MATH. SCAND. 119 (2016), 73–91

ON VANDERMONDE VARIETIES

RALF FRÖBERG and BORIS SHAPIRO

Abstract

Motivated by the famous Skolem-Mahler-Lech theorem we initiate in this paper the study of a natural class of determinantal varieties, which we call *Vandermonde varieties*. They are closely related to the varieties consisting of all linear recurrence relations of a given order possessing a non-trivial solution vanishing at a given set of integers. In the regular case, i.e., when the dimension of a Vandermonde variety is the expected one, we present its free resolution, obtain its degree and the Hilbert series. Some interesting relations among Schur polynomials are derived. Many open problems and conjectures are posed.

1. Introduction

The results in the present paper come from an attempt to understand the famous Skolem-Mahler-Lech theorem and its consequences. Let us briefly recall its formulation. A linear recurrence relation with constant coefficients of order k is an equation of the form

$$u_n + \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + \cdots + \alpha_k u_{n-k} = 0, \quad n \geq k, \tag{1}$$

where the coefficients $(\alpha_1, \ldots, \alpha_k)$ are fixed complex numbers and $\alpha_k \neq 0$. (Equation (1) is often referred to as a linear homogeneous difference equation with constant coefficients.)

The left-hand side of the equation

$$t^k + \alpha_1 t^{k-1} + \alpha_2 t^{k-2} + \cdots + \alpha_k = 0 \tag{2}$$

is called the *characteristic polynomial* of recurrence (1). Denote the roots of (2) (listed with possible repetitions) by $x_1, \ldots, x_k$, and call them the *characteristic roots* of (1).

Notice that all x_i are non-vanishing since $\alpha_k \neq 0$. To obtain a concrete solution of (1) one has to prescribe additionally an initial k-tuple, $(u_0, \ldots, u_{k-1})$, which can be chosen arbitrarily. Then u_n, $n \geq k$, are obtained by using relation (1). A solution of (1) is called *non-trivial* if not all of its entries vanish. In case of all distinct characteristic roots a general solution of (1) can be given by

$$u_n = c_1 x_1^n + c_2 x_2^n + \cdots + c_k x_k^n,$$

Received 19 February 2014.

where $c_1, \dots, c_k$ are arbitrary complex numbers. In the general case of multiple characteristic roots a similar formula can be found in e.g. [15].

An arbitrary solution of a linear homogeneous difference (or differential) equation with constant coefficients of order k is called an *exponential polynomial of order k*. One usually substitutes $x_i \neq 0$ by e^{γ_i} and considers the obtained function in C instead of Z or N. (Other terms used for exponential polynomials are *quasipolynomials* or *exponential sums*.)

The most fundamental fact about the structure of integer zeros of exponential polynomials is the well-known Skolem-Mahler-Lech theorem formulated below. It was first proved for recurrence sequences of algebraic numbers by K. Mahler [11] in the 1930's, based upon an idea of T. Skolem [14]. Then, C. Lech [9] published the result for general recurrence sequences in 1953. In 1956 Mahler published the same result, apparently independently (but later realized to his chagrin that he had actually reviewed Lech's paper some years earlier, but had forgotten it).

THEOREM 1 (The Skolem-Mahler-Lech theorem). *If $a_0, a_1, \dots$ is a solution to a linear recurrence relation, then the set of all k such that $a_k = 0$ is the union of a finite (possibly empty) set and a finite number (possibly zero) of full arithmetic progressions. (Here, a full arithmetic progression means a set of the form $\{r, r+d, r+2d, \dots\}$ with $0 < r < d$.)*

A simple criterion guaranteeing the absence of arithmetic progressions is that no quotient of two distinct characteristic roots of the recurrence relation under consideration is a root of unity, see e.g. [10]. A recurrence relation (1) satisfying this condition is called *non-degenerate*. Substantial literature is devoted to finding the upper/lower bounds for the maximal number of arithmetic progressions/exceptional roots among all/non-degenerate linear recurrences of a given order. We give more details in §3. Our study is directly inspired by these investigations.

Let L_k be the space of all linear recurrence relations (1) of order at most k with constant coefficients. Denote by $L_k^* = L_k \setminus \{\alpha_k = 0\}$ the subset of all linear recurrence of order exactly k. (L_k is the affine space with coordinates $(\alpha_1, \dots, \alpha_k)$.) To an arbitrary pair $(k; I)$ where $k \geq 2$ is a positive integer and $I = \{i_0 < i_1 < i_2 < \dots < i_{m-1}\}$, $m \geq k$, is a sequence of integers, we associate the variety $V_{k;I} \subset L_k^*$, the set of all linear recurrences of order exactly k, having a non-trivial solution vanishing at all points of I. Denote by $\overline{V_{k;I}}$ the closure of $V_{k;I}$ in L_k in the usual topology. We call $V_{k;I}$ (resp. $\overline{V}_{k;I}$) the open (resp. closed) *linear recurrence variety* associated to the pair $(k; I)$.

In what follows we will always assume that $\gcd(i_1 - i_0, \dots, i_{m-1} - i_0) = 1$ to avoid unnecessary freedom related to the time rescaling in (1). Notice that since for $m \leq k - 1$ one has $V_{k;I} = L_k^*$ and $\overline{V}_{k;I} = L_k$, this case does not require special consideration. A more important observation is that due to translation invariance of (1) for any integer ℓ and any pair $(k; I)$ the variety $V_{k;I}$ (resp. $\overline{V}_{k;I}$) coincides with the variety $V_{k;I+\ell}$ (resp. $\overline{V}_{k;I+\ell}$) where the set of integers $I+\ell$ is obtained by adding ℓ to all entries of I.

So far we defined $\overline{V}_{k;I}$ and $V_{k;I}$ as sets. However for any pair $(k; I)$ the set $\overline{V}_{k;I}$ is an affine algebraic variety, see Proposition 4. Notice that this fact is not completely obvious since if we, for example, instead of a set of integers choose as I an arbitrary subset of real or complex numbers then the similar subset of L_n will, in general, only be analytic.

Now we define the Vandermonde variety associated with a given pair $(k; I)$, $I = \{0 \le i_0 < i_1 < i_2 < \cdots < i_{m-1}\}$, $m \ge k$. Firstly, consider the set $M_{k;I}$ of (generalized) Vandermonde matrices of the form

$$M_{k;I} = \begin{pmatrix} x_1^{i_0} & x_2^{i_0} & \cdots & x_k^{i_0} \\ x_1^{i_1} & x_2^{i_1} & \cdots & x_k^{i_1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{i_{m-1}} & x_2^{i_{m-1}} & \cdots & x_k^{i_{m-1}} \end{pmatrix}, \tag{3}$$

where $(x_1, \dots, x_k) \in \mathbb{C}^k$. In other words, for a given pair $(k; I)$, we take the map $M_{k;I} \colon \mathbb{C}^k \to \mathrm{Mat}(m, k)$ given by (3), where $\mathrm{Mat}(m, k)$ is the space of all $m \times k$-matrices with complex entries and $(x_1, \dots, x_k)$ are chosen coordinates in $\mathbb{C}^k$.

We now define three slightly different but closely related versions of this variety as follows.

Version 1. Given a pair $(k; I)$ with $|I| \ge k$, define the *coarse Vandermonde variety* $Vd^{\mathbf{c}}_{k;I} \subset M_{k;I}$ as the set of all degenerate Vandermonde matrices, i.e., whose rank is smaller than k. $Vd^{\mathbf{c}}_{k;I}$ is obviously an algebraic variety whose defining ideal $\mathcal{I}_I$ is generated by all $\binom{m}{k}$ maximal minors of $M_{k;I}$. Denote the quotient ring by $\mathcal{R}_I = \mathcal{R}/\mathcal{I}_I$.

Denote by $\mathcal{A}_k \subset \mathbb{C}^k$ the standard Coxeter arrangement (of the Coxeter group A_{k-1}) consisting of all diagonals $x_i = x_j$, and by $\mathcal{BC}_k \subset \mathbb{C}^k$ the Coxeter arrangement consisting of all $x_i = x_j$ and $x_i = 0$. Obviously, $\mathcal{BC}_k \supset \mathcal{A}_k$. Notice that $Vd^{\mathbf{c}}_{k;I}$ always includes the arrangement $\mathcal{BC}_k$ if $i_0 > 0$ (some of the hyperplanes with multiplicities), which is often inconvenient. Namely, with very few exceptions this means that $Vd^{\mathbf{c}}_{k;I}$ is not equidimensional, not CM, not reduced etc. For applications to linear recurrences as well as questions in combinatorics and geometry of Schur polynomials it seems more natural to consider the localizations of $Vd^{\mathbf{c}}_{k;I}$ in $\mathbb{C}^k \setminus \mathcal{A}_k$ and in $\mathbb{C}^k \setminus \mathcal{BC}_k$.

Version 2. Define the $\mathcal{A}_k$-localization $Vd^{\mathcal{A}}_{k;I}$ of $Vd^{\mathbf{c}}_{k;I}$ as the contraction of

$Vd^{\mathbf{c}}_{k;I}$ to $\mathbf{C}^k \setminus \mathscr{A}_k$. Its is easy to obtain the generating ideal of $Vd^{\mathscr{A}}_{k;I}$. Namely, recall that given a sequence $J = (j_1 < j_2 < \cdots < j_k)$ of nonnegative integers, one defines the associated Schur polynomial $S_J(x_1, \ldots, x_k)$ as given by

$$S_J(x_1, \ldots, x_k) = \begin{vmatrix} x_1^{j_1} & x_2^{j_1} & \cdots & x_k^{j_1} \\ x_1^{j_2} & x_2^{j_2} & \cdots & x_k^{j_2} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{j_k} & x_2^{j_k} & \cdots & x_k^{j_k} \end{vmatrix} \Bigg/ W(x_1, \ldots, x_k),$$

where $W(x_1, \ldots, x_k)$ is the usual Vandermonde determinant. Given a sequence $I = (0 \le i_0 < i_1 < i_2 < \cdots < i_{m-1})$ with $\gcd(i_1 - i_0, \ldots, i_{m-1} - i_0) = 1$, consider the set of all its $\binom{m}{k}$ subsequences J_κ of length k. Here the index κ runs over the set of all subsequences of length k among $\{1, 2, \ldots, m\}$. Take the corresponding Schur polynomials $S_{J_\kappa}(x_1, \ldots, x_k)$ and form the ideal $\mathscr{I}^{\mathscr{A}}_I \subseteq \mathbf{C}[x_1, \ldots, x_k]$ generated by all $\binom{m}{k}$ such Schur polynomials $S_{J_\kappa}(x_1, \ldots, x_k)$. One can show that the Vandermonde variety $Vd^{\mathscr{A}}_{k;I} \subset \mathbf{C}^k$ is defined by $\mathscr{I}^{\mathscr{A}}_I$ *set-theoretically*, see Lemma 5. Denote the quotient ring by $\mathscr{R}^{\mathscr{A}}_I = \mathscr{R}/\mathscr{I}^{\mathscr{A}}_I$ where $\mathscr{R} = \mathbf{C}[x_1, \ldots, x_k]$. Analogously, to the coarse Vandermonde variety $Vd^{\mathscr{A}}_{k;I}$ often contains irrelevant coordinate hyperplanes which prevents it from having nice algebraic properties. For example, if $i_0 > 0$ then all coordinate hyperplanes necessarily belong to $Vd^{\mathscr{A}}_{k;I}$ ruining equidimensionality etc. On the other hand, under the assumption that $i_0 = 0$ the variety $Vd^{\mathscr{A}}_{k;I}$ often has quite reasonable properties presented below.

Version 3. Define the $\mathscr{BC}_k$-localization $Vd^{BC}_{k;I}$ of $Vd^{\mathbf{c}}_{k;I}$ as the contraction of $Vd^{\mathbf{c}}_{k;I}$ to $\mathbf{C}^k \setminus \mathscr{BC}_k$. Again it is straightforward to find the generating ideal of $Vd^{BC}_{k;I}$. Namely, given a sequence $J = (0 \le j_1 < j_2 < \cdots < j_k)$ of nonnegative integers define the reduced Schur polynomial $\hat{S}_J(x_1, \ldots, x_k)$ as given by

$$\hat{S}_J(x_1, \ldots, x_k) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1^{j_2-j_1} & x_2^{j_2-j_1} & \cdots & x_k^{j_2-j_1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{j_k-j_1} & x_2^{j_k-j_1} & \cdots & x_k^{j_k-j_1} \end{vmatrix} \Bigg/ W(x_1, \ldots, x_k).$$

In other words, $\hat{S}_J(x_1, \ldots, x_k)$ is the usual Schur polynomial corresponding to the sequence $(0, j_2 - j_1, \ldots, j_k - j_1)$. Given a sequence $I = (0 \le i_0 < i_1 < i_2 < \cdots < i_{m-1})$ with $\gcd(i_1 - i_0, \ldots, i_{m-1} - i_0) = 1$, consider as before the set of all its $\binom{m}{k}$ subsequences J_κ of length k, where the index κ runs over the set of all subsequences of length k. Take the corresponding reduced

Schur polynomials $\hat{S}_{J_\kappa}(x_1, \dots, x_k)$ and form the ideal $\mathscr{I}_I^{BC} \subseteq \mathbb{C}[x_1, \dots, x_k]$ generated by all $\binom{m}{k}$ such Schur polynomials $\hat{S}_{J_\kappa}(x_1, \dots, x_k)$. One can easily see that the Vandermonde variety $Vd_{k;I}^{BC} \subset \mathbb{C}^k$ is defined set-theoretically by $\mathscr{I}_I^{BC}$. Denote the quotient ring by $\mathscr{R}_I^{BC} = \mathscr{R}/\mathscr{I}_I^{BC}$.

CONJECTURE 2. *If* $\dim(Vd_{k;I}^{BC}) \geq 2$ *then* $\mathscr{I}_I^{BC}$ *is a radical ideal.*

Notice that considered as sets the restrictions to $\mathbb{C}^k \setminus \mathscr{BC}_k$ of all three varieties $Vd_{k;I}^{\mathfrak{c}}$, $Vd_{k;I}^{\mathscr{A}}$, $Vd_{k;I}^{BC}$ coincide with what we call the *open Vandermonde variety* $Vd_{k;I}^{op}$ which is the subset of all matrices of the form $M_{k;I}$ with three properties:

(i) rank is smaller than k;

(ii) all x_i's are non-vanishing;

(iii) all x_i's are pairwise distinct.

Thus set-theoretically all the differences between the three Vandermonde varieties are concentrated on the hyperplane arrangement $\mathscr{BC}_k$. Also from the above definitions it is obvious that $Vd_{k;I}^{op}$ and $Vd_{k;I}^{BC}$ are invariant under addition of an arbitrary integer to I. The relation between the linear recurrence variety $V_{k;I}$ and the open Vandermonde variety $Vd_{k;I}^{op}$ is quite straight-forward. Namely, consider the standard Vieta map

$$Vi \colon \mathbb{C}^k \to L_k \tag{4}$$

sending an arbitrary k-tuple $(x_1, \dots, x_k)$ to the polynomial $t^k + \alpha_1 t^{k-1} + \alpha_2 t^{k-2} + \cdots + \alpha_k$ whose roots are $x_1, \dots, x_k$. Inverse images of the Vieta map are exactly the orbits of the standard S_k-action on $\mathbb{C}^k$ by permutations of coordinates. Thus, the Vieta map sends a homogeneous and symmetric polynomial to a weighted homogeneous polynomial.

Define the *open linear recurrence variety* $V_{k;I}^{op} \subseteq V_{k;I}$ of a pair $(k; I)$ as consisting of all recurrences in $V_{k;I}$ with all characteristic roots distinct. The following statement is obvious.

LEMMA 3. *The map Vi restricted to* $Vd_{k;I}^{op}$ *gives an unramified* $k!$*-covering of the set* $V_{k;I}^{op}$.

Unfortunately at the present moment the following natural question is still open.

PROBLEM 1. Is it true that that $\overline{V_{k;I}^{op}} = V_{k;I}$ for any pair $(k; I)$, where $\overline{V_{k;I}^{op}}$ is the set-theoretic closure of $V_{k;I}^{op}$ in L_k^*? If 'not', then under what additional assumptions?

Our main results are as follows. Using the Eagon-Northcott resolution of determinantal ideals, we determine the resolution, and hence the Hilbert series and degree of $\mathscr{R}_I^{\mathscr{A}}$ in Theorem 6. We give an alternative calculation of this degree using the Giambelli-Thom-Porteous formula in Proposition 8. In the simplest non-trivial case, when $m = k + 1$, we get more detailed information about $Vd_{k;I}^{\mathscr{A}}$. We prove that its codimension is 2, and that $\mathscr{R}_I^{\mathscr{A}}$ is Cohen-Macaulay. We also discuss minimal sets of generators of $\mathscr{I}_I$, and determine when we have a complete intersection in Theorem 9. (The proof of this theorem gives some interesting relations between Schur polynomials, see Theorem 10.) In this case the variety has the expected codimension, which is not always the case for $m > k + 1$. In fact our computer experiments suggest that then the codimension rather seldom is the expected one. In case $k = 3, m = 5$, we show that having the expected codimension is equivalent to $\mathscr{R}_I^{\mathscr{A}}$ being a complete intersection, and that $\mathscr{I}_I$ is generated by three complete symmetric functions. Exactly the problem (along with many other similar questions) when three complete symmetric functions constitute a regular sequence was considered in a paper [4], where the authors formulated a detailed conjecture. We slightly strengthen their conjecture below.

For the $\mathscr{B}\mathscr{C}_k$-localized variety $Vd_{k;I}^{BC}$ we have only proofs when $k = 3$, but we present Conjectures 15 and 16, supported by many calculations. We end the paper with a section which describes the connection of our work with the fundamental problems in linear recurrence relations.

Acknowledgements. The authors want to thank Professor Maxim Kazarian (Steklov Institute of Mathematical Sciences) for his help with the Giambelli-Thom-Porteous formula, Professor Igor Shparlinski (Macquarie University) for highly relevant information on the Skolem-Mahler-Lech theorem and Professors Nicolai Vorobjov (University of Bath) and Michael Shapiro (Michigan State University) for discussions. We are especially grateful to Professor Winfried Bruns (University of Osnabrück) for pointing out important information on determinantal ideals. We also thank the anonymous referee for a careful reading, and especially for noting a mistake in Example 7.

2. Results and conjectures on Vandermonde varieties

We start by proving that $\overline{V}_{k;I}$ is an affine algebraic variety, see the Introduction.

Proposition 4. *For any pair $(k; I)$ the set $\overline{V}_{k;I}$ is an affine algebraic variety. Therefore, $V_{k;I} = \overline{V}_{k;I}|_{L_k^*}$ is a quasi-affine variety.*

Proof. We will show that for any pair $(k; I)$ the variety $\overline{V}_{k;I}$ of linear recurrences is constructible. Since it is by definition closed in the usual topology of $L_k \simeq \mathbb{C}^k$, it is algebraic. The latter fact follows from [12], I.10 Corollary 1, claiming that if $Z \subset X$ is a constructible subset of a variety, then the Zarisky closure and the strong closure of Z are the same. Instead of showing that

$\overline{V}_{k;I}$ is constructible, we prove that $V_{k;I} \subset L_k^*$ is constructible. Namely, we can use an analog of Lemma 3 to construct a natural stratification of $V_{k;I}$ into the images of quasi-affine sets under appropriate Vieta maps. Namely, let us stratify $V_{k;I}$ as $V_{k;I} = \bigcup_{\lambda \vdash k} V_{k;I}^{\lambda}$, where $\lambda \vdash k$ is an arbitrary partition of k and $V_{k;I}^{\lambda}$ is the subset of $V_{k;I}$ consisting of all recurrence relations of length k which has a non-trivial solution vanishing at each point of I and whose characteristic polynomial determines the partition λ of its degree k. In other words, if $\lambda = (\lambda_1, \dots, \lambda_s)$, $\sum_{j=1}^{s} \lambda_j = k$, then the characteristic polynomial should have s distinct roots of multiplicities $\lambda_1, \dots, \lambda_s$ resp. Notice that any of these $V_{k;I}^{\lambda}$ can be empty including the whole $V_{k;I}$ in which case there is nothing to prove. Let us now show that each $V_{k;I}^{\lambda}$ is the image under the appropriate Vieta map of a set similar to the open Vandermonde variety. Recall that if $\lambda = (\lambda_1, \dots, \lambda_s)$, $\sum_{j=1}^{s} \lambda_j = k$, and $x_1, \dots, x_s$ are the distinct roots with the multiplicities $\lambda_1, \dots, \lambda_s$ respectively of the linear recurrence (1) then the general solution of (1) has the form

$$u_n = P_{\lambda_1}(n)x_1^n + P_{\lambda_2}(n)x_2^n + \cdots + P_{\lambda_s}(n)x_s^n,$$

where $P_{\lambda_1}(n), \dots, P_{\lambda_s}(n)$ are arbitrary polynomials in the variable n of degrees $\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_{s-1} - 1$ resp. Now, for a given $\lambda \vdash k$ consider the set of matrices

$$M_{k;I}^{\lambda} = \begin{pmatrix} x_1^{i_0} & i_0 x_1^{i_0} & \cdots & i_0^{\lambda_1 - 1} x_1^{i_0} & \cdots & x_s^{i_0} & i_0 x_s^{i_0} & \cdots & i_0^{\lambda_s - 1} x_s^{i_0} \\ x_1^{i_1} & i_1 x_1^{i_1} & \cdots & i_1^{\lambda_1 - 1} x_1^{i_1} & \cdots & x_s^{i_1} & i_1 x_s^{i_1} & \cdots & i_1^{\lambda_s - 1} x_s^{i_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_1^{i_{m-1}} & i_{m-1} x_1^{i_{m-1}} & \cdots & i_{m-1}^{\lambda_1 - 1} x_1^{i_{m-1}} & \cdots & x_s^{i_{m-1}} & i_1 x_s^{i_{m-1}} & \cdots & i_{m-1}^{\lambda_s - 1} x_s^{i_{m-1}} \end{pmatrix}.$$

In other words, we are taking the fundamental solution $x_1^n, nx_1^n, \dots, n^{\lambda_1 - 1}x_1^n$, $x_2^n, nx_2^n, \dots, n^{\lambda_2 - 1}x_2^n, \dots, x_s^n, nx_s^n, \dots, n^{\lambda_s - 1}x_s^n$ of (1) under the assumption that the characteristic polynomial gives a partition $\lambda \vdash k$ and we are evaluating each function in this system at $i_0, i_1, \dots, i_{m-1}$, resp. We now define the variety $Vd_{k;I}^{\lambda}$ as the subset of matrices of the form $M_{k;I}^{\lambda}$ such that: (i) the rank of such a matrix is smaller than k; (ii) all x_i are distinct; (iii) all x_i are non-vanishing. Obviously, $Vd_{k;I}^{\lambda}$ is a quasi-projective variety in $\mathbb{C}^s$. Define the analog $Vi_{\lambda}\colon \mathbb{C}^s \to L_k$ which sends an s-tuple $(x_1, \dots, x_s) \in \mathbb{C}^s$ to the polynomials $\prod_{j=1}^{s}(x - x_j)^{\lambda_j} \in L_k$. One can easily see that Vi_{λ} maps $Vd_{k;I}^{\lambda}$ onto $V_{k;I}^{\lambda}$. Applying this construction to all partitions $\lambda \vdash k$ we will obtain that $V_{k;I} = \bigcup_{\lambda \vdash k} V_{k;I}^{\lambda}$ is constructible, which finishes the proof.

The remaining part of the paper is devoted to the study of the Vandermonde varieties $Vd_{k;I}^{\mathscr{A}}$ and $Vd_{k;I}^{BC}$. We start with the $\mathscr{A}_k$-localized variety $Vd_{k;I}^{\mathscr{A}}$. Notice that if $m = k$ the variety $Vd^{\mathscr{A}}_{k;I} \subset \mathsf{C}^k$ is an irreducible hypersurface given by the equation $S_I = 0$ and its degree equals $\sum_{j=0}^{k-1} i_j - \binom{k}{2}$ see the definition in Version 2 above). We will need the following alternative description of the ideal $\mathscr{I}^{\mathscr{A}}_I$ in the general case. Namely, using the Jacobi-Trudi identity for the Schur polynomials, we get the following statement.

LEMMA 5. *For any pair $(k; I)$, $I = \{i_0 < i_1 < \cdots < i_{m-1}\}$, the ideal $\mathscr{I}^{\mathscr{A}}_I$ is generated by all $k \times k$-minors of the $m \times k$-matrix*

$$H_{k;I} = \begin{pmatrix} h_{i_0-(k-1)} & h_{i_0-(k-2)} & \cdots & h_{i_0} \\ h_{i_1-(k-1)} & h_{i_1-(k-2)} & \cdots & h_{i_1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_{m-1}-(k-1)} & h_{i_{m-1}-(k-2)} & \cdots & h_{i_{m-1}} \end{pmatrix}. \tag{5}$$

Here h_i denotes the complete symmetric function of degree i, $h_i = 0$ if $i < 0$, $h_0 = 1$.

PROOF. It follows directly from the standard Jacobi-Trudi identity for the Schur polynomials, see e.g. [16].

In particular, Lemma 5 shows that $Vd^{\mathscr{A}}_{k;I}$ is a determinantal variety in the usual sense. When working with $Vd^{\mathscr{A}}_{k;I}$ and unless the opposite is explicitly mentioned, we will assume that $I = \{0 < i_1 < \cdots < i_{m-1}\}$, i.e. that $i_0 = 0$ and that additionally $\gcd(i_1, \ldots, i_{m-1}) = 1$. Let us first study some properties of $Vd^{\mathscr{A}}_{k;I}$ in the so-called *regular case*, i.e. when its dimension coincides with the expected one.

Namely, consider the set $\Omega_{m,k} \subset \mathrm{Mat}(m, k)$ of all $m \times k$-matrices having positive corank. It is well-known that $\Omega_{m,k}$ has codimension equal to $m-k+1$. Since $Vd^{\mathsf{C}}_{k;I}$ coincides with the pullback of $\Omega_{m,k}$ under the map $M_{k;I}$ and $Vd^{\mathscr{A}}_{k;I}$ is closely related to it (with trivial pathology on $\mathscr{A}_k$ removed), the expected codimension of $Vd^{\mathscr{A}}_{k;I}$ equals $m - k + 1$. We call a pair $(k; I)$ *$\mathscr{A}$-regular* if $k \leq m \leq 2k - 1$ (implying that the expected dimension of $Vd^{\mathscr{A}}_{k;I}$ is positive) and the actual codimension of $Vd^{\mathscr{A}}_{k;I}$ coincides with its expected codimension. We now describe the Hilbert series of the quotient ring $\mathscr{R}^{\mathscr{A}}_I$ in the case of an arbitrary regular pair $(k; I)$ using the well-known resolution of determinantal ideals of Eagon-Northcott [6].

To explain the notation in the following theorem, we introduce two gradings, tdeg and deg, on $\mathsf{C}[t_0, \ldots, t_{k-1}]$. The first one is the usual grading induced by $\mathrm{tdeg}(t_i) = 1$ for all i, and a second one is induced by $\deg(t_i) = -i$. In the next theorem M denotes a monomial in $\mathsf{C}[t_0 \ldots, t_{k-1}]$.

THEOREM 6. *In the above notation*

(a) *the Hilbert series* $\mathrm{Hilb}_I^{\mathscr{A}}(t)$ *of* $\mathscr{R}_I^{\mathscr{A}} = \mathscr{R}/\mathscr{I}_I^{\mathscr{A}}$ *is given by*

$$\mathrm{Hilb}_I^{\mathscr{A}}(t) = \frac{1 - t^{-\binom{k}{2}} \sum_{i=0}^{m-k} (-1)^{i+1} \Big(\sum_{J \subseteq I, |J|=k+i} t^{s_J} \sum_{\mathrm{tdeg}\, M=i} t^{\deg(M)} \Big)}{(1-t)^m},$$

where $s_J = \sum_{i_j \in J} i_j$.

(b) *The degree of* $\mathscr{R}_I^{\mathscr{A}}$ *is* $T^{(m-k+1)}(1)(-1)^{m-k+1}/(m-k+1)!$, *where* $T(t)$ *is the numerator in* (a).

PROOF. According to [6] provided that $\mathscr{I}_I^{\mathscr{A}}$ has the expected codimension $m-k+1$, it is known to be Cohen-Macaulay and it has a resolution of the form

$$0 \to F_{m-k+1} \to \cdots \to F_1 \to \mathscr{R} \to \mathscr{R}_I^{\mathscr{A}} \to 0, \tag{6}$$

where F_j is free module over $\mathscr{R} = \mathsf{C}[x_1, \dots, x_k]$ of rank $\binom{m}{k+j-1}\binom{k+j-2}{k-1}$. We denote the basis elements of F_j by JM, where $J \subseteq \{i_0, \dots, i_{m-1}\}$, $|J| = k + j - 1$, and M is an arbitrary monomial in $\{t_0, \dots, t_{k-1}\}$ of degree $j-1$. Here, in our situation, J has degree $\sum_{i_j \in J} i_j$ and M has degree $\deg M - \binom{k}{2}$. Observe that this resolution is never minimal. Indeed, for any sequence $I = \{0 = i_0 < i_1 < \cdots < i_{m-1}\}$, we only need the Schur polynomials coming from subsequences starting with 0, so $\mathscr{I}_I^{\mathscr{A}}$ is generated by at most $\binom{m-1}{k-1}$ Schur polynomials instead of totally $\binom{m}{k}$; see also discussions preceding the proof of Theorem 9 below. Now, if $\mathscr{J}$ is an arbitrary homogeneous ideal in $\mathscr{R} = \mathsf{C}[x_0, \dots, x_{m-1}]$ and $\mathscr{R}/\mathscr{J}$ has a resolution

$$0 \to \bigoplus_{i=1}^{\beta_r} \mathscr{R}(-n_{r,i}) \to \cdots \to \bigoplus_{i=1}^{\beta_1} \mathscr{R}(-n_{1.i}) \to \mathscr{R} \to \mathscr{R}/\mathscr{J} \to 0,$$

then the Hilbert series of $\mathscr{R}/\mathscr{J}$ is given by

$$\frac{1 - \sum_{i=1}^{\beta_1} t^{n_{1,i}} + \cdots + (-1)^r \sum_{i=1}^{\beta_r} t^{n_{r,i}}}{(1-t)^m}.$$

For the resolution (6), all terms coming from JM with $i_0 \notin J$ cancel. If the Hilbert series is given by $T(t)/(1-t)^k = P(t)/(1-t)^{\dim(\mathscr{R}/\mathscr{I}_I)}$, then the degree of the corresponding variety equals $P(1)$. We have $T(t) = (1-t)^{m-k+1}P(t)$, so after differentiating the latter identity $m-k+1$ times we get

$$P(1) = T^{(m-k+1)}(1)(-1)^{m-k+1}/(m-k+1)!.$$

EXAMPLE 7. For the case 3×5 with $I = \{0, i_1, i_2, i_3, i_4\}$, if the ideal $Vd^{\mathscr{A}}_{k;I}$ has the right codimension, we get that its Hilbert series equals $T(t)/(1-t)^5$, where

$$T(t) = 1 - t^{-3}(t^{i_1+i_2} + t^{i_1+i_3} + t^{i_1+i_4} + t^{i_2+i_3} + t^{i_2+i_4} + t^{i_3+i_4}) \\ + (t^{-4} + t^{-5})(t^{i_1+i_2+i_3} + t^{i_1+i_2+i_4} + t^{i_1+i_3+i_4} + t^{i_2+i_3+i_4}) \\ + (t^{-5} + t^{-6} + t^{-7})t^{i_1+i_2+i_3+i_4}$$

and the degree of $Vd^{\mathscr{A}}_{k;I}$ equals

$$i_1i_2i_3 + i_1i_2i_4 + i_1i_3i_4 + i_2i_3i_4 - 3(i_1i_2 + i_1i_3 + i_1i_4 + i_2i_3 + i_2i_4 + i_3i_4) \\ + 7(i_1 + i_2 + i_3 + i_4) - 15.$$

An alternative way to calculate $\deg(Vd^{\mathscr{A}}_{k;I})$ is to use the Giambelli-Thom-Porteous formula, see e.g. [8]. The next result corresponded to the authors by M. Kazarian explains how to do that.

PROPOSITION 8. *Assume that $Vd^{\mathscr{A}}_{k;I}$ has the expected codimension $m-k+1$. Then its degree (taking multiplicities of the components into account) is equal to the coefficient of t^{m-k+1} in the Taylor expansion of the series*

$$\frac{\prod_{j=1}^{m-1}(1+i_jt)}{\prod_{j=1}^{k-1}(1+jt)}.$$

More explicitly,

$$\deg(Vd^{\mathscr{A}}_{k;I}) = \sum_{j}^{m-k+1} \sigma_j(I)u_{m-k+1-j},$$

where σ_j is the jth elementary symmetric function of the entries $(i_1, \dots, i_{m-1})$ and $u_0, u_1, u_2, \dots$ are the coefficients in the Taylor expansion of $\prod_{j=1}^{k-1}\frac{1}{1+jt}$, i.e. $u_0 + u_1t + u_2t^2 + \cdots = \prod_{j=1}^{k-1}\frac{1}{1+jt}$. In particular, $u_0 = 1$, $u_1 = -\binom{k}{2}$, $u_2 = \binom{k+1}{3}\frac{3k-2}{4}$, $u_3 = -\binom{k+2}{4}\binom{k}{2}$, $u_4 = \binom{k+3}{5}\frac{15k^3-15k^2-10k+8}{48}$.

PROOF. In the Giambelli formula setting, we consider a "generic" family of $n \times \ell$-matrices $A = \|a_{p,q}\|$, $1 \le p \le n$, $1 \le q \le \ell$, whose entries are homogeneous functions of degrees $\deg(a_{p,q}) = \alpha_p - \beta_q$ in parameters $(x_1, \dots, x_k)$ for some fixed sequences $\beta = (\beta_1, \dots, \beta_\ell)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$. Denote by Σ^r the subvariety in the parameter space $\mathbb{C}^k$ determined by the condition that the matrix A has rank at most $\ell - r$, that is, the linear operator $A\colon \mathbb{C}^\ell \to \mathbb{C}^n$

has at least a r-dimensional kernel. Then the expected codimension of the subvariety Σ^r is equal to

$$\operatorname{codim}(\Sigma^r) = r(n - \ell + r).$$

In case when the actual codimension coincides with the expected one its degree is computed as the following $r \times r$-determinant:

$$\deg(\Sigma^r) = \det \|c_{n-\ell+r-i+j}\|_{1 \le i,\ j \le r}, \tag{7}$$

where the entries c_i's are defined by the Taylor expansion

$$1 + c_1 t + c_2 t^2 + \cdots = \frac{\prod_{p=1}^{n}(1+\alpha_p t)}{\prod_{q=1}^{\ell}(1+\beta_q t)}.$$

There is a number of situations where this formula can be applied. Depending on the setting, the entries α_p, β_q can be rational numbers, formal variables, first Chern classes of line bundles or formal Chern roots of vector bundles of ranks n and ℓ, respectively. In the situation of Theorem 6 we should use the presentation (5) of $Vd^{\mathcal{A}}_{k;I}$ from Lemma 5. Then we have $n = m$, $\ell = k$, $r = 1$, $\alpha = I = (0, i_1, \ldots, i_{m-1})$, $\beta = (k-1, k-2, \ldots, 0)$. Under the assumptions of Theorem 6 the degree of the Vandermonde variety $Vd^{\mathcal{A}}_{k;I}$ will be given by the 1×1-determinant of the Giambelli-Thom-Porteous formula (7), that is, the coefficient c_{m-k+1} of t^{m-k+1} in the expansion of

$$1 + c_1 t + c_2 t^2 + \cdots = \frac{\prod_{j=0}^{m-1}(1+i_j t)}{\prod_{j=1}^{k}(1+(k-j)t)} = \frac{\prod_{j=1}^{m-1}(1+i_j t)}{\prod_{j=1}^{k-1}(1+jt)},$$

which gives exactly the stated formula for $\deg(Vd^{\mathcal{A}}_{k;I})$.

In the simplest non-trivial case $m = k + 1$ one can obtain more detailed information about $Vd^{\mathcal{A}}_{k;I}$. Notice that for $m = k + 1$ the $k + 1$ Schur polynomials generating the ideal $\mathcal{I}^{\mathcal{A}}_{I}$ are naturally ordered according to their degree. Namely, given an arbitrary $I = \{0 < i_1 < i_1 < \cdots < i_k\}$ with $\gcd(i_1, \ldots, i_k) = 1$ denote by S_j, $j = 0, \ldots, k$ the Schur polynomial obtained by removal of the (j)-th row of the matrix $M_{k;I}$. (Pay attention that here we enumerate the rows starting from 0.) Then, obviously, $\deg S_k < \deg S_{k-1} < \cdots < \deg S_0$. Using presentation (5) we get the following.

THEOREM 9. *For any integer sequence $I = \{0 = i_0 < i_1 < i_2 < \cdots < i_k\}$ of length $k + 1$ with* $\gcd(i_1, \ldots, i_k) = 1$ *the following facts are valid:*

(i) $\operatorname{codim}(Vd^{\mathcal{A}}_{k;I}) = 2$;

(ii) *the quotient ring* $\mathscr{R}_I^{\mathscr{A}}$ *is Cohen-Macaulay;*

(iii) *the Hilbert series* $\mathrm{Hilb}_I^{\mathscr{A}}(t)$ *of* $\mathscr{R}_I^{\mathscr{A}}$ *is given by the formula*

$$\mathrm{Hilb}_I^{\mathscr{A}}(t) = \left(1 - \sum_{j=1}^{k} t^{N-i_j-\binom{k}{2}} + \sum_{j=1}^{k-1} t^{N-j-\binom{k}{2}}\right) \Big/ (1-t)^k,$$

where $N = \sum_{j=1}^{k} i_j$;

(iv) $\deg(Vd_{k;I}^{\mathscr{A}}) = \sum_{1\le j<\ell\le k} i_j i_\ell - \binom{k}{2}\sum_{j=1}^{k} i_j + \binom{k+1}{3}(3k-2)/4$;

(v) *the ideal* $\mathscr{I}_I^{\mathscr{A}}$ *is always generated by k generators* $S_k, \dots, S_1$ (*i.e., the last generator* S_0 *always lies in the ideal generated by* $S_k, \dots, S_1$). *Moreover, if for some* $1 \le n \le k-2$ *one has* $i_n \le k-n$, *then* $\mathscr{I}_I^{\mathscr{A}}$ *is generated by* $k-n$ *elements* $S_k, \dots, S_{n+1}$. *In particular, it is generated by two elements* S_k, S_{k-1} (*i.e., is a complete intersection*) *if* $i_{k-2} \le k-1$.

The theorem gives some relations between Schur polynomials.

THEOREM 10. *Let the generators be* $S_k = s_{i_{k-1}-k+1, i_{k-2}-k+2, \dots, i_1-1}$, $S_{k-1}, \dots, S_0 = s_{i_k-(k-1), i_{k-1}-(k-2), \dots, i_1}$ *in degree increasing order. For* $s = 0, 1 \dots, k-1$ *we have*

$$h_{i_k-s}S_k - h_{i_{k-1}-s}S_{k-1} + \cdots + (-1)^{k-1}h_{i_1-s}S_1 + (-1)^k h_{-s}S_0 = 0.$$

Here $h_i = 0$ *if* $i < 0$.

To prove Theorems 9 and 10 notice that since Schur polynomials are irreducible [5], in the case $m = k+1$ the ideal $\mathscr{I}_I^{\mathscr{A}}$ always has the expected codimension 2, unless it coincides with the whole ring $\mathbb{C}[x_1, \dots, x_k]$. Therefore vanishing of any two Schur polynomials lowers the dimension by two. (Recall that we assume that $\gcd(i_1, \dots, i_k) = 1$.) On the other hand, as we mentioned in the introduction the codimension of $Vd_{k;I}^{\mathscr{A}}$ in this case is at most 2. For $m = k+1$ one can present a very concrete resolution of the quotient ring $\mathscr{R}_I^{\mathscr{A}}$.

Namely, given a sequence $I = \{0 = i_0 < i_1 < \cdots < i_k\}$ we know that the ideal $\mathscr{I}_I^{\mathscr{A}}$ is generated by the $k+1$ Schur polynomials $S_\ell = s_{a_k, a_{k-1}, \dots, a_1}$, $\ell = 0, \dots, k$, where

$$(a_k, \dots, a_1) = (i_k, i_{k-1}, \dots, i_{\ell+1}, \hat{i}_\ell, i_{\ell-1}, \dots, i_0) - (k-1, k-2, \dots, 1, 0).$$

Obviously, S_ℓ has degree $\sum_{j=1}^{k} i_j - i_\ell - \binom{k}{2}$ and by the Jacobi-Trudi identity is given by

$$S_\ell = \begin{vmatrix} h_{i_0-(k-1)} & h_{i_0-(k-2)} & \cdots & h_{i_0} \\ h_{i_1-(k-1)} & h_{i_1-(k-2)} & \cdots & h_{i_1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_{\ell-1}-(k-1)} & h_{i_{\ell-1}-(k-2)} & \cdots & h_{i_{\ell-1}} \\ h_{i_{\ell+1}-(k-1)} & h_{i_{\ell+1}-(k-2)} & \cdots & h_{i_{\ell+1}} \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_{k-1}-(k-1)} & h_{i_{k-1}-(k-2} & \cdots & h_{i_{k-1}} \\ h_{i_k-(k-1)} & h_{i_k-(k-2)} & \cdots & h_{i_k} \end{vmatrix}.$$

Here (as above) h_j denotes the complete symmetric function of degree j in $x_1, \ldots, x_k$. (We set $h_j = 0$ if $j < 0$ and $h_0 = 1$.) Consider the $(k+1) \times k$-matrix $H = H_{k;I}$ given by

$$H = \begin{pmatrix} h_{i_0-(k-1)} & h_{i_0-(k-2)} & \cdots & h_{i_0} \\ h_{i_1-(k-1)} & h_{i_1-(k-2)} & \cdots & h_{i_1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{i_{k-1}-(k-1)} & h_{i_{k-1}-(k-2)} & \cdots & h_{i_{k-1}} \\ h_{i_k-(k-1)} & h_{i_k-(k-2)} & \cdots & h_{i_k} \end{pmatrix}.$$

Let H_ℓ be the $(k+1) \times (k+1)$-matrix obtained by extending H with ℓ-th column of H. Notice that $\det(H_\ell) = 0$, and expanding it along the last column we get for $0 \le \ell \le k-1$ the relation

$$0 = \det(H_\ell) = h_{i_k-(k-\ell)} S_k - h_{i_{k-1}-(k-\ell)} S_{k-1} + \cdots + (-1)^{k-1} h_{i_1-(k-\ell)} S_1.$$

For $\ell = k$ we get

$$h_{i_k} S_k - h_{i_{k-i}} S_{k-1} + \cdots + (-1)^k h_{i_0} S_0 = 0,$$

which implies that S_0 always lie in the ideal generated by the remaining $S_1, \ldots, S_k$.

We now prove Theorem 9.

Proof. Set $N = \sum_{j=1}^k i_j$. For an arbitrary $I = \{0, i_1, \ldots, i_k\}$ with $\gcd(i_1, \ldots, i_k) = 1$ we get the following resolution of the quotient ring $\mathscr{R}_I^{\mathscr{A}} = \mathscr{R}/\mathscr{I}_I^{\mathscr{A}}$

$$0 \longrightarrow \bigoplus_{\ell=1}^{k} \mathscr{R}(-N + \binom{k}{2} + \ell) \longrightarrow \bigoplus_{\ell=1}^{k-1} \mathscr{R}\left(-N + i_\ell + \binom{k}{2}\right) \\ \longrightarrow \mathscr{R} \longrightarrow \mathscr{R}_I^{\mathscr{A}} \longrightarrow 0,$$

where $\mathscr{R} = \mathsf{C}[x_1, \dots, x_k]$. Simple calculation with this resolution implies that the Hilbert series $\mathrm{Hilb}_I^{\mathscr{A}}(t)$ of $\mathscr{R}_I^{\mathscr{A}}$ is given by

$$\mathrm{Hilb}_I^{\mathscr{A}}(t) = \left(1 - \sum_{\ell=1}^{k} t^{N-i_\ell-\binom{k}{2}} + \sum_{\ell=1}^{k-1} t^{N-\binom{k}{2}-\ell}\right) \Big/ (1-t)^k$$

and the degree of $Vd_{k;I}^{\mathscr{A}}$ is given by

$$\deg(Vd_{k;I}^{\mathscr{A}}) = \sum_{1\le r<s\le k} i_r i_s - \binom{k}{2}\sum_{r=1}^{k} i_r + \binom{k+1}{3}(3k-2)/4.$$

Notice that the latter resolution might not be minimal, since the ideal might have fewer than k generators. To finish proving Theorem 9 notice that if conditions of (v) are satisfied then a closer look at the resolution reveals that the Schur polynomials $S_0, \dots, S_{k-n}$ lie in the ideal generated by $S_{k-n+1}, \dots, S_k$.

In connection with Theorems 6 and 9 the following question is completely natural.

PROBLEM 2. Under the assumptions $i_0 = 0$ and $\gcd(i_1, \dots, i_{m-1}) = 1$ which pairs $(k; I)$ are $\mathscr{A}$-regular?

Theorem 9 shows that for $m = k + 1$ the condition $\gcd(i_1, \dots, i_k) = 1$ guarantees regularity of any pair $(k; I)$ with $|I| = k + 1$. On the other hand, our computer experiments with Macaulay suggest that for $m > k$ regular cases are rather seldom. In particular, we were able to prove the following.

THEOREM 11. *If $m > k$ a necessary (but insufficient) condition for $Vd_{k;I}^{\mathscr{A}}$ to have the expected codimension is $i_1 = 1$.*

PROOF. If $i_1 \ge 2$, then $i_{k-2} \ge k-1$. This means that the ideal is generated by Schur polynomials $s_{a_0,\dots,a_{k-1}}$ with $a_{k-2} \ge 1$. Multiplying these up to degree n gives linear combinations of Schur polynomials $s_{b_1,\dots,b_{k-1}}$ with $b_{k-2} \ge 1$. Thus we miss all Schur polynomials with $b_{k-2} = 0$. The number of such Schur polynomials equals the number of partitions of n in at most $k-2$ parts. The number of partitions of n in exactly $k-2$ parts is approximated with $n^{k-3}/((k-2)!(k-1)!)$. Thus the number of elements of degree n in the ring is at least cn^{k-3} for some positive c, so the ring has dimension $\ge k-2$. The expected dimension is $\le k-3$, which is a contradiction.

So far a complete (conjectural) answer to Problem 2 is only available in the first non-trivial case $k = 3, m = 5$. Namely, for a 5-tuple $I = \{0, 1, i_2, i_3, i_4\}$ to be regular one needs the corresponding Vandermonde variety $Vd^{\mathcal{A}}_{3;I}$ to be a complete intersection. This is due to the fact that in this situation the ideal $\mathcal{I}^{\mathcal{A}}_I$ is generated by the Schur polynomials S_4, S_3, S_2 of the least degrees in the above notation. Notice that $S_4 = h_{i_2-2}, S_3 = h_{i_3-2}, S_2 = h_{i_4-2}$. Thus $Vd^{\mathcal{A}}_{3;I}$ has the expected codimension (equal to 3) if and only if $\mathbb{C}[x_1, x_2, x_3]/\langle h_{i_2-2}, h_{i_3-2}, h_{i_4-2}\rangle$ is a complete intersection or, in other words, $h_{i_2-2}, h_{i_3-2}, h_{i_4-2}$ is a regular sequence. Exactly this problem (along with many other similar questions) was considered in the intriguing paper [4] where the authors formulated the following claim, see Conjecture 2.17 of [4].

CONJECTURE 12. *Let $A = \{a, b, c\}$ with $a < b < c$. Then h_a, h_b, h_c in three variables is a regular sequence if and only if the following conditions are satisfied:*

(1) $abc \equiv 0 \mod 6$;

(2) $\gcd(a + 1, b + 1, c + 1) = 1$;

(3) *For all $t \in \mathbb{N}$ with $t > 2$ there exists $d \in A$ such that $d + 2 \not\equiv 0, 1 \mod t$.*

In fact, our experiments allow us to strengthen the latter conjecture in the following way.

CONJECTURE 13. *In the above set-up if the sequence h_a, h_b, h_c with $a > 1$ in three variables is not regular, then h_c lies in the ideal generated by h_a and h_b. (If $(a, b, c) = (1, 4, 3k + 2)$, $k \geq 1$, then h_a, h_b, h_c neither is a regular sequence, nor $h_c \in (h_a, h_b)$.)*

We note that if we extend the set-up of [4] by allowing Schur polynomials $s(r, s, t)$ instead of just complete symmetric functions then if $t > 0$ in all three of them the sequence is never regular. Conjectures 12 and 13 provide a criterion which agrees with our calculations of $\dim(Vd^{\mathcal{A}}_{3;I})$. Finally, we made experiments checking how $\dim(Vd^{\mathcal{A}}_{k;I})$ depends on the last entry i_{m-1} of $I = \{0, 1, i_2, \ldots, i_{m-1}\}$ while keeping the first $m - 1$ entries fixed.

CONJECTURE 14. *For any given $I = (0, 1, i_2, \ldots, i_{m-1})$ the dimension $\dim(Vd^{\mathcal{A}}_{k;I})$ depends periodically on i_{m-1} for all i_{m-1} sufficiently large.*

Notice that Conjecture 14 follows from Conjecture 12 in the special case $k = 3$, $m = 5$. Unfortunately, we do not have a complete description of the length of this period in terms of the fixed part of I and it might be quite tricky.

For the $\mathcal{BC}_k$-localized variety $Vd^{BC}_{k;I}$ we have, except for $k = 3$, only conjectures, supported by many calculations.

CONJECTURE 15. *For any integer sequence $I = \{0 = i_0 < i_1 < i_2 < \cdots < i_k\}$ of length $k + 1$ with $\gcd(i_1, \ldots, i_k) = 1$ the following facts are valid.*

(i) $\operatorname{codim}(Vd^{BC}_{k;I}) = 2$;

(ii) *the quotient ring* $\mathcal{R}^{BC}_I$ *is Cohen-Macaulay;*

(iii) *there is a* $\mathcal{C}[x_1, \dots, x_n] = R$*-resolution of* $\mathcal{R}^{BC}_I$ *of the form*

$$0 \to \bigoplus_{j=0}^{k-1} R\left[-N + j + \binom{k}{2}\right] \to \bigoplus_{j=1}^{k} R\left[-N + i_j + \binom{k}{2}\right] \oplus R[-N + ki_1] \to R \to \mathcal{R}^{BC}_I \to 0$$

(iv) *the Hilbert series* $\operatorname{Hilb}^{BC}_I(t)$ *of* $\mathcal{R}^{BC}_I$ *is given by the formula*

$$\operatorname{Hilb}^{BC}_I(t) = \left(1 - \sum_{j=1}^{k} t^{N-j-\binom{k}{2}} - t^{N-ki_1} + \sum_{j=1}^{k-1} t^{N-i_1-j} - \binom{k}{2}\right) / (1-t)^k$$

where $N = \sum_{j=1}^{k} i_j$;

(v) $$\deg(Vd^{BC}_{k;I}) = \sum_{1 \le j < \ell \le k} i_j i_\ell - \binom{k}{2} \sum_{j=1}^{k} i_j + \binom{k+1}{3}(3k-2)/4 - \binom{k}{2} i_1(i_1 - 1);$$

(vi) *the ideal* $\mathcal{I}^{BC}_I$ *is always generated by k generators. It is generated by two elements* (*i.e., is a complete intersection*) *if* $i_1 \le k - 1$.

CONJECTURE 16. *Let* $S_k, \dots, S_1$ *be as in Theorem 10 and* $G_0 = s_{i_k-i_1-k+1,\dots,i_2-i_1-1}$. *Then, for* $s = 0, \dots, k-1$ *we have*

$$h_{i_k-i_1-s} S_k - h_{i_{k-1}-i_1-s} S_{k-1} + \cdots + (-1)^{k-2} h_{i_2-i_1-s} S_2 + (-1)^{k-1} h_{-s} S_1 + (-1)^k s_{i_1-1,\dots,(i_1-1)^{k-1},k-1-s} G_0 = 0.$$

Here $h_i = 0$ *if* $i < 0$ *and* $h_{i,\dots,i,j} = 0$ *if* $j > i$, *and* $(i_1 - 1)^{k-1}$ *means* $i_1 - 1, \dots, i_1 - 1$ ($k - 1$ *times*).

That the ring is CM follows from the fact that the ideal is generated by the maximal minors of a $t \times m$-matrix in the ring of Laurent polynomials. To prove the theorem it suffices to prove the relations between the Schur polynomials. Unfortunately we have managed to do that only for $k = 3$.

3. Final remarks

Here we briefly explain the source of our interest in Vandermonde varieties. In 1977, J. H. Loxton and A. J. van der Poorten formulated an important conjecture (Conjecture 1′ of [10]) claiming that there exists a constant μ_k such that any integer recurrence of order k either has at most μ_k integer zeros or has infinitely many zeros.

This conjecture was first settled by W. M. Schmidt in 1999, see [13] and also by J. H. Evertse and H. P. Schlickewei, see [7].

The upper bound for μ_k obtained in [13] was

$$\mu_k < e^{e^{e^{3k \log k}}},$$

which was later improved by the same author to

$$\mu_k < e^{e^{e^{20k}}}.$$

Apparently the currently best known upper bound for μ_k was obtained in [1] and is given by

$$\mu_k < e^{e^{k^{\sqrt{11k}}}}.$$

Although the known upper bounds are at least double exponential it seems plausible that the realistic upper bounds should be polynomial. The only known nontrivial lower bound for μ_k was found in [2] and is given by

$$\mu_k \geq \binom{k+1}{2} - 1.$$

One should also mention the non-trivial exact result of F. Beukers showing that for sequences of rational numbers obtained from recurrence relations of length 3 one has $\mu_3 = 6$, see [3].

The initial idea of this project was to try to obtain upper/lower bounds for μ_k by studying algebraic and geometric properties of Vandermonde varieties but they seem to be quite complicated. Let us finish with some further problems and comments on them, that we got with an extensive computer search. Many questions related to the Skolem-Mahler-Lech theorem translate immediately into questions about $V_{k;I}$. For example, one can name the following formidable challenges.

PROBLEM 3. For which pairs $(k; I)$ the variety $V_{k;I}$ is empty/non-empty? More generally, what is the dimension of $V_{k;I}$?

We made a complete computer search for $\mathcal{R}_I^{\mathcal{A}}$ and some variants where we removed solutions on the coordinate planes and axes, and looked for arithmetic sequences, for $(0, i_1, i_2, i_3)$, $0 < i_1 < i_2 < i_3$, $i_3 \leq 13$ (so $k = 3, m = 4$). The only cases when $V_{k;I}$ was empty were $I = (0, 1, 3, 7)$ and $I = (0, 1, 3, 9)$ and their "duals" $(0, 4, 6, 7)$ and $(0, 6, 8, 9)$. We suspect that our exceptions are the only possible. For $k = 3, m = 5$ we investigated $I = (0, i_1, i_2, i_3, i_4)$,

$0 < i_1 < i_2 < i_3 < i_4$, $i_4 \leq 9$. For $i_1 = 1$ about half of the cases had the expected dimension. For $(k, m) = (3, 6)$, $i_5 \leq 10$, for $(k, m) = (4, 6)$, $i_5 \leq 9$ and for $(k, m) = (5, 8)$, $i_7 \leq 10$, most cases were of expected dimension. The corresponding calculations for $\mathscr{R}_I^{BC}$, $(k, m) = (3, 5)$, $i_4 \leq 9$, showed that about half of the cases had expected codimension.

PROBLEM 4. For which pairs $(k; I)$ any solution of a linear recurrence vanishing at I must have an additional integer root outside I? More specifically, for which pairs $(k; I)$ any solution of a linear recurrence vanishing at I must vanish infinitely many times in Z? In other words, for which pairs $(k; I)$ the set of all integer zeros of the corresponding solution of any recurrence relation from $V_{k;I}$ must necessarily contain an arithmetic progression?

For example, in case $k = 3$, $m = 4$ we found that the first situation occurs for 4-tuples $(0, 1, 4, 6)$ and $(0, 1, 4, 13)$ which both force a non-trivial solution of a third order recurrence vanishing at them to vanish at the 6-tuple $(0, 1, 4, 6, 13, 52)$, which is the basic example in [3]. The second situation occurs if in a 4-tuple $I = \{0, i_1, i_2, i_3\}$ two differences between its entries coincide, see [3]. But this condition is only sufficient and no systematic information is available. Notice that for any pair $(k; I)$ the variety $\overline{V}_{k;I}$ is weighted-homogeneous where the coordinate α_i, $i = 1, \ldots, k$ has weight i. (This action corresponds to the scaling of the characteristic roots of (2).)

We looked for cases containing an arithmetic sequence with difference at most 10 and we found cases which gave arithmetic sequences with difference 2, 3, 4 and 5, and a few cases which didn't give any arithmetic sequences.

PROBLEM 5. Is it true that if an $(k+1)$-tuple I consists of two pieces of arithmetic progression with the same difference then any exponential polynomial vanishing at I contains an arithmetic progression of integer zeros?

PROBLEM 6. If the answer to the previous question is positive is it true that there are only finitely many exceptions from this rule leading to only arithmetic progressions?

Finally a problem similar to that of J. H. Loxton and A. J. van der Poorten can be formulated for real zeros of exponential polynomials instead of integer. Namely, the following simple lemma is true.

LEMMA 17. *Let $\lambda_1, \ldots, \lambda_n$ be a arbitrary finite set of (complex) exponents having all distinct real parts then an arbitrary exponential polynomial of the form $c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \cdots + c_n e^{\lambda_n z}$, $c_i \in \mathsf{C}$, has at most finitely many real zeros.*

PROBLEM 7. Does there exist an upper bound on the maximal number real for the set of exponential polynomials given in the latter lemma in terms of n only?

PROBLEM 8. What about non-regular cases? Describe their relation to the existence of additional integer zeros and arithmetic progressions as well as additional Schur polynomials in the ideals.

REFERENCES

1. Allen, P. B., *On the multiplicity of linear recurrence sequences*, J. Number Theory 126 (2007), no. 2, 212–216.
2. Bavencoffe, E., and Bézivin, J.-P., *Une famille remarquable de suites récurrentes linéaires*, Monatsh. Math. 120 (1995), no. 3-4, 189–203.
3. Beukers, F., *The zero-multiplicity of ternary recurrences*, Compositio Math. 77 (1991), no. 2, 165–177.
4. Conca, A., Krattenthaler, C., and Watanabe, J., *Regular sequences of symmetric polynomials*, Rend. Semin. Mat. Univ. Padova 121 (2009), 179–199.
5. Dvornicich, R., and Zannier, U., *Newton functions generating symmetric fields and irreducibility of Schur polynomials*, Adv. Math. 222 (2009), no. 6, 1982–2003.
6. Eagon, J. A. and Northcott, D. G., *Ideals defined by matrices and a certain complex associated with them*, Proc. Roy. Soc. Ser. A 269 (1962), 188–204.
7. Evertse, J.-H., and Schlickewei, H. P., *A quantitative version of the absolute subspace theorem*, J. Reine Angew. Math. 548 (2002), 21–127.
8. Fulton, W., *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. 65 (1992), no. 3, 381–420.
9. Lech, C., *A note on recurring series*, Ark. Mat. 2 (1953), 417–421.
10. Loxton, J. H. and Van der Poorten, A. J., *On the growth of recurrence sequences*, Math. Proc. Cambridge Philos. Soc. 81 (1977), no. 3, 369–376.
11. Mahler, K., *Eine arithmetische Eigenschaft der Taylor-Koeffizienten rationaler Funktionen*, Proc. Akad. Wet. Amsterdam 38 (1935), 50–60.
12. Mumford, D., *The red book of varieties and schemes*, expanded ed., Lecture Notes in Mathematics, vol. 1358, Springer-Verlag, Berlin, 1999.
13. Schmidt, W. M., *The zero multiplicity of linear recurrence sequences*, Acta Math. 182 (1999), no. 2, 243–282.
14. Skolem, T., *Einige Sätze über gewisse Reihenentwicklungen und exponentiale Beziehungen mit Anwendung auf diophantische Gleichungen*, Skr. Norske Vidensk.-Akad. Oslo, 1, Mat.-Naturvidensk. Kl. 6 (1933), 1–61.
15. Stanley, R. P., *Enumerative combinatorics. Vol. I*, The Wadsworth & Brooks/Cole Mathematics Series, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986.
16. Tamvakis, H., *The theory of Schur polynomials revisited*, Enseign. Math. (2) 58 (2012), no. 1-2, 147–163.

DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
SE-106 91, STOCKHOLM
SWEDEN
E-mail: ralff@math.su.se

DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
SE-106 91, STOCKHOLM
SWEDEN
E-mail: shapiro@math.su.se