MATH. SCAND. 119 (2016), 92–112

MARSTRAND'S APPROXIMATE INDEPENDENCE OF SETS AND STRONG DIFFERENTIATION OF THE INTEGRAL

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Abstract

A constructive proof is given for the existence of a function belonging to the product Hardy space $H^1(\mathsf{R} \times \mathsf{R})$ and the Orlicz space $L(\log L)^{\epsilon}(\mathsf{R}^2)$ for all $0 < \epsilon < 1$, for all whose integral is not strongly differentiable almost everywhere on a set of positive measure. It consists of a modification of a non-negative function created by J. M. Marstrand. In addition, we generalize the claim concerning "approximately independent sets" that appears in his work in relation to hyperbolic-crosses. Our generalization, which holds for any sets with boundary of sufficiently low complexity in any Euclidean space, has a version of the second Borel-Cantelli Lemma as a corollary.

1. Introduction

Given a real-valued function $f \in L^1_{\mathrm{loc}}(\mathsf{R}^d)$, $d \geq 2$, the strong derivative of the integral of f is defined in [11] and [5]. We adopt the notation from the latter and we consider differentiation with respect to *rectangles* (d-dimensional rectangular boxes) with sides parallel to the coordinate axes. The set of all such rectangles will be denoted by $\mathscr{R}$. For $x \in \mathsf{R}^d$, the *strong upper derivative* and the *strong lower derivative* of $\int f$ at x are defined by

$$\overline{D}\left(\int f, x\right) := \sup\left\{\limsup_{n\to\infty} \frac{1}{|R_n|} \int_{R_n} f(y)\, dy : \{R_n\}_{n\in\mathsf{N}} \subset \mathscr{R},\, R_n \to x\right\}$$

and

$$\underline{D}\left(\int f, x\right) := \inf\left\{\liminf_{n\to\infty} \frac{1}{|R_n|} \int_{R_n} f(y)\, dy : \{R_n\}_{n\in\mathsf{N}} \subset \mathscr{R},\, R_n \to x\right\},$$

respectively, where $|A|$ denotes the d-dimensional Lebesgue measure of a measurable set A in R^d and $R_n \to x$ means that $\{R_n\}_{n\in\mathsf{N}}$ satisfies: $x \in \bigcap_{n\in\mathsf{N}} R_n$ and $\lim_{n\to\infty} \operatorname{diam}(R_n) = 0$. If $\overline{D}(\int f, x)$ and $\underline{D}(\int f, x)$ coincide and are finite, then $\lim_{n\to\infty} |R_n|^{-1} \int_{R_n} f(y)\, dy$ exists for any $\{R_n\}_{n\in\mathsf{N}} \subset \mathscr{R}$ with $R_n \to x$, is

Received 13 January 2014.

denoted by $D(\int f, x)$ and is referred to as the *strong derivative* of $\int f$ at x. In this case we say that $\int f$ is *strongly differentiable* at x. Since every cube with sides parallel to the axes belongs to $\mathscr{R}$, if $\int f$ is strongly differentiable at a point x, then $D(\int f, x)$ agrees with the derivative of $\int f$ with respect to cubes at x. Thus, the classical differentiation theorem of Lebesgue implies that the equality $D(\int f, x) = f(x)$ holds for almost every point x in the set where $\int f$ is strongly differentiable.

The one-parameter *real Hardy space* $H^1(\mathsf{R}^d)$ [3] can be defined as the space of distributions f in $\mathscr{S}'(\mathsf{R}^d)$ such that $\sup_{t>0} \left|t^{-d}(f * \varphi)(t^{-1}x)\right|$ is integrable, for some fixed $\varphi \in \mathscr{S}(\mathsf{R}^d)$ with non-vanishing integral. The *product Hardy space* $H^1(\mathsf{R}^{d_1} \times \mathsf{R}^{d_2})$ [4] can be defined as the space of distributions f in $\mathscr{S}'(\mathsf{R}^{d_1+d_2})$ such that, for some fixed $\varphi \in \mathscr{S}(\mathsf{R}^{d_1})$, $\psi \in \mathscr{S}(\mathsf{R}^{d_2})$ with non-vanishing integrals,

$$\sup_{t_j>0}\left|t_1^{-d_1}t_2^{-d_2}\iint \varphi(t_1^{-1}y_1)\psi(t_2^{-1}y_2)f(x_1-y_1, x_2-y_2)\,dy_1\,dy_2\right|$$

is in $L^1(\mathsf{R}^{d_1+d_2})$, where the points x in $\mathsf{R}^{d_1} \times \mathsf{R}^{d_2}$ are represented as $x = (x_1, x_2)$, with $x_j \in \mathsf{R}^{d_j}$, $j = 1, 2$.

For each $0 < \epsilon < 1$, the *Orlicz space* $L(\log L)^\epsilon(\mathsf{R}^d)$ [7], also denoted $L^{\Phi_\epsilon}(\mathsf{R}^d)$, can be defined as the set of real-valued, measurable functions f on R^d such that

$$\int_{\mathsf{R}^d} \Phi_\epsilon\left(\frac{f(x)}{\lambda}\right) dx \le 1,$$

for some $\lambda > 0$, where $\Phi_\epsilon(t) := |t|\,(\log(1+|t|))^\epsilon$, $t \in \mathsf{R}$. The *Luxemburg norm* on $L^{\Phi_\epsilon}(\mathsf{R}^d)$ is defined by

$$\|f\|_{\Phi_\epsilon} := \inf\left\{\lambda > 0 : \int \Phi_\epsilon\left(\frac{f(x)}{\lambda}\right) dx \le 1\right\}.$$

Endowed with the norm $\|\cdot\|_{\Phi_\epsilon}$, $L^{\Phi_\epsilon}(\mathsf{R}^d)$ is a complete space.

While the integral of functions in $L^p_{\text{loc}}(\mathsf{R}^d)$, $p > 1$, is strongly differentiable a.e. [6] and this property also holds for the integral of functions which are locally in $L \log L(\mathsf{R}^2)$ [6], it fails for certain classes of functions satisfying slightly weaker integrability conditions [10]. In particular, it fails in $L^1_{\text{loc}}(\mathsf{R}^d)$. Since many results concerning boundedness of singular operators can be extended from $L^p(\mathsf{R}^d)$, $p > 1$, to the Hardy spaces $H^1(\mathsf{R}^d)$ [12], the question arose as to whether the strong differentiation of the integral would hold in $H^1(\mathsf{R}^d)$. This was answered negatively by Stokolos [15], who gave an example of a function f in the real Hardy space $H^1(\mathsf{R}^2)$ such that $|\overline{D}(\int f, x)| = |\underline{D}(\int f, x)| = \infty$ for almost every x in the unit square. We show that the answer is also negative for the space $H^1(\mathsf{R}\times\mathsf{R})\cap\left(\bigcap_{0<\epsilon<1} L(\log L)^\epsilon(\mathsf{R}^2)\right)$. In particular, $\mathscr{R}$ is not a differentiation basis (see definition in [5], [13], or [14]) for any Orlicz space $L(\log L)^\epsilon(\mathsf{R}^2)$ with $0<\epsilon<1$.

THEOREM 1.1. *There exists a function f in $H^1(\mathsf{R}\times\mathsf{R})\cap L(\log L)^\epsilon(\mathsf{R}^2)$ for all $0<\epsilon<1$, such that*

$$\left|\overline{D}\left(\int f, x\right)\right| = \left|\underline{D}\left(\int f, x\right)\right| = \infty \tag{1}$$

for almost every x on $\left[-\frac{1}{2},\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]$.

The proof of this theorem is in Section 3. In fact, we will, by modifying the example created by Marstrand [8], construct a function f that belongs to $H^1_{\text{rect}}(\mathsf{R}\times\mathsf{R})$ [1], the proper subspace of $H^1(\mathsf{R}\times\mathsf{R})$ which consists of sums of rectangular atoms with coefficients in ℓ^1. Then we show that f is in $L(\log L)^\epsilon(\mathsf{R}^2)$ for all $0<\epsilon<1$. The almost everywhere part relies on a variant of the second Borel-Cantelli lemma which extends the version used in [8]. This is a corollary of the theorem below, proved in Section 2, which illustrates how geometric properties can yield consequences of a probabilistic nature. In the next result and throughout this text, the notation $\alpha\sim\beta$, for $\alpha,\beta\in[0,\infty)$, means that there exist constants c, C such that $c\alpha\le\beta\le C\alpha$.

THEOREM 1.2. *Let $S_0\subset\mathsf{R}^d$ be the unit cube centered at the origin and let $\{A_n\}_{n\in\mathsf{N}}$ be a family of subsets of S_0 satisfying $|A_n|>0$ and $\delta_n:=\dim_{\text{upper box}}(\partial\overline{A_n})<d$ for all n. There is a sequence $\{m_n\}_{n\in\mathsf{N}}$ of positive integers such that if, for each n, we partition S_0 into m_n^d cubes of same the size, and place inside each a homothetic copy of A_n, then denoting by Λ_n the union of these homothetic copies, we have, for any finite subset $F\subset\mathsf{N}$,*

$$\left|\bigcap_{n\in F}\Lambda_n\right|\sim\prod_{n\in F}|\Lambda_n|\,. \tag{2}$$

This result generalizes Marstrand's statement [8, p. 210], where he claims, without proof, the approximately independence (in the probabilistic sense) of homothetic copies of certain "hyperbolic-cross" shaped sets:

$$\left\{(x_1,x_2)\in\mathsf{R}^2 : |x_1x_2|\le 1,\ x_1^2+x_2^2\le(n+1)(\log(n+1))^2\right\},\quad n\in\mathsf{N}. \tag{3}$$

Furthermore, we show that if the sets A_n are finite unions of dyadic cubes, then (2) holds with an equality.

We would like to thank A. M. Stokolos, who translated for me his paper [15] (only available to me in Russian); and G. Dafni, my doctoral supervisor.

2. Approximately independent sets

Before we begin, let us fix some notation. By a *cube* we mean a closed cube with sides parallel to the coordinate axes. Given a cube Q, we denote its side length by $\ell(Q)$ and its interior by Q°. Adopting the terminology used in [12], we say that two cubes P and Q *intersect* if $P^\circ \cap Q^\circ \neq \emptyset$ and are *disjoint* if $P^\circ \cap Q^\circ = \emptyset$. For a set A in R^d, we denote its closure by $\overline{A}$ and its upper box-counting dimension by $\dim_{\text{upper box}}(A)$, where the latter can be defined [2] as

$$\limsup_{m\to\infty} \frac{\log\left(\#\left\{j \in \mathsf{Z}^d : \left[\frac{j_1-1}{m}, \frac{j_1}{m}\right] \times \cdots \times \left[\frac{j_d-1}{m}, \frac{j_d}{m}\right] \cap A \neq \emptyset\right\}\right)}{\log(m)}.$$

REMARK 2.1. It can be shown that, for any bounded set $A \subset \mathsf{R}^d$, the following are equivalent:

(i) $\dim_{\text{upper box}}(\partial\overline{A}) \leq \delta_A$;

(ii) for any cube S in R^d containing A, there exist a constant $C_{A,S} > 0$ and an integer $\mathcal{N}_{A,S}$ satisfying:

$$\#\left\{j \in \{1, \ldots, m^d\} : S^\circ_{m,j} \cap \partial\overline{A} \neq \emptyset\right\} \leq C_{A,S}\, m^{\delta_A} \quad \forall m \geq \mathcal{N}_{A,S}, \tag{4}$$

where, for each $m > 0$, $\{S_{m,j}\}_{j=1}^{m^d}$ is a partition of S into m^d equal sized cubes.

LEMMA 2.1. *Consider a cube $S \subset \mathsf{R}^d$, centered at the origin, and a set $A \subset S$ such that $|A| > 0$ and $\dim_{\text{upper box}}(\partial\overline{A}) \leq \delta_A < d$ and let $\epsilon > 0$. For any integer m satisfying*

$$m \geq \max\left\{\mathcal{N}_{A,S}, \left(\frac{C_{A,S}\,|S|}{\epsilon\,|A|}\right)^{1/(d-\delta_A)}\right\},$$

where $\mathcal{N}_{A,S}$ and $C_{A,S}$ are as in Remark 2.1, and for any measurable set $E \subset S$, the following holds: if we partition S into m^d equal sized cubes $S_{m,j}$ with center $o_{m,j}$, $j = 1, \ldots, m^d$ and denote by $E_{m,j}$ the homothetic copies of E, namely

$$E_{m,j} := o_{m,j} + \frac{1}{m}E, \quad j = 1, \ldots, m^d, \tag{5}$$

then

$$(1-\epsilon)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|} \leq \frac{\left|A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j}\right)\right|}{|A|} \leq (1+\epsilon)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|}. \tag{6}$$

PROOF. A counting argument yields

$$\#\{j : S_{m,j} \subset \overline{A}\} \leq \frac{|A|}{|S_{m,1}|} = \frac{m^d}{|S|}\,|A|\,,$$

while Remark 2.1 gives us

$$\#\{j : S^\circ_{m,j} \cap \partial\overline{A} \neq \emptyset\} \leq C_{A,S}m^{\delta_A}.$$

If $|S_{m,j} \cap A| > 0$, then either $|S_{m,j} \cap A| = |S_{m,j}|$ or $0 < |S_{m,j} \cap A| < |S_{m,j}|$. Since $|S_{m,j} \cap A| = |S_{m,j}|$ is equivalent to $S_{m,j} \subset \overline{A}$, and since $0 < |S_{m,j} \cap A| < |S_{m,j}|$ implies $S^\circ_{m,j} \cap \partial\overline{A} \neq \emptyset$, it follows that

$$\mathfrak{N}_m = \mathfrak{N}_m(A, S) := \#\{j : |S_{m,j} \cap A| > 0\} \leq \frac{m^d}{|S|}\,|A| + C_{A,S}m^{\delta_A}. \tag{7}$$

Because the choice of m implies $C_{A,S}\,|S|\,m^{\delta_A - d} \leq \epsilon\,|A|$, we get

$$\mathfrak{N}_m \frac{|S|}{m^d} \leq (1+\epsilon)\,|A|\,. \tag{8}$$

As $E_{m_k,j} \subset S_{m,j}$ for each $1 \leq j \leq m^d$, the number of $E_{m,j}$'s satisfying $|A \cap E_{m,j}| > 0$ is at most $\mathfrak{N}_m$. So the proportion of A that lies inside $\bigcup_{j=1}^{m^d} E_{m,j}$ is

$$\frac{\left|A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j}\right)\right|}{|A|} \leq \frac{\mathfrak{N}_m \left|\frac{1}{m}E\right|}{|A|} = \frac{\mathfrak{N}_m\,|E|}{m^d\,|A|} \leq (1+\epsilon)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|},$$

where the last inequality follows by (8). Similarly,

$$\frac{\left|A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j}\right)\right|}{|A|} \geq \frac{(\#\{j : S_{m,j} \subset \overline{A}\})\left|\frac{1}{m}E\right|}{|A|} \geq \frac{\left(\frac{m^d}{|S|}\,|A| - \frac{C_{A,S}m^{\delta_A}}{|A|}\right)\left|\frac{1}{m}E\right|}{|A|}$$

$$= \left(1 - \frac{C_{A,S}\,|S|}{m^{d-\delta_A}\,|A|}\right)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|} \geq (1-\epsilon)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|}.$$

The example below illustrates a type of set for which the box-counting dimension of the closure is equal to the dimension of the ambient space and (6) holds for infinitely many integers m.

EXAMPLE 2.1. Let $\alpha \in (0, 1)$ and let $F = F_\alpha$ be the "fat" Cantor set constructed on $[0, 1]$ as the Cantor ternary set except that the 2^{k-1} intervals removed at step k have length $\alpha/3^k$ instead of $1/3^k$ (see for example [9, p. 64]).

When $\alpha = p/q \in \mathsf{Q}$, the endpoints of the intervals that remained after the k first steps of the building of F have the form $n/(2^k 3^k q)$ for some integer $0 \le n \le 2^k 3^k q$. Thus, when we partition [0, 1] into $m := 2^k 3^k q$ intervals of the same length, the sum of the lengths of the intervals of that partition which intersect F is exactly the measure of the union of the closed intervals that remained on [0, 1] after the k-th step of the construction of F, i.e.

$$\frac{1}{m}\#\left\{ j \in \mathsf{N} : \left[\frac{j-1}{m}, \frac{j}{m}\right] \cap F \neq \emptyset \right\} = 1 - \left(\frac{\alpha}{3} + 2\frac{\alpha}{3^2} + \cdots + 2^{k-1}\frac{\alpha}{3^k}\right).$$

Defining $A := F - 1/2$, it follows that, when we partition $S := [-1/2, 1/2]$ into m intervals $S_{m,j} := [(j-1)/m, j/m] - 1/2$, $j = 1, \ldots, m$, we obtain

$$\frac{1}{m}\mathfrak{N}_m = 1 - \frac{\alpha}{3}\sum_{i=1}^{k-1}\left(\frac{2}{3}\right)^i \to 1 - \alpha = |A| \quad \text{as } k \to \infty,$$

where $\mathfrak{N}_m$ is as in (7). Thus, given $\epsilon > 0$, $\exists k_0 \in \mathsf{N}$ such that (8) holds with $m = 2^k 3^k q$ for all $k \ge k_0$. So the argument used to prove Lemma 2.1 yields (6).

In higher dimensions, if a subset A of $S \subset \mathsf{R}^d$ satisfies $|A| > 0$ and

$$\liminf_{m\to\infty}\left(\frac{|S|}{m^d}\mathfrak{N}_m\right) = |A|, \tag{9}$$

then (6) holds for infinitely many integers m. What (9) says is that we can approximate the volume of A with a regular grid of boxes. When $\dim_{\text{upper box}}(\partial\overline{A}) \le \delta_A < d$, (9) holds since (7) implies that $|S|\,\mathfrak{N}_m m^{-d}$ converges to $|A|$ as $m \to \infty$.

However, as shown by the example below, the result of Lemma 2.1 fails if $\dim_{\text{upper box}}(\partial\overline{A}) = d$.

EXAMPLE 2.2. Let $G := F_\alpha - 1/2$, where F_α is as in Example 2.1 with $\alpha = 3/4$. We define a set A (by filling the gaps in G) as follows

$$A := G \cup \left\{\bigcup_{m=1}^{\infty}\left[\bigcup_{j=1}^{m}\left(-\frac{1}{2} + \frac{j-1/2}{m} + \frac{1}{2^m m}G\right)\right]\right\},$$

and note that $A \subset S := [-1/2, 1/2]$ and $1/4 \le |A| \le 1/2$. Moreover, $\mathfrak{N}_m = m$ for any $m \in \mathsf{N}$, since, by construction,

$$\left|\left[-\frac{1}{2} + \frac{j-1}{m}, -\frac{1}{2} + \frac{j}{m}\right] \cap A\right| \ge \left|\frac{1}{2^m m}G\right| > 0 \quad \forall 1 \le j \le m, \forall m \in \mathsf{N}.$$

Fix $m \in \mathsf{N}$ and let $E := 2^{-m} G$. Then (6) fails for all $0 < \epsilon < 1$. Indeed, using the notation in (5), $E_{m,j} = -\frac{1}{2} + \frac{j-1/2}{m} + \frac{1}{2^m m} G \subset A, \forall j$. So $|A \cap E_{m,j}| = |E_{m,j}| = 2^{-m} m^{-1} |G|, \forall j$, and it follows that

$$\left| A \cap \left(\bigcup_{j=1}^{m} E_{m,j} \right) \right| = \sum_{j=1}^{m} |A \cap E_{m,j}| = m \frac{1}{2^m m} |G| = |E| = \left| \bigcup_{j=1}^{m} E_{m,j} \right|. \tag{10}$$

By the choice of A, S and ϵ, we have $\frac{1}{|A|} > \frac{1+\epsilon}{|S|}$, which, combined with (10), implies that (6) does not hold.

Recall that in a probability space $(\Omega, \mathscr{F}, P)$, two events $E_1, E_2 \in \mathscr{F}$ are said to be independent if $P(E_1 \cap E_2) = P(E_1)P(E_2)$. Letting Ω be S; $\mathscr{F}$ be the σ-algebra of Lebesgue measurable subsets of S; and $P(E_1) := |E_1| / |S|$ for $E_1 \subset S$ measurable, Lemma 2.1 shows that for certain measurable sets $A \subset S$, there exist arbitrarily large integers m such that, for any measurable set $E \subset S$,

$$P\left(A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j} \right) \right) \sim P(A) P\left(\bigcup_{j=1}^{m^d} E_{m,j} \right),$$

where the $E_{m,j}$'s are as in (5). We call this property "approximately independence" and we extend it to infinitely many sets as is (2).

PROOF OF THEOREM 1.2. We will construct a sequence $\{m_n\}_{n \in \mathsf{N}} \subset \mathsf{N}$, such that when we partition S_0 into m_n^d cubes $S_{m_n,j}$, $j = 1, \ldots, m_n^d$, of same the size, let $o_{m_n,j}$ denote the center of $S_{m_n,j}$, and set

$$\Lambda_n := \bigcup_{j=1}^{m_n^d} \left(o_{m_n,j} + \frac{1}{m_n} A_n \right), \quad n \in \mathsf{N}, \tag{11}$$

we obtain (2). It suffices to show that we can choose $\{m_n\}_{n \in \mathsf{N}}$ such that

$$\prod_{i \in F} \left(1 - \frac{1}{4i^2} \right) |\Lambda_i| \le \left| \bigcap_{i \in F} \Lambda_i \right| \le \prod_{i \in F} (1 + 2^{-(i-1)}) |\Lambda_i|, \quad \forall F \subset \{1, \ldots, n\}, \tag{12}$$

holds for all $n \in \mathsf{N}$. Indeed, using the representation $\sin \frac{\pi}{2} = \frac{\pi}{2} \prod_{j \in \mathsf{N}} \left(1 - \frac{1}{4j^2}\right)$ and the inequality $1 + t \le e^t \ \forall t \in [0, 1]$, we get from (12) that, for any finite set $F \subset \mathsf{N}$,

$$\frac{2}{\pi}\prod_{i\in F}|\Lambda_i| = \prod_{j\in\mathsf{N}}\Big(1-\frac{1}{4j^2}\Big)\prod_{i\in F}|\Lambda_i| \le \prod_{j\in F}\Big(1-\frac{1}{4j^2}\Big)\prod_{i\in F}|\Lambda_i|$$

$$\le \Big|\bigcap_{i\in F}\Lambda_i\Big| \le \prod_{i\in F}(1+2^{-(i-1)})\,|\Lambda_i| \le \prod_{i\in F}e^{2^{-(i-1)}}\,|\Lambda_i| \le e^2\prod_{i\in F}|\Lambda_i|\,.$$

To construct $\{m_n\}_{n\in\mathsf{N}}$, we use induction. Choose $m_1 = 1$. Then $\Lambda_1 = A_1$ and

$$(1-4^{-1})\,|\Lambda_1| \le |\Lambda_1| \le (1+2^{-(1-1)})\,|\Lambda_1|\,.$$

Now, assume that the integers $m_1,\dots,m_n$ are chosen such that (12) holds. By definition, Λ_k is composed of m_k^d homothetic copies of A_k. So $\dim_{\text{upper box}}(\partial\overline{\Lambda_k}) = \delta_k$, since $\dim_{\text{upper box}}$ is bi-Lipschitz invariant and finitely stable [2, p. 48]. For any finite subset $F \subset \{1,\dots,n\}$, the boundary of the closure of $\Gamma_F := \bigcap_{i\in F}\Lambda_i$ satisfies

$$\dim_{\text{upper box}}(\partial\overline{\Gamma_F}) \le \gamma_n := \max\{\delta_k : 1\le k\le n\},$$

because $\partial\overline{\Gamma_F} \subset \bigcap_{i\in F}\partial\overline{\Lambda_i}$ and $\dim_{\text{upper box}}$ is finitely stable [2]. We claim that if

$$C_n := \sum_{k=1}^{n} C_{\Lambda_k,S_0} \quad\text{and}\quad \mathcal{N}_n := \sum_{k=1}^{n}\mathcal{N}_{\Lambda_k,S_0}, \tag{13}$$

then it is possible to take $C_{\Gamma_F,S_0} = C_n$ and $\mathcal{N}_{\Gamma_F,S_0} = \mathcal{N}_n$ in (4). Indeed, if we take $m \ge \mathcal{N}_n$ and partition S_0 into m^d cubes $S_{m,j}$, $j = 1,\dots,m^d$, then the number of cubes $S_{m,j}$ which intersect $\partial\overline{\Lambda_k}$ is not greater than $C_{\Lambda_k,S_0}m^{\delta_k}$, $1\le k\le n$. Since $\partial\overline{\Gamma_F} \subset \bigcup_{k=1}^{n}\partial\overline{\Lambda_k}$, the number of cubes $S_{m,j}$ which intersect $\partial\overline{\Gamma_F}$ is not greater than $\sum_{k=1}^{n}C_{\Lambda_k,S_0}m^{\delta_k} \le C_n m^{\gamma_n}$, and we conclude that our claim holds.

We choose m_{n+1} to be an integer such that

$$m_{n+1} \ge \max\Bigg\{\mathcal{N}_n \max_{\substack{I\subset\{1,\dots,n\}\\ |\bigcap_{i\in I}\Lambda_i|>0}}\Bigg\{\Bigg(2^n C_n\Big|\bigcap_{i\in I}\Lambda_i\Big|^{-1}\Bigg)^{1/(d-\gamma_n)}\Bigg\}\Bigg\}, \tag{14}$$

and we will show that, for any subset $F \subset \{1,\dots,n\}$ such that $\big|\bigcap_{i\in F}\Lambda_i\big| > 0$,

$$\prod_{i\in F\cup\{n+1\}}\Big(1+\frac{1}{4i^2}\Big)\,|\Lambda_i| \le \Big|\bigcap_{i\in F\cup\{n+1\}}\Lambda_i\Big| \le \prod_{i\in F\cup\{n+1\}}(1+2^{-(i-1)})\,|\Lambda_i|$$

holds. The case when $\big|\bigcap_{i\in F}\Lambda_i\big| = 0$ is trivial.

Fix $F \subset \{1, \dots, n\}$ such that $\Gamma_F := \bigcap_{i \in F} \Lambda_i$ has positive measure. We intend to use Lemma 2.1 with

$$S = S_0, \quad A = \Gamma_F, \quad \epsilon = 2^{-n}, \quad E = A_{n+1}, \quad m = m_{n+1}. \tag{15}$$

But first let us verify that the hypotheses are satisfied. We have:

(i) $A \subset S = S_0$ and S_0 is a cube centered at the origin;

(ii) A satisfies (4) with $C_{A,S} = C_n$ and $\mathcal{N}_{A,S} = \mathcal{N}_n$, since Γ_F does;

(iii) $|A| = |\Gamma_F| > 0$, by the choice of F;

(iv) $m = m_{n+1} \geq \max\left\{\mathcal{N}_n, \left(\frac{2^n C_n}{|\Gamma_F|}\right)^{1/(d-\gamma_n)}\right\} = \max\left\{\mathcal{N}_{A,S}, \left(\frac{C_{A,S}|S|}{\epsilon|A|}\right)^{1/(d-\gamma_n)}\right\}$.

So we can apply Lemma 2.1 to obtain

$$\begin{aligned}(1-\epsilon)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|}\,|A| &\leq \left|A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j}\right)\right| \\ &\leq (1+\epsilon)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|}\,|A|\,.\end{aligned} \tag{16}$$

Note that

$$\bigcup_{j=1}^{m^d} E_{m,j} = \bigcup_{j=1}^{m_{n+1}^d}\left(o_{m_{n+1},j} + \frac{1}{m_{n+1}}A_{n+1}\right) = \Lambda_{n+1}.$$

This, combined with (15) and (16), implies

$$(1-\epsilon)\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|\frac{|A|}{|S|} = (1-2^{-n})\,|\Lambda_{n+1}|\,|\Gamma_F| \geq \left[1-\frac{1}{4(n+1)^2}\right]|\Lambda_{n+1}|\,|\Gamma_F|\,,$$

$$\left|A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j}\right)\right| = |\Gamma_F \cap \Lambda_{n+1}| = \left|\left(\bigcap_{i \in F} \Lambda_i\right) \cap \Lambda_{n+1}\right|,$$

and

$$(1+\epsilon)\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|\frac{|A|}{|S|} = (1+2^{-n})\,|\Lambda_{n+1}|\,|\Gamma_F|\,.$$

Thus,

$$\prod_{i\in F\cup\{n+1\}}\left(1-\frac{1}{4i^2}\right)|\Lambda_i|$$
$$\le\left[1-\frac{1}{4(n+1)^2}\right]|\Lambda_{n+1}|\left|\bigcap_{i\in F}\Lambda_i\right| = \left[1-\frac{1}{4(n+1)^2}\right]|\Lambda_{n+1}|\,|\Gamma_F|$$
$$\le\left|\left(\bigcap_{i\in F}\Lambda_i\right)\cap\Lambda_{n+1}\right|\le(1+2^{-n})\,|\Lambda_{n+1}|\,|\Gamma_F|$$
$$=(1+2^{-n})\,|\Lambda_{n+1}|\left|\bigcap_{i\in F}\Lambda_i\right|\le\prod_{i\in F\cup\{n+1\}}(1+2^{-(i-1)})\,|\Lambda_i|\,,$$

where the first and last inequalities are due to the induction hypothesis (12). We conclude that (12) holds for every $n\in\mathsf{N}$.

COROLLARY 2.1. *Under the hypotheses of Theorem 1.2, if, in addition, the series* $\sum_n |S_0\cap A_n^c|$ *diverges, then there is a sequence* $\{m_n\}_{n\in\mathsf{N}}\subset\mathsf{N}$ *such that when we partition* S_0 *into* m_n^d *cubes* $S_{m_n,j}$, $j=1,\dots,m_n^d$, *of the same size and let* $o_{m_n,j}$ *denote the center of* $S_{m_n,j}$ *and*

$$K_n:=\bigcup_{j=1}^{m_n^d}\left[o_{m_n,j}+\frac{1}{m_n}(S_0\cap A_n^c)\right],\quad n\in\mathsf{N},$$

the following holds:

$$\left|\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}K_n\right|=1,$$

i.e. almost every point of S_0 *is contained in infinitely many* K_n*'s.*

PROOF. Indeed, define Λ_n, $n\in\mathsf{N}$, as in (11) and note that

$$S_0\cap K_n^c = S_0\cap\left\{\bigcup_{j=1}^{m_n^d}\left[o_{m_n,j}+\frac{1}{m_n}(S_0\cap A_n^c)\right]\right\}^c$$
$$=S_0\cap\left\{\bigcap_{j=1}^{m_n^d}\left[o_{m_n,j}+\frac{1}{m_n}(S_0\cap A_n^c)\right]^c\right\}=\bigcup_{j=1}^{m_n^d}\left(o_{m_n,j}+\frac{1}{m_n}A_n\right)=\Lambda_n.$$

Applying Theorem 1.2 to the family $\{A_n\}_{n\in\mathsf{N}}$, we obtain $\left|\bigcap_{n=k}^{k+l}\Lambda_n\right|\le e^2\cdot\prod_{n=k}^{k+l}|\Lambda_n|$ for any $k,l\in\mathsf{N}$. Letting $l\to\infty$, we get $\left|\bigcap_{n=k}^{\infty}\Lambda_n\right|\le e^2\prod_{n=k}^{\infty}|\Lambda_n|$.

We now use this inequality in what is nearly the standard proof of the second Borel-Cantelli lemma:

$$\begin{aligned}1-\left|\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}K_n\right| &= \left|\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\Lambda_n\right| = \lim_{m\to\infty}\left|\bigcap_{n=m}^{\infty}\Lambda_n\right| \\ &\le \lim_{m\to\infty}\left[e^2\prod_{n=m}^{\infty}|\Lambda_n|\right] = e^2\lim_{m\to\infty}\prod_{n=m}^{\infty}(1-|K_n|) \\ &\le e^2\lim_{m\to\infty}\prod_{n=m}^{\infty}e^{-|K_n|} = e^2\lim_{m\to\infty}\exp\left(-\sum_{n=m}^{\infty}|K_n|\right)=0,\end{aligned}$$

where the last equality holds because $\sum_n |K_n| = \sum_n |S_0 \cap A_n^c| = \infty$.

As mentioned above, if we restrict ourselves to sets that are finite unions dyadic cubes, i.e. cubes in the collection

$$\mathscr{D} := \left\{z + 2^{-k}[0,1]^d : k \in \mathsf{Z},\ z \in 2^{-k}\mathsf{Z}^d\right\},$$

then we have equality in (2). The example in [15] is built in the dyadic setting and has motivated us to prove the claims below.

Claim 2.1. *Let $S = [-2^{k-1}, 2^{k-1}]^d$ for some $k \in \mathsf{Z}$ and let $A \subset S$ be a finite union of dyadic cubes. Then, there exists $i_0 \in \mathsf{N}$ such that, for $i \ge k - i_0$, and $m = 2^i$, when we partition S into m^d equal sized cubes $S_{m,j}$ with center $o_{m,j}$, $j = 1, \dots, m^d$, the following holds: for any measurable set $E \subset S$, we have* (6) *with $\epsilon = 0$.*

Proof. By hypothesis, we can write $A = \bigcup_{i=1}^n Q_i$, for some $n \in \mathsf{N}$ and some disjoint cubes $Q_i \in \mathscr{D}$. Choose

$$i_0 := \min_{1\le i\le n}\left\{\log_2(\ell(Q_i))\right\}.$$

For any $i \ge k - i_0$, if we set $m := 2^i$ and partition S into m^d cubes $S_{m,j}$, $j = 1, \dots, m^d$, of the same size, then $S_{m,j} \in \mathscr{D}$ and $\ell(S_{m,j}) \le 2^{i_0}$. Since each Q_i is a dyadic cube of side length 2^j for some $j \ge i_0$, it follows that each Q_i is a disjoint union of some of the $S_{m,j}$'s. Therefore so is A. Hence

$$\mathfrak{N}_m = \#\left\{j \in \{1,\dots,m^d\} : |S_{m,j} \cap A| > 0\right\} = |S_{m,1}|^{-1}\,|A| = m^d\,|S|^{-1}\,|A|\,.$$

Thus

$$\left|A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j}\right)\right| = \mathfrak{N}_m\left|\frac{1}{m}E\right| = |S|^{-1}\,|A|\,|E| = \left|\bigcup_{j=1}^{m^d} E_{m,j}\right|\,|S|^{-1}\,|A|\,. \tag{17}$$

Dividing (17) by $|A|$, we get (6) with $\epsilon = 0$.

CLAIM 2.2. *Let* $S_0 = [-1/2, 1/2]^d$ *and let* $\{A_n\}_{n\in\mathsf{N}}$ *be a family of measurable subsets of* S_0 *such that every* A_n *is a finite union of dyadic cubes. There is a sequence of integers* $\{k_n\}_{n\in\mathsf{N}}$ *satisfying: if, for each* n, *we partition* S_0 *into* $m_n^d := 2^{k_n d}$ *cubes* $S_{m_n,j}$, $j = 1, \dots, m_n^d$, *of the same size and let* $o_{m_n,j}$ *denote the center of* $S_{m_n,j}$ *and* $\Lambda_n := \bigcup_{j=1}^{m_n^d} \left(o_{m_n,j} + \frac{1}{m_n} A_n\right)$, *then for any finite subset* $F \subset \mathsf{N}$,

$$\left|\bigcap_{n\in F} \Lambda_n\right| = \prod_{n\in F} |\Lambda_n|\,. \tag{18}$$

PROOF. By induction. Choose $k_1 = 0$. Then $m_1 = 1$ and $\Lambda_1 = A_1$.

Now, assume that $k_1, \dots, k_n$ are chosen such that, with the above notation,

$$\left|\bigcap_{i\in F} \Lambda_i\right| = \prod_{i\in F} |\Lambda_i| \quad \forall F \subset \{1, \dots, n\}. \tag{19}$$

We will choose k_{n+1} such that

$$\left|\bigcap_{i\in F\cup\{n+1\}} \Lambda_i\right| = \prod_{i\in F\cup\{n+1\}} |\Lambda_i| \quad \forall F \subset \{1, \dots, n\}. \tag{20}$$

Fix $F \subset \{1, \dots, n\}$. By construction, for each $1 \le i \le n$, the set Λ_i is a finite union of disjoint dyadic cubes. So, for each $1 \le i \le n$, we can write $\Lambda_i = \bigcup_{l\in I_i} Q_{i,l}$, for some disjoint dyadic cubes $Q_{i,l}$. We choose

$$m_{n+1} := 2^{-i_n},$$

where $i_n := \min\{\log_2(\ell(Q_{i,l})) : l \in I_i,\, 1 \le i \le n\}$. When we partition S into m_{n+1}^d cubes $S_{m_{n+1},j}$, $j = 1, \dots, m_{n+1}^d$, with $\ell(S_{m_{n+1},j}) = 2^{i_n}$, each $S^\circ_{m_{n+1},j}$ is either contained in $\bigcap_{i\in F} \Lambda_i$ or in its complement. Thus

$$\#\left\{ j : \left| S_{m_{n+1},j} \cap \left(\bigcap_{i\in F} \Lambda_i\right)\right| > 0\right\} = |S_{m_{n+1},1}|^{-1}\left|\bigcap_{i\in F} \Lambda_i\right| = m_{n+1}^d \left|\bigcap_{i\in F} \Lambda_i\right|.$$

So

$$\left|\left(\bigcap_{i\in F} \Lambda_i\right) \cap \Lambda_{n+1}\right| = \left(m_{n+1}^d \left|\bigcap_{i\in F} \Lambda_i\right|\right)\left|\frac{1}{m_{n+1}} A_{n+1}\right| = |\Lambda_{n+1}|\left|\bigcap_{i\in F} \Lambda_i\right|.$$

This and the induction hypothesis (19) yield (20). Thus (18) holds.

3. A counterexample

We divide the proof of Theorem 1.1 into two parts. In the first part we construct a function f in $H^1_{\mathrm{rect}}(\mathsf{R}\times\mathsf{R})\cap L(\log L)^\epsilon(\mathsf{R}^2)$ for all $0<\epsilon<1$; in the second, we show that f satisfies (1). An analogous reasoning, with a rotation of X_n about the orignin replacing X_n, shows that $\underline{D}(\int f, p) = -\infty$ for almost every p in S.

PROOF OF THEOREM 1.1 – PART I. We begin by choosing sequences of positive numbers, $\{\alpha_n\}_n$, $\{\lambda_n\}_n$ and $\{\gamma_n\}_n$, which satisfy the following:

$$\sum_n \frac{\lambda_n}{\alpha_n^4} < \infty, \quad \sum_n \gamma_n < \infty, \tag{21}$$

$$\sum_n \frac{\log \alpha_n}{\alpha_n^2} = \infty, \quad \lim_{n\to\infty} \frac{\lambda_n}{\alpha_n^2} = \infty, \tag{22}$$

$$\frac{\lambda_n^{-1}\alpha_n^4}{\lambda_{n+1}^{-1}\alpha_{n+1}^4} \le 1 \tag{23}$$

and

$$\frac{\lambda_n}{\kappa_\epsilon \gamma_n \alpha_n^4}\left(\log\left(1+\frac{\lambda_n}{\kappa_\epsilon\gamma_n}\right)\right)^\epsilon \le 1 \quad \forall 0<\epsilon<1, \tag{24}$$

for some constant $\kappa_\epsilon > 0$, depending on ϵ, but independent of n. A suitable choice is described at the end of this section.

We define $S := \left[-\frac{1}{2}, \frac{1}{2}\right]\times\left[-\frac{1}{2}, \frac{1}{2}\right]$ and we let $\{m_n\}_{n=1}^\infty \subset \mathsf{N}$ be a sequence. The m_n's are required to satisfy certain properties that will be specified later.

We partition S into m_n^2 squares $S_{n,j} \in \mathscr{R}$, $j = 1, \ldots, m_n^2$, of side length $1/m_n$. At the center $o_{n,j}$ of each $S_{n,j}$ we place a smaller square

$$Q_{n,j} := \left\{x \in \mathsf{R}^2 : \|o_{n,j} - x\|_\infty \le \frac{1}{2m_n\lceil\alpha_n\rceil^2}\right\},$$

where here, and in what follows, $\lceil a\rceil := \min\{n \in \mathsf{Z} : n \ge a\}$ for $a \in \mathsf{R}$, and $\|\cdot\|_\infty$ denotes the maximum norm $\|x\|_\infty := \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathsf{R}^2$.

For each $j = 1, \ldots, m_n^2$, we partition $Q_{n,j}$ into 4 squares $Q_{n,j,k} \in \mathscr{R}$, $1 \le k \le 4$, of side length $1/(2m_n\lceil\alpha_n\rceil^2)$ and we label the interiors of these 4 squares as black or white in a chessboard pattern with the upper right square being white, as in Figure 1. The union of all *white* squares in all squares $Q_{n,j}$'s, $1 \le j \le m_n^2$, will be denoted by $\mathscr{W}_n$; that of all *black* squares in all $Q_{n,j}$'s,

$1 \le j \le m_n^2$, by $\mathscr{B}_n$. Now we define

$$f_n := \lambda_n \chi_{\mathscr{W}_n} - \lambda_n \chi_{\mathscr{B}_n}, \quad f := \sum_{n=1}^{\infty} f_n,$$

where χ_E denotes the characteristic function of a set E. Note that $\sum_n |f_n|$ is integrable. Thus the set $W := \{x : \sum_n |f_n(x)| = \infty\}$ has measure zero, a fact the we will use in Part II below.

To see that f is in $H^1(\mathsf{R} \times \mathsf{R})$, we write $f = \sum_{n=1}^{\infty} \sum_{j=1}^{m_n^2} \gamma_n m_n^{-2} a_{n,j}$, where

$$a_{n,j}(x) := m_n^2 \gamma_n^{-1} f_n(x) \chi_{Q_{n,j}}(x), \quad 1 \le j \le m_n^2,\ n \in \mathsf{N}.$$

The $a_{n,j}$'s are rectangular atoms [1] in $H^1(\mathsf{R} \times \mathsf{R})$ and, by (21), the series $\sum_n \big(\sum_{j=1}^{m_n^2} \gamma_n m_n^{-2}\big)$ converges. Hence

$$\sum_{n=1}^{\infty} \sum_{j=1}^{m_n^2} \gamma_n m_n^{-2} a_{n,j} \in H^1_{\mathrm{rect}}(\mathsf{R} \times \mathsf{R}) \subset H^1(\mathsf{R} \times \mathsf{R}).$$

Now, to show that f belongs to $L^{\Phi_\epsilon}(\mathsf{R}^2)$, we write $f = \sum_{n=1}^{\infty} \gamma_n g_n$, where

$$g_n(x) := \gamma_n^{-1} f_n(x) = \sum_{j=1}^{m_n^2} m_n^{-2} a_{n,j}, \quad n \in \mathsf{N}.$$

Since $(L^{\Phi_\epsilon}(\mathsf{R}^2), \|\cdot\|_{\Phi_\epsilon})$ is complete and the coefficients γ_n's satisfy $\sum_n |\gamma_n| < \infty$, to show that $f \in L^{\Phi_\epsilon}(\mathsf{R}^2)$, it suffices to prove that for each $\epsilon \in (0, 1)$ we can find a constant $\kappa_\epsilon > 0$, independent of n, such that

$$\|g_n\|_{\Phi_\epsilon} \le \kappa_\epsilon \quad \text{for all } n \in \mathsf{N}. \tag{25}$$

In fact, we claim that (25) holds for any κ_ϵ for which (24) holds. Indeed, to form each g_n, we gathered all the rectangular atoms that compose f_n. So

$$|g_n| = \gamma_n^{-1} \lambda_n \chi_{\mathscr{W}_n \cup \mathscr{B}_n},$$

and this yields

$$\begin{aligned}
\int \Phi_\epsilon\left(\frac{g_n(x)}{\kappa_\epsilon}\right) dx &= \int \frac{|g_n(x)|}{\kappa_\epsilon}\left[\log\left(1 + \frac{|g_n(x)|}{\kappa_\epsilon}\right)\right]^\epsilon dx \\
&= \frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon}\left[\log\left(1 + \frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon}\right)\right]^\epsilon |\mathrm{supp}(f_n)| \\
&\le \frac{\lambda_n}{\kappa_\epsilon \gamma_n \alpha_n^4}\left[\log\left(1 + \frac{\lambda_n}{\kappa_\epsilon \gamma_n}\right)\right]^\epsilon \le 1,
\end{aligned}$$

for all $n \in \mathsf{N}$, where the last inequality follows from (24). This shows that κ_ϵ is an uniform (on n) upper bound for the Luxemburg norms $\|g_n\|_{\Phi_\epsilon}$, proving our claim.

PROOF OF THEOREM 1.1 – PART II. The result relies on the construction of a sequence $\{K_n\}_{n\in\mathsf{N}}$ of subsets of S such that

$$\left|\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} K_n\right| = 1, \tag{26}$$

and therefore almost every point in S belongs to $W^c \cap \left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} K_n\right)$.

For each $n \in \mathsf{N}$, we define the set (compare with (3))

$$X_n := \left\{(x_1, x_2) \in \mathsf{R}^2 : 0 \le x_1x_2 \le \frac{1}{4\lceil\alpha_n\rceil^2}, \frac{1}{2\lceil\alpha_n\rceil^2} \le \|(x_1, x_2)\|_\infty \le \frac{1}{2}\right\}.$$

Since ∂X_n is union of two rectifiable curves, $\dim_{\text{upper box}}(\partial X_n) = 1$.

By construction, the dilation of X_n by $1/m_n$ is contained in the square of side length $1/m_n$ centered at the origin. In Figure 1, we represent a set $o_{n,j} + m_n^{-1}X_n$ in gray and the squares $Q_{n,j,k}$, $1 \le k \le 4$, in black and white at the center. So $o_{n,j} + m_n^{-1}X_n \subset S_{n,j}$ for all $1 \le j \le m_n^2$. In addition, the area of X_n satisfies (in our proof here, we only need the lower bound for $|X_n|$)

$$\begin{aligned}\frac{\log\lceil\alpha_n\rceil}{2\lceil\alpha_n\rceil^2} &= 2\int_{1/2\lceil\alpha_n\rceil}^{1/2}\frac{1}{4\lceil\alpha_n\rceil^2 t}\,dt \le |X_n| \\ &\le 2\left(\int_0^{1/2\lceil\alpha_n\rceil} t\,dt + \int_{1/2\lceil\alpha_n\rceil}^{1/2}\frac{1}{4\lceil\alpha_n\rceil^2 t}\,dt\right) \le \frac{\log\lceil\alpha_n\rceil}{\lceil\alpha_n\rceil^2}.\end{aligned} \tag{27}$$

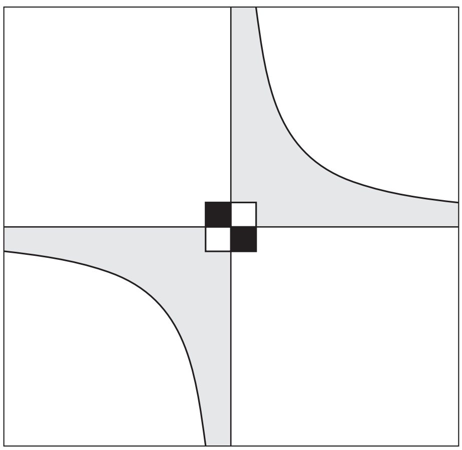

FIGURE 1

Fixed $n \in \mathsf{N}$ and $j \in \{1, \ldots, m_n^2\}$, every point $p = (p_1, p_2)$ in the set $o_{n,j} + m_n^{-1} X_n$ lies in a rectangle $R_p \in \mathscr{R}$ satisfying $p \in R_p$,

$$(28) \qquad |R_p| = \frac{1}{4m_n^2 \lceil \alpha_n \rceil^2} \quad \text{and} \quad |R_p \cap \mathscr{W}_n| - |R_p \cap \mathscr{B}_n| = \frac{1}{4}|Q_{n,j}|.$$

Indeed, let $p \in o_{n,j} + m_n^{-1} X_n$. We will construct R_p. By symmetry, it suffices to consider p with $0 \le p_2 - (o_{n,j})_2 \le p_1 - (o_{n,j})_1$. One of the two cases happens:

(i) If $0 \le p_2 - (o_{n,j})_2 \le 1/(2m_n \lceil \alpha_n \rceil^2)$, then we define

$$R_p := o_{n,j} + \left(\left[0, \frac{1}{2m_n}\right] \times \left[0, \frac{1}{2m_n \lceil \alpha_n \rceil^2}\right] \right)$$

and we observe that (28) holds.

(ii) If $p_2 - (o_{n,j})_2 > 1/(2m_n \lceil \alpha_n \rceil^2)$, then $p_1 - (o_{n,j})_1 > 1/(2m_n \lceil \alpha_n \rceil^2)$ as well, and we choose

$$R_p := o_{n,j} + \left(\left[0, p_1 - (o_{n,j})_1\right] \times \left[0, \frac{1}{4m_n^2 \lceil \alpha_n \rceil^2 (p_1 - (o_{n,j})_1)}\right] \right).$$

With this choice, $p \in R_p$, since $(p_2 - (o_{n,j})_2)(p_1 - (o_{n,j})_1) \le 1/(2m_n \lceil \alpha_n \rceil)^2$. Also, R_p satisfies (28).

Similarly, for every $p \in o_{n,j} + m_n^{-1} \rho(X_n)$, where ρ is the rotation by $\pi/2$ radians about the origin, there exists $S_p \in \mathscr{R}$ such that

$$p \in S_p, \quad |S_p| = \frac{1}{4m_n^2 \lceil \alpha_n \rceil^2} \quad \text{and} \quad |S_p \cap \mathscr{B}_n| - |S_p \cap \mathscr{W}_n| = \frac{1}{4}|Q_{n,j}|.$$

How does $\lambda_n |Q_{n,1}|$ compare with $\sum_{i=1}^{\infty} \lambda_{n+i} \left|Q_{n+i,1}\right|$? The answer given is below and will be used when we deal with the strong upper derivative of the integral of f. If

$$(29) \qquad m_n \ge 2^4 m_{n-1} \quad \forall n,$$

then $m_{n+i} \geq 2^4 m_{n+i-1} \geq \cdots \geq 2^{4i} m_n \geq 2^i(2^3 m_n)$, $\forall n$. This and (23) yield

$$\begin{aligned}
\sum_{i=1}^{\infty} \lambda_{n+i}|Q_{n+i,1}| &= \frac{\lambda_n|Q_{n,1}|}{4} \sum_{i=1}^{\infty} \frac{4\lambda_{n+i}|Q_{n+i,1}|}{\lambda_n|Q_{n,1}|} \\
&= \frac{\lambda_n|Q_{n,1}|}{4} \sum_{i=1}^{\infty} \frac{4\lambda_{n+i}(4m_n^2\lceil\alpha_n\rceil^4)}{\lambda_n(4m_{n+i}^2\lceil\alpha_{n+i}\rceil^4)} \\
&\leq \frac{\lambda_n|Q_{n,1}|}{4} \sum_{i=1}^{\infty} \frac{2^2\lambda_{n+i}m_n^2(2\alpha_n)^4}{\lambda_n m_{n+i}^2\alpha_{n+i}^4} \\
&= \frac{\lambda_n|Q_{n,1}|}{4} \sum_{i=1}^{\infty} \left(\frac{\lambda_n^{-1}\alpha_n^4}{\lambda_{n+i}^{-1}\alpha_{n+i}^4}\right)\left(\frac{2^3 m_n}{m_{n+i}}\right)^2 \\
&\leq \frac{\lambda_n|Q_{n,1}|}{4} \sum_{i=1}^{\infty} (2^{-i})^2 = \frac{\lambda_n|Q_{n,1}|}{12} \quad \forall n.
\end{aligned}$$

Thus (29) implies

(30)
$$\frac{\lambda_n|Q_{n,1}|}{4} - \sum_{i=1}^{\infty} \frac{\lambda_{n+i}|Q_{n+i,1}|}{2} \geq \left(\frac{1}{4} - \frac{1}{24}\right)\lambda_n|Q_{n,1}| = \frac{5}{24}\lambda_n|Q_{n,1}| \quad \forall n.$$

For each n, we define

(31) $$A_n := S \cap X_n^c \quad \text{and} \quad \Lambda_n := \bigcup_{j=1}^{m_n^2}\left[o_{n,j} + \frac{1}{m_n}A_n\right].$$

Each A_n is contained in S and satisfies $|A_n| > 0$ and $\dim_{\text{upper box}}(\partial\overline{A_n}) = 1$. Moreover, since $|S \cap A_n^c| = |X_n|$, estimate (27) yields

(32) $$|S \cap A_n^c| \geq \frac{\log\lceil\alpha_n\rceil}{2\lceil\alpha_n\rceil^2} \geq \frac{\log\alpha_n}{2(2\alpha_n)^2}.$$

Also, for each n, we define

$$K_n := \bigcup_{j=1}^{m_n^2}\left(c_{n,j} + \frac{1}{m_n}X_n\right)$$

and note that $K_n = \bigcup_{j=1}^{m_n^2}\left[c_{n,j} + \frac{1}{m_n}(S \cap A_n^c)\right]$ and $S \cap K_n^c = \Lambda_n$.

Now we will construct a sequence $\{m_n\}_{n\in\mathsf{N}}$ such that both (30) and

$$\left|\bigcap_{i\in F}\Lambda_i\right| \le \prod_{i\in F}(1+2^{-(i-1)})\,|\Lambda_i| \quad \forall F\subset\{1,\dots,n\} \tag{33}$$

hold for all $n\in\mathsf{N}$, where the sets Λ_i are defined in (31). We must choose $\{m_n\}_{n\in\mathsf{N}}$ satisfying (29) and (14). Condition (14) appears in the proof of Theorem 1.2, which we apply to $\{A_n\}_{n\in\mathsf{N}}$. We build $\{m_n\}_{n\in\mathsf{N}}$ by the recurrence relation

$$m_1=1,\quad m_n=\left\lceil \max\left\{\mathcal{N}_n,\ \frac{2^{n-1}C_{n-1}}{\theta_{n-1}},\ 2^4\right\}\right\rceil m_{n-1}\lceil\alpha_{n-1}\rceil^2 \quad \text{for } n>1,$$

where C_n and $\mathcal{N}_n$ are as in (13), $\theta_n := \min_I\left\{\left|\bigcap_{i\in I}\Lambda_i\right|\right\}$ and the minimum is taken over all finite collections $I\subset\{1,\dots,n\}$ satisfying $\left|\bigcap_{i\in I}\Lambda_i\right|>0$. By construction, with this sequence $\{m_n\}_{n\in\mathsf{N}}$, both (29) and (14) hold. Hence both (30) and (33) hold for all $n\in\mathsf{N}$.

From (32) and (22), we get

$$\sum_{n=1}^{\infty}|S\cap A_n^c| \ge \frac{1}{8}\sum_{n=1}^{\infty}\frac{\log\alpha_n}{\alpha_n^2}=\infty.$$

This, together with (33), implies (26), as shown in Corollary 2.1.

For fixed $p\in W^c\cap\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}K_n\right)$, we will show that $\overline{D}(\int f, p)=+\infty$. An analogous reasoning, with $\rho(X_n)$ replacing X_n, shows that $\underline{D}(\int f, p)=-\infty$. Indeed, let $\{n_i\}_{i\in\mathsf{N}}$ be such that $p\in K_{n_i}$ $\forall i\in\mathsf{N}$. Then, it suffices to show that

$$\lim_{i\to\infty}\left[\sum_{k=1}^{\infty}\frac{1}{|R_{n_i}(p)|}\int_{R_{n_i}(p)}f_k(x)\,dx\right]=\infty.$$

For each $i\in\mathsf{N}$, p lies in one of the homothetic copies of X_{n_i}, say $p\in S_{n_i,j}\cap K_{n_i}$. By (28), p lies in a rectangle $R_{n_i}(p)\in\mathcal{R}$ satisfying

$$|R_{n_i}(p)|=\frac{1}{4m_{n_i}^2\lceil\alpha_{n_i}\rceil^2}\quad\text{and}\quad |R_{n_i}(p)\cap\mathcal{W}_{n_i}|-|R_{n_i}(p)\cap\mathcal{B}_{n_i}|=\frac{1}{4}|Q_{n_i,1}|. \tag{34}$$

Moreover, for any $k\ge 1$, $|R_{n_i}(p)\cap\mathcal{B}_{n_i+k}|-|R_{n_i}(p)\cap\mathcal{W}_{n_i+k}|$ cannot be greater than the area of 2 of the 4 black or white squares that compose each $Q_{n_i+k,j}$, $1\le j\le m_{n_i+k}^2$, i.e.

$$|R_{n_i}(p)\cap\mathcal{B}_{n_i+k}|-|R_{n_i}(p)\cap\mathcal{W}_{n_i+k}|\le 2\left(\frac{|Q_{n_i+k,1}|}{4}\right)\quad\forall k\in\mathsf{N}. \tag{35}$$

From (34), (35) and (30), we get

$$\begin{aligned}
&\int_{R_{n_i}(p)} f_{n_i}(x)\,dx + \sum_{k=1}^{\infty}\int_{R_{n_i}(p)} f_{n_i+k}(x)\,dx \\
&\quad \geq \lambda_{n_i}\big(|R_{n_i}(p)\cap \mathscr{W}_{n_i}| - |R_{n_i}(p)\cap \mathscr{B}_{n_i}|\big) \\
&\qquad - \sum_{k=1}^{\infty}\lambda_{n_i+k}\big(|R_{n_i}(p)\cap \mathscr{B}_{n_i+k}| - |R_{n_i}(p)\cap \mathscr{W}_{n_i+k}|\big) \\
&\quad \geq \frac{\lambda_{n_i}}{4}|Q_{n_i,1}| - \sum_{k=1}^{\infty}\lambda_{n_i+k}\frac{|Q_{n_i+k,1}|}{2} \\
&\quad \geq \frac{5}{24}\lambda_{n_i}|Q_{n_i,1}| = \frac{5}{24}\frac{\lambda_{n_i}}{m_{n_i}^2\lceil\alpha_{n_i}\rceil^4} \quad \forall i \in \mathsf{N}.
\end{aligned}$$

Then

$$\frac{1}{|R_{n_i}(p)|}\sum_{k=0}^{\infty}\int_{R_{n_i}(p)} f_{n_i+k}(x)\,dx \geq C\frac{1}{(m_{n_i}^2\lceil\alpha_{n_i}\rceil^2)^{-1}}\frac{\lambda_{n_i}}{m_{n_i}^2\lceil\alpha_{n_i}\rceil^4} \sim \frac{\lambda_{n_i}}{\alpha_{n_i}^2} \to \infty,$$

as $i \to \infty$, by (22). It remains to control $|R_{n_i}(p)|^{-1}\sum_{k=1}^{n_i-1}\int_{R_{n_i}(p)} f_k(x)dx$, $i \in \mathsf{N}$. By construction, for every i and every $k \in \{1,\dots,n_i-1\}$, m_{n_i} is an integer multiple of $4m_k\lceil\alpha_k\rceil^2$. This and the fact that the black and white squares $Q_{k,l,v}$, $1 \leq v \leq 4$, that compose each $Q_{k,l}$, $1 \leq l \leq m_k^2$, have side length $1/(2m_k\lceil\alpha_k\rceil^2)$, yield

$$S_{n_i,j}\cap Q_{m,l,v} \neq \emptyset \Leftrightarrow S_{n_i,j}^{\circ} \subset Q_{m,l,v}$$

$\forall 1 \leq k \leq n_i - 1$, $1 \leq l \leq m_k^2$, $1 \leq v \leq 4$. Hence either $R_{n_i}(p) \cap \big(\operatorname{supp}\big(\sum_{k=1}^{n_i-1} f_k\big)\big) = \emptyset$ or $R_{n_i}(p) \subset Q_{k,l,v}$ for some $1 \leq k \leq n_i - 1$, $1 \leq l \leq m_k^2$, $1 \leq v \leq 4$. In any of these cases,

$$\frac{1}{|R_{n_i}(p)|}\int_{R_{n_i}(p)} f_k(x)dx = f_k(p) \quad \forall 1 \leq k \leq n_i - 1,$$

which implies that

$$\left|\sum_{k=1}^{n_i-1}\frac{1}{|R_{n_i}(p)|}\int_{R_{n_i}(p)} f_k(x)\,dx\right| \leq \sum_{k=1}^{n_i-1}|f_k(p)| \leq \sum_{k=1}^{\infty}|f_k(p)| < \infty \quad \forall i \in \mathsf{N},$$

where the last inequality holds due to the choice of p in W^c. Therefore

$$\frac{1}{|R_{n_i}(p)|}\int_{R_{n_i}(p)} f(x)\,dx \geq -\sum_{k=1}^{\infty}|f_k(p)| + \frac{1}{|R_{n_i}(p)|}\sum_{k=n_i}^{\infty}\int_{R_{n_i}(p)} f_k(x)\,dx \to \infty$$

as $i \to \infty$. Thus $\overline{D}(\int f, p) = +\infty$.

Here we present a choice of positive numbers satisfying (21)–(24). For each $n \in \mathsf{N}$, let

$$\alpha_n := 4n^{1/2}\log(4n)(\log(\log(4n)))^{1/2}, \tag{36}$$

$$\lambda_n := n(\log(4n))^2(\log(\log(4n)))^2, \tag{37}$$

$$\gamma_n := \frac{1}{4^4 n\log(4n)(\log(\log(4n)))^2}. \tag{38}$$

In addition, let

$$\kappa_\epsilon := \max\left\{2^5, 9^\epsilon \max_{n\in\mathsf{N}}\left\{\frac{(\log(\log(4n)))^2}{(\log(4n))^{1-\epsilon}}\right\}\right\}. \tag{39}$$

To see that the sequences $\{\alpha_n\}_n$, $\{\lambda_n\}_n$ and $\{\gamma_n\}_n$, defined above, satisfy (21) and (22), it suffices to observe that

$$\frac{\lambda_n}{\alpha_n^4} \sim \frac{1}{n(\log n)^2}, \quad \gamma_n \sim \frac{1}{n(\log n)(\log(\log n))^2},$$
$$\frac{\log\alpha_n}{\alpha_n^2} \sim \frac{1}{n(\log n)(\log(\log n))} \quad \text{and} \quad \frac{\lambda_n}{\alpha_n^2} \sim \log(\log n).$$

A direct substitution yields (23). The proof of (24) requires a bit more work. From (36)–(39) we obtain

$$1 + \frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon} \leq \frac{2\gamma_n^{-1}\lambda_n}{2^5} = (4n)^2(\log(4n))^3(\log(\log(4n)))^4 \leq (4n)^9. \tag{40}$$

Plugging (40) into the left-handside of (24), we get

$$\frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon}\left[\log\left(1+\frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon}\right)\right]^\epsilon \frac{1}{\alpha_n^4} \leq \frac{(\log(\log(4n)))^2}{\kappa_\epsilon\log(4n)}[9\log(4n)]^\epsilon$$
$$= \frac{9^\epsilon(\log(\log(4n)))^2}{\kappa_\epsilon(\log(4n))^{1-\epsilon}} \leq 1,$$

where the last inequality follows from the choice of κ_ϵ.

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