MATH. SCAND. 119 (2016), 113–132

CONVOLUTION IN WEIGHTED LORENTZ SPACES OF TYPE Γ

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Abstract

We characterize boundedness of the convolution operator between weighted Lorentz spaces $\Gamma^p(v)$ and $\Gamma^q(w)$ for the range of parameters $p, q \in [1, \infty]$, or $p \in (0, 1)$ and $q \in \{1, \infty\}$, or $p = \infty$ and $q \in (0, 1)$. We provide Young-type convolution inequalities of the form

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Gamma^p(v)}\|g\|_Y, \quad f \in \Gamma^p(v), g \in Y,$$

characterizing the optimal rearrangement-invariant space Y for which the inequality is satisfied.

1. Introduction

Let f and g be locally integrable functions on R^d, $d \in \mathsf{N}$. The *convolution* $f * g$ is given by

$$(f * g)(x) := \int_{\mathsf{R}^d} f(y)g(x - y)\,\mathrm{d}y, \quad x \in \mathsf{R}^d.$$

If the function g is fixed, we define the convolution operator T_g by

$$T_g f := f * g. \tag{1}$$

This paper has the following purpose. First, given weights v, w and exponents p, q, to characterize when the operator T_g is bounded between the weighted Lorentz spaces $\Gamma^p(v)$ and $\Gamma^q(w)$, in terms of the kernel g. Second, to prove related Young-type inequalities in the form

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Gamma^p(v)}\|g\|_Y, \quad f \in \Gamma^p(v), g \in Y,$$

and to characterize the optimal (i.e. essentially largest) rearrangement-invariant space Y such that this inequality holds. (For definitions see Section 2.)

* The author thanks the referee as well as Professors Sorina Barza, Luboš Pick and Javier Soria for their remarks and comments which helped to improve the final version of the article.

Received 10 January 2014.

A variety of results can be labelled as Young-type convolution inequalities. Their common ancestor is the classical Young inequality reading

$$\|f * g\|_q \le \|f\|_p \|g\|_r, \quad f \in L^p, g \in L^r,$$

where $1 \le p, q, r \le \infty$ and $1+\frac{1}{q} = \frac{1}{p}+\frac{1}{r}$. Results of a similar form have been obtained for many classes of function spaces other than the Lebesgue spaces in the original Young inequality. In [15], [8], [19], [2] the Lorentz spaces $L_{p,q}$ were considered and the following inequality was proved: for $1 < p, q, r < \infty$ and $0 < a, b, c \le \infty$ such that $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$, one has

$$\|f * g\|_{L_{q,a}} \le C\|f\|_{L_{p,b}} \|g\|_{L_{r,c}}, \quad f \in L_{p,b}, g \in L_{r,c}.$$

An analogous problem for convolution of periodic functions on the real line was studied in [14].

In the papers [12], [11], inequalities of the type

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_X \|g\|_Y, \quad f \in X, g \in Y,$$

were obtained for X being the weighted Lorentz space $\Lambda^p(v)$ or the Lorentz-type class $S^p(v)$, defined in terms of oscillation. The proof technique there was based on the use of the O'Neil convolution inequality

$$(f * g)^{**}(t) \le t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s, \quad t > 0, \tag{2}$$

(see [15, Lemma 2.5]) and various weighted Hardy-type inequalities. This method also granted that the obtained rearrangement-invariant space Y was optimal.

An analogous technique will be used here. After presenting the definitions and auxiliary results in Section 2, in Section 3 we will characterize, in terms of g, v, w, p, q, validity of the inequality

$$\left\| t \mapsto \left(t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s \right) \right\|_{L^q(w)} \le C\|f\|_{\Gamma^p(v)},$$

for all $f \in \Gamma^p(v)$, with C being a constant independent of f. The conditions obtained in this way will be, by the O'Neil inequality (2), sufficient for boundedness $T_g\colon \Gamma^p(v) \to \Gamma^q(w)$. To show their necessity as well, we will make use of a reverse O'Neil inequality (see Lemma 2.1) holding for positive radially decreasing functions. This is included in Section 4, where the results are presented in the form of Young-type inequalities

$$\|f * g\|_{\Gamma^q(w)} \le C\|f\|_{\Gamma^p(v)} \|g\|_Y, \quad f \in \Gamma^p(v), g \in Y.$$

The result may indeed be formulated so, since, as observed in Section 3, the conditions on g characterizing the optimal constant C in (2) have the form of a norm of g in a rearrangement-invariant space Y. Its optimality will be proved as well.

Let us note here that although we will consider just R^d as the underlying space in this paper, the results can be easily modified for periodic functions on the real line, as it was done e.g. in [12].

2. Preliminaries

Throughout the text, the following notation is used: the positive integer d will denote the dimension of the space R^d. By $\mathscr{M}(\Omega)$ we denote the set of all measurable functions on Ω with values in $[-\infty, \infty]$. We will work with the choice $\Omega = \mathsf{R}^d$ or $\Omega = (0, \infty)$. Similarly, $\mathscr{M}_+(\Omega)$ stands for the set of all nonnegative functions from $\mathscr{M}(\Omega)$. Next, we denote by $\mathscr{M}_+^{\odot}(\mathsf{R}^d)$ the set of all functions $f \in \mathscr{M}_+(\mathsf{R}^d)$ such that there exists a nonincreasing $f_0 \in \mathscr{M}_+(0, \infty)$ such that $f(x) = f_0(|x|)$ holds for a.e. $x \in \mathsf{R}^d$, i.e. $\mathscr{M}_+^{\odot}(\mathsf{R}^d)$ is the set of nonnegative radially decreasing functions on R^d.

The notation $A \lesssim B$ means that $A \leq CB$ where C is a positive constant independent of relevant quantities. Unless specified else, this C in fact always depends only on exponents p and q, if they are involved. If $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$. The *optimal constant* C in an inequality $A \leq CB$ is the least C such that the inequality holds. By writing inequalities in the form

$$A(f) \lesssim B(f), \quad f \in X,$$

we mean that $A(f) \lesssim B(f)$ is satisfied for all $f \in X$.

If $f \in \mathscr{M}(\mathsf{R}^d)$, we define the *nonincreasing rearrangement* of f by

$$f^*(t) := \inf\big\{s > 0 : |\{x \in \mathsf{R}^d : |f(x)| > s\}| \leq t\big\}, \quad t > 0,$$

and the *Hardy-Littlewood maximal function* of f by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad t > 0.$$

For the definition of a *rearrangement-invariant* (*r.i.*) *norm* and an *r.i. space* see [1]. We will also use the terms *r.i. quasi-norm* and *r.i. lattice*, as defined e.g. in [12]. Here we consider R^d to be the underlying measure space, unless specified else.

A *weight* is a function from $\mathscr{M}_+(0, \infty)$. We write $W(t) := \int_0^t w(s)\,\mathrm{d}s$ for $t > 0$. By $L^1_{\mathrm{loc}}(\mathsf{R}^d)$ we denote the locally integrable functions on R^d. If

$q \in (0, \infty]$ and w is a weight, then $L^q(w)$ denotes the Lebesgue L^q-space over $(0, \infty)$ with the measure $w(t)\,\mathrm{d}t$.

Let $p \in (0, \infty]$, and v be a weight. The *weighted Lorentz spaces* are defined in the following way:

$$\Lambda^p(v) := \left\{ f \in \mathscr{M}(\mathsf{R}^d) : \|f\|_{\Lambda^p(v)} := \left(\int_0^\infty (f^*(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0, \infty),$$

$$\Lambda^\infty(v) := \left\{ f \in \mathscr{M}(\mathsf{R}^d) : \|f\|_{\Lambda^\infty(v)} := \operatorname*{ess\,sup}_{t>0} f^*(t)v(t) < \infty \right\}, \quad p = \infty,$$

$$\Gamma^p(v) := \left\{ f \in \mathscr{M}(\mathsf{R}^d) : \|f\|_{\Gamma^p(v)} := \left(\int_0^\infty (f^{**}(t))^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0, \infty),$$

$$\Gamma^\infty(v) := \left\{ f \in \mathscr{M}(\mathsf{R}^d) : \|f\|_{\Gamma^\infty(v)} := \operatorname*{ess\,sup}_{t>0} f^{**}(t)v(t) < \infty \right\}, \quad p = \infty.$$

If we assume that $V(t) > 0$ for all $t > 0$, the functional $\|\cdot\|_{\Gamma^p(v)}$ is at least a quasinorm, for $p \in [1, \infty]$ it is a norm. The key property here is the sublinearity of the maximal function, i.e.

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad f, g \in \mathscr{M}(\mathsf{R}^d), t > 0. \tag{3}$$

(See e.g. [1, p. 54].) In contrast, the Λ-"spaces" are not even linear sets in general. Functional properties of Λ and Γ are discussed in detail e.g. in [4], [9].

Let us list several auxiliary results. First, the O'Neil inequality (2) has also a converse form, as shown in the following lemma. The proof of this multi-dimensional version may be found e.g. in [10], the corresponding one-dimensional result was mentioned already in [15], its proof is shown e.g. in [12].

LEMMA 2.1. *Let $f, g \in \mathscr{M}_+^{\odot}(\mathsf{R}^d)$. Then for every $t \in (0, \infty)$ we have*

$$t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(y) g^*(y)\,\mathrm{d}y \le C_d (f * g)^{**}(t),$$

where C_d is a constant depending on the dimension d of the underlying space R^d but independent of f, g and t.

To handle inequalities involving the maximal function on both sides, it is possible to use the result of [5, Theorem 4.4]. It reads as follows:

LEMMA 2.2. *Let $p, q \in (1, \infty)$ and let v, w be weights. Define the weight ψ by*

$$\psi(t) := \frac{t^{p'+p-1} V(t) \int_t^\infty v(s) s^{-p}\, \mathrm{d}s}{\big(V(t) + t^p \int_t^\infty v(s) s^{-p}\, \mathrm{d}s\big)^{p'+1}}, \quad t > 0. \tag{4}$$

Let R be a positive linear operator on $\mathcal{M}_+(0,\infty)$ and S be the Stieltjes operator given by

$$Sh(t) := \int_0^\infty \frac{h(s)}{s+t}\, \mathrm{d}s, \quad t > 0. \tag{5}$$

Then

$$\bigg(\int_0^\infty (R(f^{**})(t))^q w(t)\, \mathrm{d}t\bigg)^{\frac{1}{q}} \le K_1 \bigg(\int_0^\infty (f^{**}(t))^p v(t)\, \mathrm{d}t\bigg)^{\frac{1}{p}}, \quad f \in \mathcal{M}(\mathsf{R}^d),$$

if and only if

$$\bigg(\int_0^\infty (RSh(t))^q w(t)\, \mathrm{d}t\bigg)^{\frac{1}{q}} \le K_2 \bigg(\int_0^\infty h^p(t) \psi^{1-p}(t)\, \mathrm{d}t\bigg)^{\frac{1}{p}}, \quad h \in \mathcal{M}_+(0,\infty).$$

Moreover, we have $K_1 \simeq K_2$.

The proposition below is a particular case of [17, Lemma 1.2].

PROPOSITION 2.3. *Let $h \in \mathcal{M}$. Then there exists a sequence of nonnegative measurable functions γ_n with compact support in $(0,\infty)$ such that for a.e. $t > 0$ they satisfy*

$$\int_t^\infty \gamma_n(s)\, \mathrm{d}s \uparrow h^*(t), \quad n \to \infty.$$

The next result follows by integration by parts (cf. [18, Lemma, p. 176]).

PROPOSITION 2.4. *Let $1 < q < p < \infty$ and $r := \frac{pq}{p-q}$. Let v, w be weights. Then we have the following inequalities*

$$\begin{aligned}\int_0^\infty W^{\frac{r}{p}}(t) w(t) \bigg(\int_t^\infty v\bigg)^{\frac{r}{p'}} \mathrm{d}t &\le \frac{q}{p'} \int_0^\infty W^{\frac{r}{q}}(t) \bigg(\int_t^\infty v\bigg)^{\frac{r}{q'}} v(t)\, \mathrm{d}t \\ &\le \int_0^\infty W^{\frac{r}{p}}(t) w(t) \bigg(\int_t^\infty v\bigg)^{\frac{r}{p'}} \mathrm{d}t.\end{aligned}$$

3. Inequalities related to the boundedness of the convolution operator

In this section we are going to characterize validity of the inequality

$$\left\| t \mapsto \left(t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s \right) \right\|_{L^q(w)} \le C_{(6)} \|f\|_{\Gamma^p(v)}, \tag{6}$$

for all $f \in \Gamma^p(v)$, by certain conditions on the kernel function g, the weights v, w and exponents p, q. By doing this, we obtain sufficient conditions for the boundedness of T_g between $\Gamma^p(v)$ and $\Gamma^q(w)$. Indeed, thanks to the O'Neil inequality (2), if (6) holds, then $T_g\colon \Gamma^p(v) \to \Gamma^q(w)$.

We start with (6) with the parameters satisfying $1 < p, q < \infty$. The lemma below shows that (6) is equivalent to two certain weighted Hardy inequalities.

LEMMA 3.1. *Let $p, q \in (1, \infty)$ and let v, w be weights. Let the weight ψ be defined by* (4). *Let $g \in L^1_{\mathrm{loc}}(\mathsf{R}^d)$. Then the inequality* (6) *holds if and only if*

$$\left(\int_0^\infty \left(\int_0^t h(s)\,\mathrm{d}s \right)^q (g^{**}(t))^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \le C_{(7)} \left(\int_0^\infty h^p \psi^{1-p} \right)^{\frac{1}{p}}, \tag{7}$$

for all $h \in \mathcal{M}_+(0, \infty)$, and

$$\left(\int_0^\infty \left(\int_t^\infty h(s)\,\mathrm{d}s \right)^q w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} \le C_{(8)} \left(\int_0^\infty h^p (g^{**})^{-p} \psi^{1-p} \right)^{\frac{1}{p}}, \tag{8}$$

for all $h \in \mathcal{M}_+(0, \infty)$. Moreover, the optimal constants satisfy $C_{(6)} \simeq C_{(7)} + C_{(8)}$.

PROOF. Assume that there exists a nonnegative measurable function γ compactly supported in $(0, \infty)$ and such that

$$g^*(t) = \int_t^\infty \frac{\gamma(s)}{s}\,\mathrm{d}s, \qquad t > 0. \tag{9}$$

By the Fubini theorem, for any $t > 0$ we obtain

$$\begin{aligned}
&t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s \qquad (10)\\
&= f^{**}(t) \int_0^t \int_s^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x\,\mathrm{d}s + \int_t^\infty f^*(s) \int_s^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x\,\mathrm{d}s \\
&= f^{**}(t) \int_0^t \gamma(x)\,\mathrm{d}x + \int_0^t f^*(s)\,\mathrm{d}s \int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x + \int_t^\infty f^*(s) \int_s^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

$$= f^{**}(t)\int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x \int_0^t f^*(s)\,\mathrm{d}s + \int_t^\infty \frac{\gamma(x)}{x}\int_t^x f^*(s)\,\mathrm{d}s\,\mathrm{d}x$$

$$= f^{**}(t)\int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \gamma(x) f^{**}(x)\,\mathrm{d}x.$$

Now define the positive linear operator $R\colon \mathscr{M}_+(0,\infty) \to \mathscr{M}_+(0,\infty)$ by

$$Rf(t) := f(t)\int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \gamma(x) f(x)\,\mathrm{d}x.$$

By Lemma 2.2, the inequality (6) holds if and only if

$$\left(\int_0^\infty (RSh(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \le C_{(11)}\left(\int_0^\infty h^p(t)\psi^{1-p}(t)\,\mathrm{d}t\right)^{\frac{1}{p}}, \tag{11}$$

for all $h \in \mathscr{M}_+(0,\infty)$, where S is the Stieltjes operator (5). Moreover, $C_{(6)} \simeq C_{(11)}$ for the optimal constants. Recall that for any $h \in \mathscr{M}_+$ one has

$$\int_0^\infty \frac{h(s)}{s+t}\,\mathrm{d}s \le \frac{1}{t}\int_0^t h(s)\,\mathrm{d}s + \int_t^\infty \frac{h(s)}{s}\,\mathrm{d}s \le 2\int_0^\infty \frac{h(s)}{s+t}\,\mathrm{d}s, \quad t > 0.$$

Let $h \in \mathscr{M}_+(0,\infty)$ and $t > 0$. We express $RSh(t)$ using g^{**} instead of γ, as follows:

$$\begin{aligned}
RSh(t) &\simeq \frac{1}{t}\int_0^t h(s)\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \frac{h(s)}{s}\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x \\
&\quad + \int_t^\infty \frac{\gamma(x)}{x}\int_0^x h(s)\,\mathrm{d}s\,\mathrm{d}x + \int_t^\infty \gamma(x)\int_x^\infty \frac{h(s)}{s}\,\mathrm{d}s\,\mathrm{d}x \\
&= \frac{1}{t}\int_0^t h(s)\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \frac{h(s)}{s}\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x \\
&\quad + \int_0^t h(s)\,\mathrm{d}s \int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x + \int_t^\infty h(s)\int_s^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_t^\infty \frac{h(s)}{s}\int_t^s \gamma(x)\,\mathrm{d}x\,\mathrm{d}s \\
&= \frac{1}{t}\int_0^t h \int_0^t \gamma + \int_t^\infty \frac{h(s)}{s}\int_0^s \gamma(x)\,\mathrm{d}x\,\mathrm{d}s + g^*(t)\int_0^t h + \int_t^\infty h g^*.
\end{aligned}$$

Since $g \in L^1_{\mathrm{loc}}(\mathsf{R}^d)$, one has $0 \le x g^*(x) \le x g^{**}(x) = \int_0^x g^*(y)\,\mathrm{d}y \xrightarrow{x\to 0+} 0$. Next, in a.e. point $t > 0$ the derivative of g^* exists and is equal to $-\frac{\gamma(t)}{t}$. Hence, integration by parts gives, for a.e. $t > 0$,

$$\int_0^t \gamma(x)\,\mathrm{d}x = \left[-xg^*(x)\right]_{x=0}^t + \int_0^t g^*(x)\,\mathrm{d}x = -tg^*(t) + \int_0^t g^*(x)\,\mathrm{d}x. \tag{12}$$

Applying this on the equivalent expression of $RSh(t)$ we calculated above, we obtain that, for a.e. $t > 0$,

$$\begin{aligned} RSh(t) &\simeq \frac{1}{t}\int_0^t h \int_0^t g^* + \int_t^\infty \frac{h(s)}{s}\int_0^s g^*(x)\,\mathrm{d}x\,\mathrm{d}s \\ &= g^{**}(t)\int_0^t h + \int_t^\infty hg^{**}. \end{aligned}$$

Using this expression in (11), we observe that (11) is equivalent to (7) and (8) and the optimal constants satisfy $C_{(11)} \simeq C_{(7)} + C_{(8)}$, i.e. $C_{(6)} \simeq C_{(7)} + C_{(8)}$.

So far we proved the lemma for g satisfying (9). Now consider an arbitrary $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. By Proposition 2.3 we find a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of measurable nonnegative functions with compact support in $(0,\infty)$ such that for a.e. $t > 0$ we have

$$g_n^*(t) := \int_t^\infty \frac{\gamma_n(x)}{x}\,\mathrm{d}x \uparrow g^*(t), \quad n \to \infty. \tag{13}$$

We also have $g_n^{**}(t) \uparrow g^{**}(t)$ for all $t > 0$. Using these approximations and the fact that the lemma holds for every g_n^*, we get that $C_{(6)} \simeq C_{(7)} + C_{(8)}$ for the optimal constants in the case of general g.

An a priori characterization of (6) for $p, q \in (1,\infty)$ hence reads as follows.

THEOREM 3.2. *Let $1 < p < \infty$. Let v, w be weights. Let ψ be given by* (4) *and $\Psi(t) := \int_0^t \psi$ for $t > 0$.*

(i) *Let $1 < p \le q < \infty$. Then the inequality* (6) *holds if and only if*

$$A_{(14)} := \sup_{t>0}\left(\int_t^\infty (g^{**}(s))^q w(s)\,\mathrm{d}s\right)^{\frac{1}{q}} \Psi^{\frac{1}{p'}}(t) < \infty \tag{14}$$

and

$$A_{(15)} := \sup_{t>0} W^{\frac{1}{q}}(t)\left(\int_t^\infty (g^{**}(s))^{p'}\psi(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} < \infty. \tag{15}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(14)} + A_{(15)}$.

(ii) *Let* $1 < q < p < \infty$ *and let* $r := \frac{pq}{p-q}$. *Then the inequality* (6) *holds if and only if*

$$A_{(16)} := \left(\int_0^\infty \left(\int_t^\infty (g^{**}(s))^q w(s)\,\mathrm{d}s\right)^{\frac{r}{q}} \Psi^{\frac{r}{q'}}(t)\psi(t)\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty \tag{16}$$

and

$$A_{(17)} := \left(\int_0^\infty \left(\int_t^\infty (g^{**}(s))^{p'} \psi(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty. \tag{17}$$

The optimal constant $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(16)} + A_{(17)}$.

PROOF. (i) By the weighted Hardy inequality and its dual version (see e.g. [13], [16]), the inequalities (7) and (8) hold if and only if $A_{(14)} < \infty$ and $A_{(15)} < \infty$, respectively. We also have $C_{(7)} \simeq A_{(14)}$ and $C_{(8)} \simeq A_{(15)}$ for the optimal constants. The result then follows from Lemma 3.1.

(ii) We proceed analogously to the previous case. The Hardy inequalities give that (7) holds if and only if $A_{(16)} < \infty$ and (8) holds if and only if

$$\left(\int_0^\infty W^{\frac{r}{q}}(t)\left(\int_t^\infty (g^{**}(s))^{p'}\psi(s)\,\mathrm{d}s\right)^{\frac{r}{q'}} (g^{**}(t))^{p'}\psi(t)\,\mathrm{d}t\right)^{\frac{1}{r}} < \infty.$$

This expression is by Proposition 2.4 equivalent to $A_{(17)}$. Finally, Lemma 3.1 gives the result again. Estimates on the optimal constants also follow, just as in (i).

Let us now turn our focus to the "limit cases" of the exponents p and q. First such case is the choice $q = \infty$.

THEOREM 3.3. *Let* v, w *be weights and let* $q = \infty$.

(i) *Let* $0 < p < 1$. *Then the inequality* (6) *holds if and only if*

$$A_{(18)} := \sup_{x>0} g^{**}(x)x\left(V(x) + x^p \int_x^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\right)^{-\frac{1}{p}} \operatorname*{ess\,sup}_{t\in(0,x)} w(t) < \infty. \tag{18}$$

The optimal constant $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(18)}$.

(ii) *Let* $1 < p < \infty$. *Let* ψ *be given by* (4). *Then the inequality* (6) *holds if and only if*

$$A_{(19)} := \operatorname*{ess\,sup}_{t>0} w(t)\left((g^{**}(t))^{p'}\Psi(t) + \int_t^\infty (g^{**}(s))^{p'}\psi(s)\,\mathrm{d}s\right)^{\frac{1}{p'}} < \infty. \tag{19}$$

The optimal constant $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(19)}$.

PROOF. The optimal constant $C_{(6)}$ is expressed in the following way:

$$(20)\quad C_{(6)} = \sup_{\|f\|_{\Gamma^p(v)}\le 1} \operatorname*{ess\,sup}_{t>0} w(t)\left(g^{**}(t)\int_0^t f^* + \int_t^\infty f^*g^*\right)$$

$$= \operatorname*{ess\,sup}_{t>0} w(t)\left(g^{**}(t)\sup_{\|f\|_{\Gamma^p(v)}\le 1}\int_0^t f^* + \sup_{\|f\|_{\Gamma^p(v)}\le 1}\int_t^\infty f^*g^*\right).$$

Observe also that, for any $p \in (0,\infty)$, the function $\widetilde{V}_p$ defined by

$$\widetilde{V}_p(x) := V(x) + x^p \int_x^\infty \frac{v(s)}{s^p}\,\mathrm{d}s, \qquad x > 0,$$

is increasing on $(0,\infty)$, while the function $x \mapsto \widetilde{V}_p(x)x^{-p}$ is decreasing on $(0,\infty)$.

(i) Let $0 < p < 1$. Then [6, Theorem 4.2(i)] gives

$$\sup_{\|f\|_{\Gamma^p(v)}\le 1}\int_0^t f^* \simeq \sup_{x>0}\int_0^x \chi_{[0,t]}(y)\,\mathrm{d}y\,\widetilde{V}_p^{-\frac{1}{p}}(x)$$

$$= \sup_{x\in(0,t]} x\widetilde{V}_p^{-\frac{1}{p}}(x) = t\widetilde{V}_p^{-\frac{1}{p}}(t).$$

By the same source, we have

$$\sup_{\|f\|_{\Gamma^p(v)}\le 1}\int_t^\infty f^*g^* \simeq \sup_{x>0}\int_0^x g^*(y)\chi_{[t,\infty)}(y)\,\mathrm{d}y\,\widetilde{V}_p^{-\frac{1}{p}}(x)$$

$$= \sup_{x\ge t}\int_t^x g^*(y)\,\mathrm{d}y\,\widetilde{V}_p^{-\frac{1}{p}}(x).$$

Using these calculations and (20), we now get

$$C_{(6)} \simeq \operatorname*{ess\,sup}_{t>0} w(t)\left(\int_0^t g^*(y)\,\mathrm{d}y\,\widetilde{V}_p^{-\frac{1}{p}}(t) + \sup_{x\ge t}\int_t^x g^*(y)\,\mathrm{d}y\,\widetilde{V}_p^{-\frac{1}{p}}(x)\right)$$

$$\simeq \operatorname*{ess\,sup}_{t>0} w(t)\sup_{x\ge t}\widetilde{V}_p^{-\frac{1}{p}}(x)\left(\int_0^t g^* + \int_t^x g^*\right)$$

$$= A_{(18)}.$$

(ii) Let $1 < p < \infty$. We proceed similarly as in (i). From [6, Theorem 4.2(ii)] it follows that

$$\sup_{\|f\|_{\Gamma^p(v)}\le 1}\int_0^t f^* \simeq \left(\int_0^\infty\left(\sup_{y\ge x}\frac{1}{y}\int_0^y \chi_{[0,t]}\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}$$
$$= \left(\Psi(t)+t^{p'}\int_t^\infty \frac{\psi(x)}{x^{p'}}\,\mathrm{d}x\right)^{\frac{1}{p'}}$$
$$= \left(\Psi(t)+t^{p'}\int_t^\infty\left(\sup_{y\ge x}\frac{1}{y}\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}$$

and

$$\sup_{\|f\|_{\Gamma^p(v)}\le 1}\int_t^\infty f^*g^*$$
$$\simeq \left(\int_0^\infty\left(\sup_{y\ge x}\frac{1}{y}\int_0^y g^*\chi_{[t,\infty)}\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}$$
$$= \left(\left(\sup_{y\ge t}\frac{1}{y}\int_t^y g^*\right)^{p'}\Psi(t)+\int_t^\infty\left(\sup_{y\ge x}\frac{1}{y}\int_t^y g^*\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}.$$

Together with (20), this gives

$$C_{(6)}\simeq \operatorname*{ess\,sup}_{t>0} w(t)\Bigg[\left(\left(\sup_{y\ge t}\frac{1}{y}\int_0^t g^*\right)^{p'}\Psi(t)+\int_t^\infty\left(\sup_{y\ge x}\frac{1}{y}\int_0^t g^*\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}$$
$$+\left(\left(\sup_{y\ge t}\frac{1}{y}\int_t^y g^*\right)^{p'}\Psi(t)+\int_t^\infty\left(\sup_{y\ge x}\frac{1}{y}\int_t^y g^*\right)^{p'}\psi(x)\,\mathrm{d}x\right)^{\frac{1}{p'}}\Bigg].$$

The right-hand side of the equation is equivalent to $A_{(19)}$ and the proof is finished.

Next, we proceed with the case $q=1$, covered by the following theorem.

THEOREM 3.4. *Let v, w be weights and $q=1$.*

(i) *Let $0<p\le 1$. Then the inequality* (6) *holds if and only if*

$$A_{(21)} := \sup_{t>0}\frac{g^{**}(t)tW(t)+t\int_t^\infty g^{**}(x)w(x)\,\mathrm{d}x}{\left(V(t)+t^p\int_t^\infty v(s)s^{-p}\,\mathrm{d}s\right)^{\frac{1}{p}}}<\infty. \tag{21}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)}\simeq A_{(21)}$.

(ii) *Let* $1 < p < \infty$. *Let* ψ *be given by* (4). *Then the inequality* (6) *holds if and only if*

$$A_{(22)} := \left(\int_0^\infty \left(g^{**}(t)W(t) + \int_t^\infty g^{**}(x)w(x)\,\mathrm{d}x\right)^{p'} \psi(t)\,\mathrm{d}t\right)^{\frac{1}{p'}} < \infty. \tag{22}$$

The optimal constant $C_{(6)}$ *satisfies* $C_{(6)} \simeq A_{(22)}$.

Proof. The Fubini theorem yields that (6) with $q = 1$ is equivalent to

$$\int_0^\infty f^*(t)\left(g^*(t)W(t) + \int_t^\infty g^{**}w\right)\mathrm{d}t \le C_{(6)}\left(\int_0^\infty (f^{**})^p v\right)^{\frac{1}{p}}, \tag{23}$$

for all $f \in \Gamma^p(v)$.

(i) By [6, Theorem 4.2(i)], inequality (23) holds if and only if

$$B_1 := \sup_{t>0} \frac{\int_0^t \left(g^*(x)W(x) + \int_x^\infty g^{**}(s)w(s)\,\mathrm{d}s\right)\mathrm{d}x}{\left(V(t) + t^p \int_t^\infty v(s)s^{-p}\,\mathrm{d}s\right)^{\frac{1}{p}}} < \infty.$$

Moreover, $C_{(6)} \simeq B_1$ for the optimal constant. Using the Fubini theorem we obtain

$$\begin{aligned} g^{**}(t)tW(t) + t\int_t^\infty g^{**}(x)w(x)\,\mathrm{d}x \\ = \int_0^t \left(g^*(x)W(x) + \int_x^\infty g^{**}(s)w(s)\,\mathrm{d}s\right)\mathrm{d}x \end{aligned} \tag{24}$$

for all $t > 0$. Hence, we have $B_1 = A_{(21)}$.

(ii) In this case, [6, Theorem 4.2(ii)] yields that (23) is satisfied if and only if

$$\begin{aligned} B_2 := \Bigg(\int_0^\infty \Bigg(\sup_{y \ge t} \frac{1}{y}\Bigg(\int_0^y \Bigg(g^*(x)W(x) \\ + \int_x^\infty g^{**}(s)w(s)\,\mathrm{d}s\Bigg)\mathrm{d}x\Bigg)\Bigg)^{p'} \psi(t)\,\mathrm{d}t\Bigg)^{\frac{1}{p'}} < \infty. \end{aligned}$$

One also has $C_{(6)} \simeq B_2$ for the optimal constant. Observe that the function

$x \mapsto g^*(x)W(x)+\int_x^\infty g^{**}w$ is nonincreasing, which together with (24) gives

$$\begin{aligned}\sup_{y\ge t}\frac{1}{y}\left(\int_0^y\left(g^*(x)W(x)+\int_x^\infty g^{**}w\right)\mathrm{d}x\right)\\ &=\frac{1}{t}\left(\int_0^t\left(g^*(x)W(x)+\int_x^\infty g^{**}w\right)\mathrm{d}x\right)\\ &=\frac{1}{t}\left(g^{**}(t)tW(t)+t\int_t^\infty g^{**}w\right).\end{aligned}$$

for any $t>0$. Hence, we obtain $B_2=A_{(22)}$.

To deal with the case $p=\infty$, we will make use of a more general lemma below. In its proof we follow a similar pattern as in [3, Theorem 6.4], where a particular case was treated.

LEMMA 3.5. *Let v be a weight and let $\|\cdot\|_X$ be an r.i. quasi-norm on $\mathcal{M}(0,\infty)$. Let $S\colon \mathcal{M}_+(0,\infty)\to\mathcal{M}_+(0,\infty)$ be a quasi-linear operator which, for all $f, f_n, g\in\mathcal{M}_+(0,\infty)$, $n\in\mathsf{N}$, satisfies the following conditions:*

(i) *$f\le g$ a.e. implies $Sf\le Sg$ a.e.*;

(ii) *$f_n\uparrow f$ a.e. implies $Sf_n\uparrow Sf$ a.e.*

Then the inequality

$$\|S(f^{**})\|_X\le C_{(25)}\operatorname*{ess\,sup}_{t>0}f^{**}(t)v(t),\quad f\in\Gamma^\infty(v),\tag{25}$$

holds if and only if

$$A_{(26)}:=\|S\varrho\|_X<\infty,\tag{26}$$

where

$$\varrho(t):=\left(\operatorname*{ess\,sup}_{s>0}\min\left\{1,\frac{t}{s}\right\}v(s)\right)^{-1},\quad t>0.\tag{27}$$

The optimal constant $C_{(25)}$ satisfies $C_{(25)}\simeq A_{(26)}$.

PROOF. At first, observe that, for any $f\in\mathcal{M}(\mathsf{R}^d)$,

$$\begin{aligned}\|f\|_{\Gamma^\infty(v)}&=\max\left\{\operatorname*{ess\,sup}_{s>0}v(s)\sup_{t>s}f^{**}(t),\ \operatorname*{ess\,sup}_{s>0}\frac{v(s)}{s}\sup_{t\in(0,s)}tf^{**}(t)\right\}\\ &=\operatorname*{ess\,sup}_{t>0}f^{**}(t)\max\left\{\operatorname*{ess\,sup}_{s\in(0,t)}v(s),t\operatorname*{ess\,sup}_{s>t}\frac{v(s)}{s}\right\}=\|f\|_{\Gamma^\infty(\varrho^{-1})}.\end{aligned}$$

Let us prove that (26) is sufficient for (25). Suppose that (26) holds. Thanks to the properties of S, we have the following estimate:

$$\|S(f^{**})\|_X = \left\| S\left(\frac{f^{**}\varrho}{\varrho}\right)\right\|_X \le \sup_{t>0} \frac{f^{**}(t)}{\varrho(t)} \|S\varrho\|_X$$
$$= \|f\|_{\Gamma^\infty(\varrho^{-1})} A_{(26)} = \|f\|_{\Gamma^\infty(v)} A_{(26)}.$$

Hence, (25) is satisfied and $C_{(25)} \le A_{(26)}$ for the optimal $C_{(25)}$.

Now we turn to the necessity of (26). Assume that (25) holds. Since ϱ is quasi-concave, there exists a function $f \in \mathscr{M}(\mathsf{R}^d)$ and a constant $\lambda > 0$ such that

$$\frac{1}{2}\left(\lambda + \int_0^t f^*\right) \le t\varrho(t) \le \left(\lambda + \int_0^t f^*\right), \quad t > 0. \tag{28}$$

Indeed, if ω denotes the least concave majorant of the function $t \mapsto t\varrho(t)$, then we may choose $\lambda := \lim_{s\to 0+} \omega(s)$ and $f \in \mathscr{M}(\mathsf{R}^d)$ such that $\omega(t) = \lambda + \int_0^t f^*$, $t > 0$. The inequality then follows by [1, Proposition 5.10, p. 71], since $t \mapsto t\varrho(t)$ is quasi-concave. In particular, (28) yields

$$\|f\|_{\Gamma^\infty(v)} \le 2 \operatorname*{ess\,sup}_{t>0} v(t)\varrho(t) \le 2.$$

We obtain

$$A_{(26)} \lesssim \left\|S\left(s \mapsto \tfrac{\lambda}{s} + f^{**}(s)\right)\right\|_X \lesssim \left\|S\left(s \mapsto \tfrac{\lambda}{s}\right)\right\|_X + \|S(f^{**})\|_X$$
$$\lesssim \left\|S\left(s \mapsto \tfrac{\lambda}{s}\right)\right\|_X + C_{(25)}\|f\|_{\Gamma^\infty(v)} \lesssim \left\|S\left(s \mapsto \tfrac{\lambda}{s}\right)\right\|_X + 2C_{(25)}.$$

If $\lambda = 0$, we are done, since $S(0) = 0$. Now suppose that $\lambda > 0$. Choose $\varepsilon > 0$ arbitrarily and let $g \in \mathscr{M}(\mathsf{R}^d)$ be such that $g^* = \frac{\lambda}{\varepsilon}\chi_{[0,\varepsilon]}$. Then $\|g\|_1 = \lambda$. By (28) we have $\frac{1}{t\varrho(t)} \le \frac{2}{\lambda}$ for all $t > 0$. Thus,

$$\|g\|_{\Gamma^\infty(v)} = \|g\|_{\Gamma^\infty(\varrho^{-1})} = \sup_{t>0} \frac{\int_0^t g^*}{t\varrho(t)} \le \|g\|_1 \sup_{t>0} \frac{1}{t\varrho(t)} \le 2.$$

Next, for all $s > \varepsilon$ one has $g^{**}(s) = \frac{\lambda}{s}$. Therefore we get

$$\left\|S\left(s \mapsto \tfrac{\lambda\chi_{[\varepsilon,\infty)}(s)}{s}\right)\right\|_X = \|S(\chi_{[\varepsilon,\infty)}g^{**})\|_X \le \|S(g^{**})\|$$
$$\le C_{(25)}\|g\|_{\Gamma^\infty(v)} \le 2C_{(25)}. \tag{29}$$

Since $\frac{\lambda\chi_{[\varepsilon,\infty)}(s)}{s} \uparrow \frac{\lambda}{s}$ as $\varepsilon \to 0+$ for every $s > 0$, we get $S\left(s \mapsto \frac{\lambda\chi_{[\varepsilon,\infty)}(s)}{s}\right) \uparrow S\left(s \mapsto \frac{\lambda}{s}\right)$ a.e. on $(0,\infty)$ as $\varepsilon \to 0+$. Hence, the Fatou property of $\|\cdot\|_X$ used in (29) gives

$$\left\|S\left(s \mapsto \tfrac{\lambda}{s}\right)\right\|_X \le 2C_{(25)}.$$

We have shown that $A_{(26)} \lesssim C_{(25)}$ and the proof is complete.

Making an appropriate choice of the operator S in Lemma 3.5, we obtain the following theorem.

THEOREM 3.6. *Let v, w be weights. Let $p = \infty$.*

(i) *Let $q \in (0, \infty)$. Then the inequality* (6) *is satisfied if and only if*

$$A_{(30)} := \left(\int_0^\infty \left[\frac{g^{**}(t)}{\operatorname{ess\,sup}_{s>0} \min\{\frac{1}{t}, \frac{1}{s}\} v(s)} + \int_t^\infty g^*(x)\, \mathrm{d}\left(\frac{1}{\operatorname{ess\,sup}_{s>0} \min\{\frac{1}{x}, \frac{1}{s}\} v(s)}\right)\right]^q w(t)\, \mathrm{d}t\right)^{\frac{1}{q}} < \infty. \tag{30}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(30)}$.

(ii) *Let $q = \infty$. Then the inequality* (6) *is satisfied if and only if*

$$A_{(31)} := \operatorname*{ess\,sup}_{t>0} \left[\frac{g^{**}(t)}{\operatorname{ess\,sup}_{s>0} \min\{\frac{1}{t}, \frac{1}{s}\} v(s)} + \int_t^\infty g^*(x)\, \mathrm{d}\left(\frac{1}{\operatorname{ess\,sup}_{s>0} \min\{\frac{1}{x}, \frac{1}{s}\} v(s)}\right)\right] w(t) < \infty. \tag{31}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(31)}$.

PROOF. Let us prove (i). Define the function ϱ by (27) and the function ω by

$$\omega(t) := t\varrho(t) = \frac{1}{\operatorname{ess\,sup}_{s>0} \min\{\frac{1}{t}, \frac{1}{s}\} v(s)}, \quad t > 0. \tag{32}$$

The function ω is nondecreasing and continuous on $(0, \infty)$. Thus, its derivative ω' exists a.e. on $(0, \infty)$. We may assume that $\omega(0+) := \lim_{t\to 0+} \omega(t)$ is finite, otherwise ω is constantly infinite, thus $\|\cdot\|_{\Gamma^\infty(v)} = \|\cdot\|_{\Gamma^\infty(\varrho^{-1})} \equiv 0$. Hence, we may write

$$\varrho(t) = \frac{\omega(t)}{t} = \frac{1}{t} \int_0^t \omega'(x)\, \mathrm{d}x + \frac{\omega(0+)}{t}, \quad t > 0. \tag{33}$$

Now suppose that there exists $\gamma \in \mathscr{M}_+(0, \infty)$ with compact support in $(0, \infty)$ such that (9) holds. Define

$$Sh(t) := h(t) \int_0^t \gamma(x)\, \mathrm{d}x + \int_t^\infty h(x)\gamma(x)\, \mathrm{d}x, \quad h \in \mathscr{M}_+(0, \infty).$$

Using (10), we observe that the inequality (6) is equivalent to the inequality (25) with $X := L^q(w)$ and $C_{(6)} = C_{(25)}$. Lemma 3.5 yields that (25) holds if and only if $\|S\varrho\|_{L^q(w)} < \infty$. By (12), (33) and Fubini theorem, for every $t > 0$ we get

$$\begin{aligned}
S\varrho(t) &= \varrho(t)\int_0^t \gamma(x)\,\mathrm{d}x + \int_t^\infty \varrho(x)\gamma(x)\,\mathrm{d}x \\
&= \frac{1}{t}\int_0^t \omega'(s)\,\mathrm{d}s \int_0^t \gamma(x)\,\mathrm{d}x + \frac{\omega(0+)}{t}\int_0^t \gamma(x)\,\mathrm{d}x \\
&\quad + \int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x \int_0^t \omega'(s)\,\mathrm{d}s + \int_t^\infty \frac{\gamma(x)}{x}\int_t^x \omega'(s)\,\mathrm{d}s\,\mathrm{d}x \\
&\quad + \omega(0+)\int_t^\infty \frac{\gamma(x)}{x}\,\mathrm{d}x \\
&= g^{**}(t)\int_0^t \omega'(s)\,\mathrm{d}s + g^{**}(t)\omega(0+) + \int_t^\infty g^*(s)\omega'(s)\,\mathrm{d}s \\
&= g^{**}(t)\omega(t) + \int_t^\infty g^*(s)\omega'(s)\,\mathrm{d}s.
\end{aligned}$$

Thus, we obtain $\|S\varrho\|_{L^q(w)} = A_{(30)}$. This completes the proof of (i) for g satisfying (9). For a general $g \in L^1_{\mathrm{loc}}(\mathsf{R}^d)$, we use Proposition 2.3 to approximate g by appropriate functions g_n as in (13) and then obtain the result by the limit pass $n \to \infty$. The case (ii) is proved in the same way, choosing $X := L^\infty(w)$ in Lemma 3.5.

So far we have not yet covered the case $p = 1$, $q \in (1,\infty)$. However, since $\|\cdot\|_{\Gamma^1(v)} = \|\cdot\|_{\Lambda^1(\widetilde{v})}$ with $\widetilde{v}(t) := \int_t^\infty \frac{v(s)}{s}\,\mathrm{d}s$, validity of (6) is characterized by [12, Theorem 3.2(i)]. From there we get the following result which completes our list.

PROPOSITION 3.7. *Let v, w be weights. Let $p = 1$ and $q \in (1,\infty)$. Then the inequality* (6) *holds if and only if*

$$A_{(34)} := \sup_{t>0} \frac{g^{**}(t)tW^{\frac{1}{q}}(t) + t\left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}}}{V(t) + t\int_t^\infty v(x)x^{-1}\,\mathrm{d}x} < \infty. \tag{34}$$

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(34)}$.

REMARK 3.8. The expression $A_{(14)}$, with p, q set as in Theorem 3.2(i), defines a norm of $g \in \mathscr{M}(\mathsf{R}^d)$. Similarly, the following expressions are norms: $A_{(15)}$, $A_{(16)}$, $A_{(17)}$, $A_{(18)}$, $A_{(19)}$, $A_{(21)}$, $A_{(22)}$ and $A_{(34)}$. In each case, the values of p and q correspond with the setting of the particular theorem or proposition. The subadditivity of the functional follows here from the subadditivity of the maximal function (3). For more details about r.i. spaces generated by these norms see [12].

Moreover, the expressions $A_{(30)}$ with $q \in [1, \infty)$ and $A_{(31)}$ each are equivalent to a norm of $g \in \mathscr{M}(\mathsf{R}^d)$. The expression $A_{(30)}$ with $q \in (0, 1)$ defines a quasi-norm of $g \in \mathscr{M}(\mathsf{R}^d)$. These claims may be proved by replacing the function ω from (32) by its least concave majorant (cf. [1, p. 71]) and then performing a similar procedure as in (10) to rewrite the expressions using only f^{**} and not f^*. Then it is possible to use (3) again.

4. Young-type convolution inequalities for Γ-spaces

In the previous section we obtained sufficient conditions for boundedness of T_g between $\Gamma^p(v)$ and $\Gamma^q(w)$. But more can be said. If $g \in \mathscr{M}_+^{\odot}(\mathsf{R}^d)$, then these conditions are also necessary. Moreover, the result can be given the form of a Young-type inequality. All of this is summarized in the main theorem below. Recall that we say that an r.i. lattice X *is embedded into* an r.i. lattice Y and write $X \hookrightarrow Y$, if there exists a constant $C > 0$ such that $\|f\|_Y \leq C\|f\|_X$ for all $f \in X$.

THEOREM 4.1. *Let v, w be weights. Depending on the parameters p, q, for $g \in \mathscr{M}(\mathsf{R}^d)$ define $\|g\|_Y$ by what follows:*

$$\|g\|_Y := \begin{cases} A_{(14)} + A_{(15)} & \textit{if} \quad 1 < p \leq q < \infty; \\ A_{(16)} + A_{(17)} & \textit{if} \quad 1 < q < p < \infty; \\ A_{(34)} & \textit{if} \quad 1 = p < q < \infty; \\ A_{(18)} & \textit{if} \quad 0 < p < 1, \ q = \infty; \\ A_{(19)} & \textit{if} \quad 1 < p < q = \infty; \\ A_{(21)} & \textit{if} \quad 0 < p \leq q = 1; \\ A_{(22)} & \textit{if} \quad 1 = q < p < \infty; \\ A_{(30)} & \textit{if} \quad 0 < q < p = \infty; \\ A_{(31)} & \textit{if} \quad p = q = \infty. \end{cases}$$

For each choice of p, q from the previous list define $Y := \{g \in \mathscr{M}(\mathsf{R}^d) : \|g\|_Y < \infty\}$. Then:

(i) *If $g \in Y$, then $T_g \colon \Gamma^p(v) \to \Gamma^q(w)$ and*

$$\|T_g\|_{\Gamma^p(v) \to \Gamma^q(w)} \lesssim \|g\|_Y.$$

(ii) *If* $g \in \mathscr{M}_+^{\odot}(\mathsf{R}^d)$ *and* $T_g\colon \Gamma^p(v) \to \Gamma^q(w)$, *then* $g \in Y$ *and*

$$\|g\|_Y \lesssim \|T_g\|_{\Gamma^p(v)\to\Gamma^q(w)}.$$

(iii) *The inequality*

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Gamma^p(v)}\|g\|_Y, \quad f \in \Gamma^p(v),\, g \in Y, \tag{35}$$

is satisfied. Moreover, if $\widetilde{Y}$ *is any r.i. lattice such that (35) is satisfied with* $\widetilde{Y}$ *in place of* Y, *then* $\widetilde{Y} \hookrightarrow Y$.

Proof. Let us consider the case $1 < p \le q < \infty$, the other ones are analogous.

(i) Let us define

$$R_g f(t) := t f^{**}(t) g^{**}(t) + \int_t^\infty f^*(s) g^*(s)\,\mathrm{d}s$$

for $f \in \mathscr{M}(\mathsf{R}^d)$ and $t > 0$. If $g \in Y$, then, by Theorem 3.2(i), the inequality (6) holds, with $C_{(6)} \simeq \|g\|_Y$. The O'Neil inequality (2) then gives

$$\|f * g\|_{\Gamma^q(w)} = \|(f * g)^{**}\|_{L^q(w)} \le \|R_g f\|_{L^q(w)} \lesssim \|f\|_{\Gamma^p(v)}\|g\|_Y.$$

Hence, (i) holds and so does the inequality (35).

(ii) Since $g \in \mathscr{M}_+^{\odot}(\mathsf{R}^d)$, the reverse O'Neil inequality (Lemma 2.1) implies $R_g f \lesssim (T_g f)^{**}$ on $(0,\infty)$. Observe also that $R_g f = R_g \widetilde{f}$ whenever $f^* = \widetilde{f}^*$. Using Theorem 3.2(i) we get

$$\begin{aligned}\|g\|_Y &\lesssim \sup_{\|f\|_{\Gamma^p(v)}\le 1} \|R_g f\|_{L^q(w)} = \sup_{\substack{\|f\|_{\Gamma^p(v)}\le 1\\ f\in\mathscr{M}_+^{\odot}(\mathsf{R}^d)}} \|R_g f\|_{L^q(w)}\\ &\lesssim \sup_{\substack{\|f\|_{\Gamma^p(v)}\le 1\\ f\in\mathscr{M}_+^{\odot}(\mathsf{R}^d)}} \|T_g f\|_{\Gamma^q(w)} \le \|T_g\|_{\Gamma^p(v)\to\Gamma^q(w)}.\end{aligned}$$

(iii) Let $\widetilde{Y}$ by an r.i. lattice such that

$$\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Gamma^p(v)}\|g\|_{\widetilde{Y}}, \quad f \in \Gamma^p(v),\ g \in \widetilde{Y}. \tag{36}$$

Let $h \in \widetilde{Y}$. There exists $g \in \mathscr{M}_+^{\odot}(\mathsf{R}^d)$ such that $g^* = h^*$. From (36) it follows that $T_g\colon \Gamma^p(v) \to \Gamma^q(w)$ and $\|T_g\|_{\Gamma^p(v)\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}$. Thus, (ii) gives

$$\|g\|_Y \lesssim \|T_g\|_{\Gamma^p(v)\to\Gamma^q(w)} \lesssim \|g\|_{\widetilde{Y}}.$$

Since $\|g\|_Y = \|h\|_Y$ and $\|g\|_{\widetilde{Y}} = \|h\|_{\widetilde{Y}}$, we have $\|h\|_Y \lesssim \|h\|_{\widetilde{Y}}$. Hence, we get $\widetilde{Y} \hookrightarrow Y$.

REMARK 4.2. (i) For given p, q, v, w the optimal space Y from Theorem 4.1 may be trivial, i.e. $Y = \{0\}$. In that case, T_g is not bounded between $\Gamma^p(v)$ and $\Gamma^q(w)$ for any nonnegative nontrivial kernel g (see [12, Corollary 3.3] for an analogy with $\Lambda^p(v)$ as the domain space).

(ii) The spaces Y from Theorem 4.1 are of the same type as those obtained in [12], [11] in analogous situations (with Λ and S, respectively, as the domain). Their basic functional properties were studied in [12]. Recently, in [7] these spaces appeared as associate spaces to the "generalized Γ-spaces" $G\Gamma$.

(iii) In [14, Theorem 4.1], the authors obtained a sufficient condition for the boundedness $T_g : \Gamma^p(v) \to \Gamma^q(w)$ with the following assumptions: u, v, w are weights, $1 < q < \infty$, $1 \le p, r \le \infty$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$, $\|w\|_1 = \infty$, $w \in B_q$, i.e. there exists $C > 0$ such that $\int_x^\infty w(t)t^{-q}\,\mathrm{d}t \le Cx^{-q}W(x)$ for all $x > 0$, and, moreover, there exists $D > 0$ such that the weights satisfy the pointwise inequality

$$W(t) \le D w^{\frac{1}{q'}}(t) v^{\frac{1}{p}}(t) u^{\frac{1}{r}}(t), \quad t > 0.$$

It was shown that under these conditions one has $\|f * g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Gamma^p(v)}\|g\|_{\Gamma^r(u)}$. This statement was proved in [14] using the rather strong assumptions on the weights, and it does not follow from Theorem 4.1 immediately. However, Theorem 4.1 provides a different sufficient condition for $T_g\colon \Gamma^p(v) \to \Gamma^q(w)$ with no additional assumptions on the weights and for a wider range of p and q, including the case $1 < q < p < \infty$. Moreover, this condition is also necessary provided that $g \in \mathscr{M}_+^{\odot}(\mathsf{R}^d)$.

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