g_j + n_j + 3t_j = p_a(C_j) = \frac{(N_j - 1)(N_j - 2)}{2},$$

where p_a denotes the arithmetic genus, see [15, p. 298 and p. 54]. Knowing that the total number of triple points $t = \sum_{j=1}^r t_j + s + t'$, and using the fact that $N = \sum_{j=1}^r N_j$, with some computations, we get:

$$\dim \mathrm{Gr}_F^2 H^2(U, \mathbb{C}) = \sum_{j=1}^r \left(\frac{(N_j - 1)(N_j - 2)}{2} - 1 \right) + \sum_{1 \le i < j \le r} N_i N_j + 1 - t$$

$$= \frac{(N - 1)(N - 2)}{2} - t.$$

REMARK 2.2. In the case of nodal curves, i.e. for $t = 0$, the above theorem was already proved by A. Dimca and G. Sticlaru in [12].

REMARK 2.3. As shown in the proof above, we have $W_2 H^2(U, \mathbb{C}) = 0$, and this implies

$$h^{2,0}(H^2(U)) = h^{1,1}(H^2(U)) = h^{0,2}(H^2(U)) = 0.$$

This last remark implies the following consequence.

COROLLARY 2.4. *With the above notation and assumptions, we have the following.*

(i) $h^{2,1}(H^2(U)) = h^{1,2}(H^2(U)) = \sum_{j=1}^r g_j$.

(ii) $h^{2,2}(H^2(U)) = \frac{(N-1)(N-2)}{2} - \sum_{j=1}^r g_j - t$.

(iii) $b_2(U) = \frac{(N-1)(N-2)}{2} + \sum_{j=1}^r g_j - t$, *where $b_2(U)$ denotes the second Betti number of the complement U.*

In particular, it follows that $H^2(U)$ is pure of type $(2, 2)$ when $g_j = 0$ for all j, a well known property in the case of line arrangements.

EXAMPLE 2.5. Let $C: (x^2 - y^2)(y^2 - z^2)(x^2 - z^2) = 0$. C is the union of 6 lines in $\mathbb{P}^2$. It has 4 triple points. We have $g_i = 0$ for $i = 1, \ldots, 6$, $N = 6$, and $t = 4$. Then according to the aforementioned theorem (2.1) we get $\dim \mathrm{Gr}_F^1 H^2(U, \mathbb{C}) = \dim \frac{F^1}{F^2} = 0$ and $\dim \mathrm{Gr}_F^2 H^2(U, \mathbb{C}) = \dim F^2 = 6$. Hence, $b_2(U) = 6$.

EXAMPLE 2.6. Let $C: (x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0$. C is the union of 9 lines. It has 12 triple points and no nodes. We have $g_i = 0$ for every

$i = 1, \ldots, 6$, $N = 9$, and $t = 12$. Then $\dim \mathrm{Gr}_F^1 H^2(U, \mathbb{C}) = 0$ and $\dim \mathrm{Gr}_F^2 H^2(U, \mathbb{C}) = 16 = b_2(U)$.

EXAMPLE 2.7. Let $C: xyz(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) = 0$. C is the union of 3 lines giving rise to a triangle and a smooth cubic curve. It has 3 triple points (the vertices of the triangle) and 3 nodes. We have $g_1 = g_2 = g_3 = 0$, $g_4 = 1$, $N = 6$, and $t = 3$. Then $\dim \mathrm{Gr}_F^1 H^2(U, \mathbb{C}) = 1$, $\dim \mathrm{Gr}_F^2 H^2(U, \mathbb{C}) = 7$, and $b_2(U) = 1 + 7 = 8$.

3. Koszul complexes, syzygies and spectral sequences for reduced plane curves

Let $K^*(f)$ be the Koszul complex of the partial derivatives f_x, f_y, f_z of f with the natural grading $|x| = |dx| = 1$ defined by

$$ 0 \longrightarrow \Omega^0 \xrightarrow{df\wedge} \Omega^1 \xrightarrow{df\wedge} \Omega^2 \xrightarrow{df\wedge} \Omega^3 \longrightarrow 0 $$

where $df = f_x dx + f_y dy + f_z dz$, and Ω^k denotes the global polynomial k-forms on $\mathbb{C}^3$.

It is easy to see that $H^3(K^*(f))_k = M(f)_{k-3}$. On the other hand, the homogeneous components of $H^2(K^*(f))_{2+m}$ are the syzygies

$$ R_m: af_x + bf_y + cf_z = 0, $$

where $a, b, c \in S_m$, modulo the trivial syzygies generated by $(f_j)f_i + (-f_i)f_j = 0$, where f_i and f_j denote f_x, f_y or f_z. Denote by $\mathrm{ER}(f)$ the set of these relations, called essential relations, or non-trivial relations.

By Theorem 3.1 in [8], we have

$$ (3.1) \qquad \dim \mathrm{ER}(f)_{k-2} = \dim H^2(K^*(f))_k $$

for any $2 \le k \le 2N - 3$, and $\dim \mathrm{ER}(f)_{k-2} = \tau(C)$ for $k \ge 2N - 4$. Here $\tau(C)$ denotes the sum of the Tjurina numbers of the singularities of $C : f = 0$. For instance, if C has n A_1-singularities and t D_4-singularities, then $\tau(C) = n + 4t$. We also have,

$$ (3.2) \quad t^{-3+N} HP(H^2(K^*(f))_{2N-3-k}) = HP(M(f)_{3N-6-k}) - HP(M(f_s)_k). $$

In particular,

$$ (3.3) \qquad \dim H^2(K^*(f))_{2N-3-k} = \dim M(f)_{3N-6-k} - \dim M(f_s)_k, $$

with f_s a homogeneous polynomial in S of degree N such that $C_s: f_s = 0$ is a smooth curve in $\mathbb{P}^2$.

Recall the following integers introduced (for hypersurfaces in $\mathbf{P}^n$) in [13]:

DEFINITION 3.1. For a plane curve $C: f = 0$ of degree N with isolated singularities we set

(i) the *coincidence threshold* $ct(C)$ defined as

$$ct(C) = \max\{q : \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\}.$$

(ii) the *stability threshold* $st(C)$ defined as

$$st(C) = \min\{q : \dim M(f)_k = \tau(C) \text{ for all } k \geq q\}.$$

(iii) the *minimal degree of non-trivial syzygy* $mdr(C)$ defined as

$$mdr(C) = \min\{q : H^2(K^*(f))_{q+2} \neq 0\}.$$

It is easy to see that $ct(C) = mdr(C) + N - 2$. By Proposition 1 in [2] we have $N - 2 \leq ct(C) \leq 3(N - 2)$ and by Theorem 3 in [2] we have $st(C) \leq 3N - 5$.

EXAMPLE 3.2. Let $C: f = x^p y^q + z^N$ with $p > 0$, $q > 0$ and $p + q = N > 2$. It is easy to see that $qx f_x - py f_y = 0$. Therefore, the first non-trivial syzygy is of degree one. Hence $\dim ER(f)_1 = \dim H^2(K^*(f))_3 \neq 0$. Then $mdr(C) = 1$ which yields $ct(C) = N - 1$.

Suppose now that C has only nodes as singularities, i.e. singularities of type A_1, and let $C = \bigcup_{j=1,r} C_j$ be the decomposition of C as a union of irreducible components. In this case we have $ct(C) \geq 2N - 4$, see Theorem 1.2 in [13]. Therefore, the dimensions of $M(f)_q$ are all determined for $q < 2N - 3$. The next dimension is given by

$$(3.4) \qquad \dim M(f)_{2N-3} = n(C) + \sum_{j=1}^{r} g_j = g + r - 1$$

where $n(C)$ is the number of nodes of C and

$$g = \frac{(N-1)(N-2)}{2}.$$

If C is a rational curve, i.e. $g_i = 0$ for $i = 0, \ldots, r$, then $\dim M(f)_{2N-3} = n(C) = \tau(C)$ and therefore $st(C) \leq 2N - 3$. We recall the following corollary in [13].

COROLLARY 3.3. *For a rational nodal curve C, the Hilbert-Poincaré series*

$$HP(M(f))(t) = \sum_{r} \dim M(f)_r t^r$$

is completely determined by the degree N and the number of nodes $n(C)$. In particular, $st(C) = 2N - 3$ unless C is a generic line arrangement and then $st(C) = 2N - 4$.

EXAMPLE 3.4. Let C be the degree 4 curve defined by $f = x(x^3 + y^3 + z^3)$. Then C has 3 collinear nodes. $st(C) \leq 3N - 5 = 7$ and $ct(C) \geq 2N - 4 = 4$. Indeed a computation using Singular [4] yields the following Hilbert-Poincaré series

$$HP(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4t^5 + 3(t^6 + t^7 + \cdots)$$

and hence $ct(C) = 4$ and $st(C) = 6$.

EXAMPLE 3.5. Let C be a generic line arrangement defined by $f = xyz(x + y + z) = 0$. Then C has 6 nodes, by Corollary 3.3, $HP(M(f))$ is all determined and we have $st(C) = 2N - 4 = 4$ and $ct(C) \geq 4$. Therefore

$$HP(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6(t^4 + t^5 + \cdots),$$

which implies $ct(C) = 4$.

Consider now the double complex (B, d', d'') defined by

$$B^{s,t} = \Omega^{s+t}_{(t+1)N} \qquad s, t \in \mathbb{Z},$$

$d' = d$, and $d''(\omega) = -|\omega|N^{-1}df \wedge \omega$ for a homogeneous differential form ω. Let (B^*, D_f) be the associated total complex, namely, $B^k = \bigoplus_{s+t=k} B^{s,t}$, and $D_f = d' + d''$ with $d'd'' + d''d' = 0$. Define a decreasing filtration on B^* by $F^p B^k = \bigoplus_{\substack{s \geq p \\ s+t=k}} B^{s,t}$. With this notation, we have the following result, see [7, Chapter 6]

PROPOSITION 3.6. *There exists an E_1-spectral sequence (E_r, d_r) converging to $H^{p+q-1}(U)$ such that*

$$E_1^{p,q}(f) = H^{p+q}(K^*(f))_{(q+1)N}.$$

Moreover, the filtration induced by this spectral sequence on $H^*(U)$ coincides with the pole order filtration P.

In the case of a curve $C \subset \mathbb{P}^2$ with isolated singularities, the only non-trivial cohomology groups of the Koszul complex are $H^2(K^*(f))$ and $H^3(K^*(f))$. Therefore, the non-zero terms of the E_1-spectral sequence belong to the lines $p + q = 2$ and $p + q = 3$. For the terms on the line $p + q = 3$, we have

$$\dim E_1^{p,q}(f) = \dim H^3(K^*(f))_{(q+1)N} = \dim M(f)_{(q+1)N-3}.$$

For the terms of the line $p + q = 2$,

$$\dim E_1^{p,q}(f) = \dim H^2(K^*(f))_{(q+1)N}$$
$$= \dim M(f)_{(q+2)N-3} - \dim M(f_s)_{(q+2)N-3}$$

(see [8]).

4. Curves with A_1 and D_4 singularities

We can now state our second main result.

THEOREM 4.1. *Let $C \in \mathbb{P}^2$ be a curve of degree N. Suppose C has n nodes* (A_1) *and t triple points* (D_4) *and no other singularities. Let $C = \bigcup_{j=1,r} C_j$ be the decomposition of C as a union of irreducible components, and set $g_j = g(\tilde{C}_j)$ the genus of the normalization $\tilde{C}_j$ of C_j. Then we have the following.*

(A) $0 \le \dim M(f)_{2N-3} - \tau(C) \le \sum_{j=1}^r g_j$, *where $\tau(C)$ is the sum of all Tjurina numbers of the singularities of C. Moreover, the equality* $\dim M(f)_{2N-3} - \tau(C) = \sum_{j=1}^r g_j$ *holds if and only if the Hodge filtration F and the pole order filtration P on $H^2(U)$ satisfy $F^2 H^2(U) = P^2 H^2(U)$. In particular, if all $g_i = 0$, one has $\dim M(f)_{2N-3} = \tau(C)$, i.e. $st(C) \le 2N - 3$ and $F^2 H^2(U) = P^2 H^2(U)$.*

(B) $\max(r - 1 + t - \sum_{j=1}^r g_j, r - 1) \le \dim \mathrm{ER}(f)_{N-2} \le r - 1 + t$. *In particular, $\dim \mathrm{ER}(f)_{N-2} = r - 1 + t$ if $g_j = 0$ for all j.*

PROOF. (A) Consider the spectral sequence of Proposition 3.6

$$E_1^{p,q}(f) = H^{p+q}(K^*(f))_{(q+1)N}$$

that converges to $H^{p+q-1}(U)$. By Theorem 2.4 (ii) in [13], the differential

$$d_1^t \colon E_1^{2-t,t} \longrightarrow E_1^{3-t,t}$$

is bijective for $t \ge 2$ and injective for $t = 1$.

Consider first the case when $t = 1$. Since $d_1 \colon E_1^{1,1} \to E_1^{2,1}$ is injective, then $\dim E_1^{1,1} \le \dim E_1^{2,1}$. Moreover $\dim E_1^{1,1} = \dim H^2(K^*(f))_{2N} = \dim \mathrm{ER}(f)_{2N-2}$ which is equal to $\tau(C)$ by equation (3.1). On the other hand, $\dim E_1^{2,1}(f) = \dim H^3(K^*(f))_{2N} = \dim M(f)_{2N-3}$. This proves the left hand side inequality in (A).

To prove the right hand side inequality, consider the limit term

$$E_\infty^{2,1} = \frac{P^1 H^2(U)}{P^2 H^2(U)}.$$

It is known that $P^s H^m(U) \supset F^s H^m(U)$, see [7, Chapter 6]. For $s = 1$ we have in addition $F^1 H^2(U) = H^2(U)$, then $P^1 H^2 = F^1 H^2(U) = H^2(U)$. For $s = 2$, $H^2(U) \supset P^2 H^2 \supset F^2 H^2(U)$. It follows that the map

$$\mathrm{Gr}_F^1(H^2(U)) \longrightarrow E_\infty^{2,1}$$

is an epimorphism, and hence, $\dim E_\infty^{2,1} \leq \dim \mathrm{Gr}_F^1(H^2(U))$.

By Theorem 2.1 $\dim \mathrm{Gr}_F^1(H^2(U)) = \sum_{j=1}^r g_j$. On the other hand, by Proposition 2.4 (iii) in [13] the spectral sequence degenerates at the E_2 terms, i.e. $E_\infty^{2,1} = E_2^{2,1} = coker d_1^1$. Hence, $\dim E_\infty^{2,1} = \dim M(f)_{2N-3} - \tau(C)$, and this proves the inequality in (A).

(B) To prove the second inequality, we consider the differential $d_1^0 : E_1^{2,0} \to E_1^{3,0}$. We know that $E_\infty^{2,0} = E_2^{2,0} = \ker d_1^0 \simeq H^1(U)$ and hence $\dim(\ker d_1^0) = r - 1$. In particular, $r - 1 \leq E_1^{2,0} = \dim H^2(K^*(f))_N = \dim \mathrm{ER}(f)_{N-2}$, and $\dim E_\infty^{3,0} = \dim(coker d_1^0) = g - \dim Er(f)_{N-2} + r - 1$, where $g = \frac{(N-1)(N-2)}{2} = \dim M(f)_{N-3}$. We compute now $b_2(U)$ in two different ways:

$$b_2(U) = \dim \mathrm{Gr}_F^1 H^2(U) + \dim \mathrm{Gr}_F^2 H^2(U) = \sum_{j=1}^r g_j + g - t,$$

by Theorem 2.1, and

$$\begin{aligned} b_2(U) &= \dim E_\infty^{3,0} + \dim E_\infty^{2,1} \\ &= g - \dim \mathrm{ER}(f)_{N-2} + r - 1 + \dim M(f)_{N-3} - \tau. \end{aligned}$$

On the other hand, Theorem 1 in [9] implies

$$\dim \mathrm{ER}(f)_{N-2} = \dim M(f)_{2N-3} - \dim M(f_s)_{N-3} = \dim M(f)_{2N-3} - g.$$

The last two formulas imply that $b_2(U) = 2g - \tau + r - 1$. The first formula for $b_2(U)$ now implies that $\sum_{j=1}^r g_j - t = 2g - \tau + r - 1$. If we apply part (A) of the theorem we get

$$\tau - g \leq \dim \mathrm{ER}(f)_{N-2} \leq \tau + \sum g_j - g,$$

and this proves part (B).

EXAMPLE 4.2. (i) Let C be the degree 5 curve defined by $f = xy(x+y)z^2 + x^5 + 2y^5 = 0$. In this example, C is an irreducible curve with exactly one triple point, and hence $g_1 = 3$. A Singular computation gives $\dim M(f)_7 = 6$. This gives a strict inequality in Theorem 4.1 part (A), i.e.

$$\dim M(f)_7 - \tau(C) = 2 < 3 = g_1.$$

Moreover, the inequalities of part (B) of the theorem are

$$0 \leq 0 \leq 1.$$

(ii) Let C be the degree 9 curve defined by $f = (x^3 + y^3 + z^3)^3 + (x^3 + 2y^3 + 3z^3)^3 = 0$. In this example, C is a union of 3 smooth curves with $g_i = 1$ for $i = 1, 2, 3$. It has 9 ordinary triple points, and $\dim M(f)_{16} = \tau(C) = 36$. This gives a strict inequality in Theorem 4.1 part (A), i.e.

$$\dim M(f)_{16} - \tau(C) = 0 < 3 = \sum_{i=1}^{3} g_i.$$

Moreover, the inequalities of part (B) of the theorem are

$$8 \leq 8 \leq 9 + 2 = 11.$$

REMARK 4.3. Part (A) of Theorem 4.1 can be regarded as a generalization of Corollary 3.3, and of the equation (3.4), since for nodal curves we have $F^2 H^2(U) = P^2 H^2(U)$. The above example shows that this equality may fail for curves with ordinary double and triple points. The previously known examples of curves with $F^2 H^2(U) \neq P^2 H^2(U)$ involved curves with non-ordinary singularities, see Examples 3.2, 3.3 and 3.4 in [13].

REMARK 4.4. Part (B) of Theorem 4.1 is a generalization of Theorem 4.1 in [13]. In this result, since for rational curves $\dim ER(f)_{N-2} = r - 1 + t$, then each irreducible component C_j and each triple point $P \in C$ yield one relation, and there is only one dependence relation among them. So one can ask about the possibility to write these syzygies in terms of the point P and the defining functions of C_j as in Theorem 4.1 [13].

The following example shows that Corollary 3.3 does not hold even for line arrangements with double and triple points.

EXAMPLE 4.5. Consider the following two distinct realizations of the line configuration 9_3, see [3] and [7], Example (6.4.16), p. 213. The first one is the Pappus line arrangement

$$\mathscr{A}_1: f = xyz(x - y)(y - z)(x - y - z)$$
$$\cdot (2x + y + z)(2x + y - z)(-2x + 5y - z) = 0.$$

A Singular computation yields

$$HP(M(f))(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 28t^6$$
$$+ 36t^7 + 42t^8 + 46t^9 + 48t^{10} + 48t^{11} + 47t^{12} + 45t^{13} + \cdots$$

The second one is the line arrangement $\mathscr{A}_2$ given by

$$\mathscr{A}_2 : g = xyz(x+y)(x+3z)(y+z)(x+2y+z)(x+2y+3z)(4x+6y+6z) = 0.$$

Using again the Singular software, we get

$$\begin{aligned}
HP(M(g))(t) = {} & 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 28t^6 \\
& + 36t^7 + 42t^8 + 46t^9 + 48t^{10} + 48t^{11} + 46t^{12} + 45t^{13} + \cdots
\end{aligned}$$

Both arrangements have $N = n = t = 9$ and $HP(M(f))(t) - HP(M(g))(t) = t^{12} \neq 0$. This shows once again that the curves with nodes and triple points are much more subtle than the nodal curves. It also shows that it is rather difficult to control the dimension of the homogeneous components $M(f)_r$ for $r \neq 2N - 3$.

REFERENCES

1. Bott, R., and Tu, L. W., *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York-Berlin, 1982.
2. Choudary, A. D. R., and Dimca, A., *Koszul complexes and hypersurface singularities*, Proc. Amer. Math. Soc. 121 (1994), no. 4, 1009–1016.
3. Cohen, D. C., and Suciu, A. I., *On Milnor fibrations of arrangements*, J. London Math. Soc. (2) 51 (1995), no. 1, 105–119.
4. Decker, W., Greuel, G.-M., and Pfister, G., *Singular 3-1-3 - A computer algebra system for polynomial computations*, http://www.singular.uni-kl.de, 2011.
5. Deligne, P., *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57.
6. Dimca, A., *On the Milnor fibrations of weighted homogeneous polynomials*, Compositio Math. 76 (1990), no. 1-2, 19–47.
7. Dimca, A., *Singularities and topology of hypersurfaces*, Universitext, Springer-Verlag, New York, 1992.
8. Dimca, A., *Syzygies of Jacobian ideals and defects of linear systems*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 56(104) (2013), no. 2, 191–203.
9. Dimca, A., *Monodromy of triple point line arrangements*, Singularities in geometry and topology 2011, Adv. Stud. Pure Math., vol. 66, Math. Soc. Japan, Tokyo, 2015, pp. 71–80.
10. Dimca, A. and Saito, M., *A generalization of Griffiths's theorem on rational integrals*, Duke Math. J. 135 (2006), no. 2, 303–326.
11. Dimca, A., Saito, M., and Wotzlaw, L., *A generalization of the Griffiths' theorem on rational integrals. II*, Michigan Math. J. 58 (2009), no. 3, 603–625.
12. Dimca, A., and Sticlaru, G., *Chebyshev curves, free resolutions and rational curve arrangements*, Math. Proc. Cambridge Philos. Soc. 153 (2012), no. 3, 385–397.
13. Dimca, A., and Sticlaru, G., *Koszul complexes and pole order filtrations*, Proc. Edinb. Math. Soc. (2) 58 (2015), no. 2, 333–354.
14. Griffiths, P. A., *On the periods of certain rational integrals. I, II*, Ann. of Math. (2) 90 (1969), 460–495; ibid. (2) 90 (1969), 496–541.
15. Hartshorne, R., *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

16. Peters, C. A. M., and Steenbrink, J. H. M., *Mixed Hodge structures*, Ergebnisse der Mathem-
    atik und ihrer Grenzgebiete, 3. Folge, vol. 52, Springer-Verlag, Berlin, 2008.
17. Sernesi, E., *The local cohomology of the Jacobian ring*, Doc. Math. 19 (2014), 541–565.

LABORATOIRE DE MATHÉMATIQUES J. A. DIEUDONNÉ
UMR NO 7351 CNRS UNS
UNIV. NICE SOPHIA ANTIPOLIS
06108 NICE
FRANCE
E-mail: nancy.n.abdallah@gmail.com