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ON THE SIMPLICITY OF MULTIGERMS

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Abstract

We prove several results regarding the simplicity of germs and multigerms obtained via the operations of augmentation, simultaneous augmentation and concatenation and generalised concatenation. We also give some results in the case where one of the branches is a non-stable primitive germ. Using our results we obtain a list which includes all simple multigerms from $\mathbb{C}^3$ to $\mathbb{C}^3$.

1. Introduction

In the last few years the study of classifications of singularities of map-germs $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, has given a step forward (specially when $|S| = r > 1$) by substituting the classical classification methods with operations in order to obtain multigerms from germs in lower dimensions and with fewer branches. In [5], Cooper, Mond and Wik-Atique use the operation of augmentation and define monic and binary concatenations in order to obtain all $\mathscr{A}_e$-codimension 1 corank 1 multigerms with $n \geq p - 1$ and (n, p) in Mather's nice dimensions. In [17], the authors define further operations such as a simultaneous augmentation and concatenation and the generalised concatenation (which includes both the monic and binary concatenations as particular cases) to obtain all $\mathscr{A}_e$-codimension 2 corank 1 multigerms with the same dimension restrictions. However very little is known about the simplicity of the multigerms obtained via these operations.

A multigerm $f = \{f_1, \dots, f_r\}\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ with $S = \{x_1, \dots, x_r\}$ is simple if there exists a finite number of $\mathscr{A}$-classes (classes under the action of germs of diffeomorphisms in the source and target) such that for every unfolding $F\colon (\mathbb{K}^n \times \mathbb{K}^d, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}^d, 0)$ with $F(x, \lambda) = (f_\lambda(x), \lambda)$ and $f_0 = f$, there exists a sufficiently small neighbourhood U of $S \times \{0\}$ such that for every $(y_1, \lambda), \dots, (y_r, \lambda) \in U$ with $F(y_1, \lambda) = \dots = F(y_r, \lambda)$, the multigerm $f_\lambda\colon (\mathbb{K}^n, \{y_1, \dots, y_r\}) \to (\mathbb{K}^p, f_\lambda(y_i))$ lies in one of those finite classes.

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In [5], Cooper, Mond and Wik-Atique proved that all $\mathscr{A}_e$-codimension 1 multigerms in Mather's nice dimensions are simple. Hobbs and Kirk in [7] and the third author in [25] obtain a list of all simple multigerms from $\mathbb{R}^2$ to $\mathbb{R}^3$. Kolgushkin and Sadykov in [10] and Zhitomirskii in [26] deal with simple multigerms of curves. In [15], Nishimura gives an upper bound on the multiplicity of a simple multigerm. These are probably the only references related to the simplicity of multigerms. (For the case of $\mathscr{A}$-classification of simple monogerms, many papers can be cited such as [19], [2], [14], [6], [20], [21], [12], [1], [9], etc.)

In this paper we consider the problem of knowing when a multigerm obtained by one of the operations mentioned above is simple. We also study the case when the multigerm contains a non-stable branch. Section 2 introduces the notation and the basic definitions. Section 3 deals with augmentations of monogerms. We prove that if the augmenting function g is not simple then the resulting augmentation is not simple. Section 4 deals with how simplicity is affected when you add an extra branch to a simple germ. The first subsection deals with simultaneous augmentation and concatenation. We prove that, with certain hypotheses, a simultaneous augmentation and concatenation is simple if and only if the augmentation comes from an $\mathscr{A}_e$-codimension 1 germ. In the second subsection we study the simplicity of generalised concatenations. The main result is that a non-monic generalised concatenation of stable germs F and g where F has zero-dimensional analytic stratum is not simple (Corollary 4.14). The third subsection deals with germs where one of the branches is non-stable. We classify here all the simple multigerms $h = \{f, g\}$ where f is a non-stable germ and g is a prism on a Morse function or an immersion.

We give clues to which may be the remaining simple multigerms that are not classified in this paper, namely multigerms $h = \{f, g\}$ with f and g stable and the dimensions of their analytic strata between 1 and $p - 2$, and the case where f is non-stable and g is a stable singularity more degenerate than a prism on a Morse function or an immersion. We prove some partial results and show examples of these cases.

In the last section we use our results to obtain a list which includes all simple multigerms from $\mathbb{C}^3$ to $\mathbb{C}^3$.

2. Notation

Let $\mathscr{O}_n^p$ be the vector space of monogerms with n variables and p components. When $p = 1$, $\mathscr{O}_n^1 = \mathscr{O}_n$ is the local ring of germs of functions in n-variables and $\mathscr{M}_n$ its maximal ideal. The set $\mathscr{O}_n^p$ is a free $\mathscr{O}_n$-module of rank p. A multigerm is a germ of an analytic (complex case) or smooth (real case) map $f = \{f_1, \ldots, f_r\}\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ where $S = \{x_1, \ldots, x_r\} \subset \mathbb{K}^n$,

$f_i\colon(\mathbb{K}^n, x_i) \to (\mathbb{K}^p, 0)$ and $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Let $\mathcal{M}_n\mathcal{O}^p_{n,S}$ be the vector space of such map-germs. Let $\theta_{\mathbb{K}^n,S}$ and $\theta_{\mathbb{K}^p,0}$ be the $\mathcal{O}_n$-module of germs at S of vector fields on $\mathbb{K}^n$ and $\mathcal{O}_p$-module of germs at 0 of vector fields on $\mathbb{K}^p$ respectively. We denote them by θ_n and θ_p. Let $\theta(f)$ be the $\mathcal{O}_n$-module of germs $\xi\colon(\mathbb{K}^n, S) \to T\mathbb{K}^p$ such that $\pi_p \circ \xi = f$, where $\pi_p\colon T\mathbb{K}^p \to \mathbb{K}^p$ denotes the tangent bundle over $\mathbb{K}^p$.

Define $tf\colon\theta_n \to \theta(f)$ by $tf(\chi) = df \circ \chi$ and $wf\colon\theta_p \to \theta(f)$ by $wf(\eta) = \eta \circ f$. The $\mathcal{A}_e$-tangent space of a germ f is defined as $T\mathcal{A}_e f = tf(\theta_n) + wf(\theta_p)$ and its $\mathcal{A}_e$-codimension, denoted by $\mathcal{A}_e$-cod(f), is the $\mathbb{K}$-vector space dimension of

$$NA_e(f) = \frac{\theta(f)}{T\mathcal{A}_e f}.$$

When we have the $\mathcal{A}$-tangent space $T\mathcal{A}f = tf(\mathcal{M}_n \cdot \theta_n) + wf(\mathcal{M}_p \cdot \theta_p)$ in the denominator of the previous quotient and $\mathcal{M}_n\theta(f)$ in the numerator, its dimension is called the $\mathcal{A}$-codimension. We refer to Wall's survey article [24] for general background on the theory of singularities.

DEFINITION 2.1. i) A vector field germ $\eta \in \theta_p$ is called *liftable over f* if there exists $\xi \in \theta_n$ such that $df \circ \xi = \eta \circ f$ $(tf(\xi) = wf(\eta))$. The set of vector field germs liftable over f is denoted by $\mathrm{Lift}(f)$ and is an $\mathcal{O}_p$-module.

ii) Let $\widetilde{\tau}(f) = ev_0(\mathrm{Lift}(f))$ be the evaluation at the origin of elements of $\mathrm{Lift}(f)$.

In general $\mathrm{Lift}(f) \subseteq \mathrm{Derlog}(V)$ when V is the discriminant of an analytic f and $\mathrm{Derlog}(V)$ represents the $\mathcal{O}_p$-module of vector fields tangent to V. We have an equality when $\mathbb{K} = \mathbb{C}$ and f is complex analytic.

The set $\widetilde{\tau}(f)$ is the tangent space to the well-defined manifold in the target containing 0 along which the map f is trivial (i.e. the analytic stratum). Following Mather, f is stable if and only if all its branches are stable and their analytic strata have regular intersection ([13]).

Given $f = \{f_1, \dots, f_r\}\colon(\mathbb{K}^n,S) \to (\mathbb{K}^p,0)$, let $m_0(f) = \dim_{\mathbb{K}} \mathcal{O}_{n,S}/f^*(\mathcal{M}_p)$ denote the multiplicity of the germ f. Note that

$$\dim_{\mathbb{K}} \frac{\mathcal{O}_{n,S}}{f^*(\mathcal{M}_p)} = \sum_{i=1}^{r} \dim_{\mathbb{K}} \frac{\mathcal{O}_{n,x_i}}{f_i^*(\mathcal{M}_p)}.$$

From here on we consider only corank 1 germs. We say that $f = \{f_1, \dots, f_r\}$ is of type $A_{k_1,\dots,k_r}$ if $f_i \in A_{k_i}$, $i = 1, \dots, r$. For these singularities, $m_0(f) = k_1 + \cdots + k_r + r$.

3. Simplicity of Augmentations

DEFINITION 3.1. Let $h: (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a map-germ with a 1-parameter unfolding $H: (\mathbb{K}^n \times \mathbb{K}, S \times \{0\}) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ which is stable as a map-germ, where $H(x, \lambda) = (h_\lambda(x), \lambda)$ with $h_0 = h$. Let $g: (\mathbb{K}^q, 0) \to (\mathbb{K}, 0)$ be a function-germ. Then, the *augmentation of h by H and g* is the map $A_{H,g}(h)$ given by $(x, z) \mapsto (h_{g(z)}(x), z)$. A germ that is not an augmentation is called *primitive*.

A natural question arises: given simple germs h and g, is $A_{H,g}(h)$ simple? This is not true in general as can be seen in the following

EXAMPLE 3.2. Consider the simple germ $h(x_1, x_2) = (x_1^3 + x_2^4 x_1, x_2)$ and the unfolding $H(x_1, x_2, \lambda) = (x_1^3 + x_2^4 x_1 + \lambda x_1, x_2, \lambda)$. If we augment h by the simple function $g(z) = z^4$, we obtain the non-simple germ $A_{H,g}(h)(x_1, x_2, z) = (x_1^3 + (x_2^4 + z^4)x_1, x_2, z)$ ([12]). Notice that the germ h is not primitive.

For monogerms we show that the simplicity of the augmenting function g is a necessary condition for the simplicity of the augmentation. In fact, we prove that if two augmentations $f_1(x, z) = (h_{g_1(z)}(x), z)$ and $f_2(x, z) = (h_{g_2(z)}(x), z)$ are $\mathscr{A}$-equivalent, then g_1 and g_2 are $\mathscr{K}$-equivalent. The contact group $\mathscr{K}$ is the set of germs of diffeomorphisms of $(\mathbb{K}^n \times \mathbb{K}^p, 0)$ which can be written in the form $H(x, y) = (h(x), H_1(x, y))$, with $h \in \mathrm{Diff}(\mathbb{K}^n, 0)$ and $H_1(x, 0) = 0$ for x near 0. Two map-germs g_1 and g_2 are $\mathscr{K}$-equivalent if there exists $H \in \mathscr{K}$ such that $H(x, g_1(x)) = (h(x), g_2(h(x)))$. We need a preparatory lemma.

LEMMA 3.3. *Let $G_i(z, \epsilon) = g_i(z) + \psi(g_i(z), \epsilon)$ for $i = 1, 2$ such that $\psi(0, \epsilon) = \phi(\epsilon)$ is homogeneous of degree d. If $G_1 \sim_{\mathscr{K}} G_2$, then $g_1 \sim_{\mathscr{K}} g_2$.*

PROOF. For any $G(z, \epsilon) = g(z) + \psi(g(z), \epsilon)$ satisfying the hypotheses we claim that $G(z, \epsilon) \sim_{\mathscr{K}} g(z) + \phi(\epsilon)$. In fact, let $g(z) = w$, then $G(z, \epsilon) = w + \phi(\epsilon) + w\widetilde{\psi}(w, \epsilon) = w(1 + \widetilde{\psi}(w, \epsilon)) + \phi(\epsilon)$. If we write $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$ and put $\epsilon_i = (1 + \widetilde{\psi}(w, \epsilon))^{\frac{1}{d}} \epsilon_i'$, then we obtain that $G \sim_{\mathscr{K}} w + \phi(\epsilon')$.

Since $G_1 \sim_{\mathscr{K}} G_2$, we have $g_1(z) + \phi(\epsilon) \sim_{\mathscr{K}} g_2(z) + \phi(\epsilon)$ and so their Tjurina algebras, T_i, are isomorphic. Since ϕ is homogeneous $T_i = \mathscr{O}_{z,\epsilon} \big/ \big(\big\langle \frac{\partial g_i}{\partial z}, g_i \big\rangle + \big\langle \frac{\partial \phi}{\partial \epsilon} \big\rangle\big)$. We have that

$$\frac{\mathscr{O}_z}{\left\langle \frac{\partial g_1}{\partial z}, g_1 \right\rangle} \cong \frac{T_1}{\mathscr{M}_\epsilon T_1} \cong \frac{T_2}{\mathscr{M}_\epsilon T_2} \cong \frac{\mathscr{O}_z}{\left\langle \frac{\partial g_2}{\partial z}, g_2 \right\rangle}$$

and the result follows.

We remark here that by [21], any simple germ with $n > p$ comes from a simple germ with $n = p$ by just adding quadratic terms in the remaining variables, so for the case $n \geq p$ it is enough to study the equidimensional case.

PROPOSITION 3.4. *Let* $h: (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ *with* $n \geq p-1$ *be a non-stable primitive monogerm which admits a* 1*-parameter stable unfolding* H*. Let* g_1 *and* g_2 *be augmenting functions and* f_1 *and* f_2 *the corresponding augmentations. Then*

$$f_1 \sim_{\mathscr{A}} f_2 \Rightarrow g_1 \sim_{\mathscr{K}} g_2.$$

PROOF. First suppose that $p = n$. If h is primitive, by [8], $\widetilde{\tau}(H) = \{0\}$ and so $m_0(h) = m_0(H) \geq n+2$. Since h admits a 1-parameter stable unfolding $m_0(h) \leq n+2$ (by [13] stable germs have multiplicity $\leq p+1$). Therefore $m_0(h) = n+2$. From [22, Lemma 4.10] we know that such a germ is $\mathscr{A}$-equivalent to $(x, y^{n+2} + x_1 y + \cdots + x_{n-1} y^{n-1})$ if it is $\mathscr{A}_e$-codimension 1 or to $(x, y^{n+2} + x_1 y + \cdots + x_{n-1}^k y^{n-1} + x_{n-1} y^n)$ if it is $\mathscr{A}_e$-codimension k with $k \geq 2$.

In the first case, a 1-parameter stable unfolding is $(x, \lambda, y^{n+2} + x_1 y + \cdots + x_{n-1} y^{n-1} + \lambda y^n)$. Let $f_i(x, z, y) = (x, z, y^{n+2} + x_1 y + \cdots + x_{n-1} y^{n-1} + g_i(z) y^n)$ $i = 1, 2$ be $\mathscr{A}$-equivalent augmentations. By [22, Lemma 4.7] we have that

$$G_1(x, z) = (x_1, \ldots, x_{n-1}, g_1(z)) \sim_{\mathscr{K}} G_2(x, z) = (x_1, \ldots, x_{n-1}, g_2(z))$$

and so g_1 and g_2 are $\mathscr{K}$-equivalent.

In the second case, a 1-parameter stable unfolding is $(x, \lambda, y^{n+2} + x_1 y + \cdots + x_{n-1}^k y^{n-1} + x_{n-1} y^n + \lambda y^{n-1})$. Considering $\mathscr{A}$-equivalent augmentations $(x, z, y^{n+2} + x_1 y + \cdots + x_{n-1}^k y^{n-1} + x_{n-1} y^n + g_i(z) y^{n-1})$, $i = 1, 2$, in the same way as above we have that

$$\begin{aligned} G_1(x, z) &= (x_1, \ldots, x_{n-1}^k + g_1(z), x_{n-1}) \\ &\sim_{\mathscr{K}} G_2(x, z) = (x_1, \ldots, x_{n-1}^k + g_2(z), x_{n-1}). \end{aligned}$$

Since $G_i(x, z)$ is $\mathscr{K}$-equivalent to $(x_1, \ldots, x_{n-1}, g_i(z))$ we have the desired result.

Now suppose that $p = n+1$. As in the equidimensional case, $\widetilde{\tau}(H) = \{0\}$. Therefore n is odd, say $n = 2l+1$, and $m_0(h) = m_0(H) = l+2$. From [23, Proposition 4.5], if $l \geq 2$, h is equivalent to either

$$\big(x_1, \ldots, x_{2l}, y^{l+2} + x_1 y + \cdots + x_l y^l, x_{l+1} y + \cdots + x_{2l} y^l + \tilde{h}(x, y)\big)$$

or

$$\begin{aligned} \big(x_1, \ldots, x_{2l}, y^{l+2} &+ x_1 y + \cdots + x_l y^l, \\ &x_{l+1} y + \cdots + x_{2l-1} y^{l-1} + x_{2l} y^{l+1} + y^{l+2} + \tilde{h}(x, y)\big), \end{aligned}$$

where in both cases $\tilde{h} \in \mathcal{M}_n^{l+3}\mathcal{O}_n^{n+1}$. Then f_i, $i = 1, 2$, can be either

$$\left(x_1, \dots, x_{2l}, z, y^{l+2} + \sum_{j=1}^{l} x_j y^j, x_{l+1}y + \cdots + x_{2l}y^l + g_i(z)y^{l+1} + \tilde{h}(x, y)\right)$$

or

$$\Bigg(x_1, \dots, x_{2l}, z, y^{l+2} + \sum_{j=1}^{l} x_j y^j, \\ x_{l+1}y + \cdots + x_{2l-1}y^{l-1} + g_i(z)y^l + x_{2l}y^{l+1} + y^{l+2} + \tilde{h}(x, y)\Bigg).$$

Given a corank 1 germ f_i we associate a germ G_i whose component functions define the set of $l+2$-points appearing in a stable perturbation of f_i. If f_1 is $\mathscr{A}$-equivalent to f_2 then G_1 is $\mathscr{K}$-equivalent to G_2. Following [23, Section 3.2], $G_i\colon (\mathbb{K}^{3l+2+q}, 0) \to (\mathbb{K}^{2l+2}, 0)$ with source coordinates $(x, z, y, \epsilon_2, \dots, \epsilon_{l+2})$, and we can show that G_i is $\mathscr{K}$-equivalent to $(x, y, g_i(z) + \psi(g_i(z), \epsilon_2, \dots, \epsilon_{l+2}))$ in both cases, where $\psi(0, \epsilon)$ is homogeneous. The result can now be obtained applying Lemma 3.3.

EXAMPLE 3.5. i) The augmentation $f(x, z_1, z_2) = (x^3 + (z_1^4 + z_2^4)x, z_1, z_2)$ of $h(x) = x^3$ is not simple since the augmenting function $g(z_1, z_2) = z_1^4 + z_2^4$ is not simple.

ii) The converse of the proposition is not true. If we take the primitive germ (z^2, z^5) and augment it by the simple function $g(x, y) = x^2 + y^4$, we obtain the non-simple germ $(x, y, z^2, z^5 + (x^2 + y^4)z)$ (see [9]).

REMARK 3.6. We think that Proposition 3.4 also holds for multigerms. However, we have only been able to extend the arguments in the proof for particular examples such as a multigerm consisting only of fold singularities.

4. Simplicity of multigerms

The classification techniques for multigerms developed recently consist of combining monogerms to obtain multigerms. In this sense we are interested in knowing what combinations of simple germs yield simple multigerms. Subsections 4.1 and 4.3 deal with the simplest combination of germs, which consists of adding a prism on a Morse function (when $n \geq p$) or an immersion (when $p = n + 1$) to a simple germ. In 4.1 we study the simultaneous augmentation and concatenation operation and in 4.3 we combine a primitive codimension 1 germ with a prism on a Morse function or an immersion. Subsection 4.2 studies combinations of 2 stable germs, in particular, those arising from generalised concatenations.

In what follows we discuss the codimension of a multigerm where one of the branches is a prism on a Morse function or an immersion.

We are considering corank 1 multigerms of type $A_{k_1,\dots,k_r}$, for which it is known that their corresponding orbits in the multijet space are defined by submersions in the stable case and by ICIS in the finitely determined case ([6], [11]).

We note that there is a close relation between the $\mathscr{A}$-codimension and the $\mathscr{A}_e$-codimension. This is due to Wilson's formula (for the monogerm case see [24]; see [7] too), which asserts that if the $\mathscr{A}_e$-codimension is different from 0 and f is $\mathscr{A}$-simple, then

$$\mathscr{A}_e\text{-cod}(f) = \mathscr{A}\text{-cod}(f) + r(p-n) - p,$$

where r is the number of branches.

Let $f = \{f_1, \dots, f_r\}\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, y)$ be a non-stable multigerm with $\mathscr{A}$-codimension s. Let's assume that f is k-determined and $\mathscr{A}$-simple. Suppose there exists a smooth submanifold $X \subset {}_rJ^k(\mathbb{K}^n, \mathbb{K}^p)$ such that for all $g\colon \mathbb{K}^n \to \mathbb{K}^p$ and for all $\{z_1, \dots, z_r\} \subset \mathbb{K}^n$ we have that ${}_rj^k g(z_1, \dots, z_r) \in X$ if and only if the multigerm of g in $\{z_1, \dots, z_r\}$ is $\mathscr{A}$-equivalent to f. We have:

LEMMA 4.1. $\operatorname{cod}_{{}_rJ^k(\mathbb{K}^n,\mathbb{K}^p)} X = s + (r-1)p.$

PROOF. This is proved by standard multijet and transversality techniques, for a detailed account see [16].

If the $\mathscr{A}$-codimension of f_j is i_j, $j = 1, \dots, r$, this means that each f_j defines a smooth submanifold in the appropriate jet space of respective codimension i_j. These submanifolds are defined by $i_1, \dots, i_r$ equations respectively.

If we consider the submanifold $X \subset {}_rJ^k(\mathbb{K}^n, \mathbb{K}^p)$ defined by the equations which define the multigerm (i.e. the equations which define each of the branches, which are independent since they involve different variables, plus the equations arising from all the points having the same image in the target space), we have that its codimension is $i_1 + \dots + i_r + (r-1)p$ (the $(r-1)p$ extra equations come from $f(x_1) = \dots = f(x_r)$). From the previous Lemma the codimension of such a submanifold is $s + (r-1)p$, so we deduce that the $\mathscr{A}$-codimension of the multigerm is $s = i_1 + \dots + i_r$. In the case of some type of contact between the strata of the discriminant of different branches, other equations describing these contacts should be added to define the corresponding submanifold in the multijet space and so, in that case $s \geq i_1 + \dots + i_r$.

When one of the branches of the multigerm is non-stable, it is not easy to characterize the contact between the strata of the discriminant. We need the following

DEFINITION 4.2. Let $f\colon(\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ be a non-stable germ and $F(x, \lambda) = (f_\lambda(x), \lambda)$ a stable unfolding of f, $\lambda \in \mathbb{K}^m$. Let $g\colon(\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a prism on a Morse function or an immersion such that $\{f, g\}$ is simple. We say that g is the best possible with respect to f and F if

a) g is transverse to the limit of the tangent spaces of the strata of the discriminant of f of dimension greater than 0 and

b) there exist representatives $F\colon U \times \Lambda \to \mathbb{K}^p \times \mathbb{K}^m$ and $g\colon V \to \mathbb{K}^p$ of F and g respectively such that for almost all $0 \neq \lambda \in \Lambda$, $\{f_\lambda, g\}\colon U \times V \to \mathbb{K}^p$ only has stable singularities.

Notice that if $\{F, g \times \mathrm{id}_{\mathbb{K}^m}\}$ is stable then condition b) holds.

EXAMPLE 4.3. i) The fold map $g_1(x, y) = (x, y^2)$ is the best possible with respect to $f(x, y) = (x^3 + y^2x, y)$ and $F(x, y, \lambda) = (x^3 + y^2x + \lambda x, y, \lambda)$. However, $g_2(x, y) = (x^2, y)$ is not, since taking the deformation $f_\lambda(x, y) = (x^3+y^2x+\lambda x, y)$, for $\lambda < 0$ there are two cusps of f_λ lying on the discriminant of g_2, and so $\{f_\lambda, g_2\}$ has non-stable singularities.

ii) Consider $f_\lambda(x, y) = (x^3 + y^3x + \lambda_1 x + \lambda_2 xy, y)$. Clearly, $g_1(x, y) = (x^2, y)$ is not the best possible with respect to f since for any value of $\lambda = (\lambda_1, \lambda_2)$ there are either 1 or 3 cusps of f_λ lying on the discriminant of g_1. If we take $g_2(x, y) = (x, y^2)$, there is a cuspidal curve in the bifurcation plane such that f_λ has codimension 1 singularities (namely lips and beaks), and so, for those values of λ, $\{f_\lambda, g_2\}$ has non-stable singularities. Even further, if $\lambda_1 = 0$, there is a cusp at $(x, y) = (0, 0)$ which lies on the discriminant of g_2 and again $\{f_\lambda, g_2\}$ has non-stable singularities. However, for almost all λ, $\{f_\lambda, g_2\}$ only has stable singularities and so g_2 is the best possible with respect to f and F.

iii) The fold map $g(x, y) = (x, y^2 + x)$ is the best possible with respect to the primitive germ f and F where $f_\lambda(x, y) = (x^4 + yx + \lambda x^2, y)$. Notice that $\{F, g \times \mathrm{id}_{\mathbb{K}}\}$ is not stable.

So if we have a simple germ $h = \{f, g\}$ with f non-stable, F a stable unfolding of f and g a prism on a Morse function or an immersion which is the best possible with respect to f and F, then, by the above Lemma and considerations,

$$\mathscr{A}\text{-cod}(h) = \mathscr{A}\text{-cod}(f) + \mathscr{A}\text{-cod}(g) = \mathscr{A}\text{-cod}(f) + n - p + 1.$$

The fact that this is true for example iii) above is an exceptional case since, as we will see in Corollary 4.19, a multigerm composed of a non-stable primitive germ and a fold map is almost always non-simple.

4.1. Augmentations and concatenations

We define the operation of simultaneous augmentation and monic concatenation and derive a formula for the $\mathscr{A}_e$-codimension of the resulting multigerm:

THEOREM 4.4 ([17]). *Suppose* $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ *has a* 1*-parameter stable unfolding* $F(x,\lambda) = (f_\lambda(x), \lambda)$. *Let* $g\colon (\mathbb{K}^p \times \mathbb{K}^{n-p+1}, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ *be the fold map* $(X, v) \mapsto \big(X, \sum_{j=p+1}^{n+1} v_j^2\big)$. *Then,*

i) *the multigerm* $\{A_{F,\phi}(f), g\}$, *where* $\phi\colon \mathbb{K} \to \mathbb{K}$, *has*

$$\mathscr{A}_e\text{-cod}\{A_{F,\phi}(f), g\}) \geq \mathscr{A}_e\text{-cod}(f)(\tau(\phi)+1),$$

with $\tau(\phi)$ *the Tjurina number of* ϕ. *Equality is reached when* ϕ *is quasi-homogeneous and* $\langle dZ(i^*(\mathrm{Lift}(A_{F,\phi}(f))))\rangle = \langle dZ(i^*(\mathrm{Lift}(F)))\rangle$, *where* $i\colon \mathbb{K}^p \to \mathbb{K}^{p+1}$ *is the canonical immersion* $i(X_1,\dots,X_p) = (X_1,\dots,X_p,0)$ *and* dZ *represents the last component of the target vector fields.*

ii) $\{A_{F,\phi}(f), g\}$ *has a* 1*-parameter stable unfolding.*

REMARK 4.5. We do not know an example where the condition in the previous theorem $\langle dZ(i^*(\mathrm{Lift}(A_{F,\phi}(f))))\rangle = \langle dZ(i^*(\mathrm{Lift}(F)))\rangle$ is not satisfied. However, we do not have a proof that it is true in general. A similar technical condition appears in [5, Theorem 3.8] for the $\mathscr{A}_e$-codimension of the binary concatenation and in the definition of substantial unfolding in [8].

We need the following:

LEMMA 4.6. *Suppose* $f = \{f_1,\dots,f_r\}\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ *has a* 1*-parameter stable unfolding* F, *then we have the following adjacency diagram between augmentations of* f:

$$F \longleftarrow A_{F,z^2}(f) \longleftarrow \cdots \longleftarrow A_{F,z^{k-1}}(f) \longleftarrow A_{F,z^k}(f) \longleftarrow \cdots$$

PROOF. First suppose that f can be divided into two non-stable germs h_1 and h_2. Then $F = \{H_1, H_2\}$ where H_i is a stable unfolding of h_i, $i = 1, 2$. Since $\dim \widetilde{\tau}(h_1) = \dim \widetilde{\tau}(h_2) = 0$, we have $\dim \widetilde{\tau}(H_i) \leq 1$ for $i = 1, 2$. Now, $\widetilde{\tau}(H_1)$ and $\widetilde{\tau}(H_2)$ have to be transversal because F is stable, which can only happen if $p+1 = 2$. However, when $p = 1$, there is no such germ. This means that if f has a 1-parameter stable unfolding then there is at most one branch (say f_1) which is not stable and the germ $\{f_2,\dots,f_r\}$ is stable.

Therefore, we can assume that the unfolding parameter in F appears only in F_1, i.e.

$$F(x,\lambda) = \begin{cases} (f_{1_\lambda}(x), \lambda) \\ (f_2(x), \lambda) \\ \dots \\ (f_r(x), \lambda). \end{cases} \tag{1}$$

Now consider the augmentation $A_{F,z^k}(f)(x,z) = \{(f_{1_{z^k}}(x), z), \ldots, (f_r(x), z)\}$. The germ $\{(f_{1_{(z^k+uz^{k-1})}}(x), z), \ldots, (f_r(x), z)\}$ is contained in the versal unfolding of $A_{F,z^k}(f)$ and is $\mathscr{R}$-equivalent to $A_{F,z^{k-1}}(f)$. The result follows.

THEOREM 4.7. *Suppose* $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ *has a* 1*-parameter stable unfolding* $F(x,\lambda) = (f_\lambda(x), \lambda)$. *Let* $g\colon (\mathbb{K}^p \times \mathbb{K}^{n-p+1}, 0) \to (\mathbb{K}^p \times \mathbb{K}, 0)$ *be the fold map* $(X, v) \mapsto \big(X, \sum_{j=p+1}^{n+1} v_j^2\big)$. *Suppose that* ϕ *is quasi-homogeneous,* $A_{F,\phi}(f)$ *is simple and* $\langle dZ(i^*(\mathrm{Lift}(A_{F,\phi}(f))))\rangle = \langle dZ(i^*(\mathrm{Lift}(F)))\rangle$, *then* $\mathscr{A}_e$-$\mathrm{cod}(f) = 1$ *implies that* $\{A_{F,\phi}(f), g\}$ *is simple. Furthermore, if* g *is transverse to the limits of the tangent spaces of the strata of* $A_{F,\phi}(f)$*, then the converse is also true.*

PROOF. From Theorem 4.4, $\mathscr{A}_e$-$\mathrm{cod}(\{A_{F,\phi}(f), g\}) = \mathscr{A}_e$-$\mathrm{cod}(f)(\tau(\phi) + 1)$.

Suppose first that $\mathscr{A}_e$-$\mathrm{cod}(f) = 1$. We know that the stratum codimension of $\{A_{F,\phi}(f), g\}$ is greater than or equal to $\mathscr{A}_e$-$\mathrm{cod}(A_{F,\phi}(f)) + 1 = \mathscr{A}_e$-$\mathrm{cod}(f)\tau(\phi) + 1 = \tau(\phi) + 1 = \mathscr{A}_e$-$\mathrm{cod}(\{A_{F,\phi}(f), g\})$. The stratum codimension can never be greater than the $\mathscr{A}_e$-codimension, so they must be equal. Having this, the only way for $\{A_{F,\phi}(f), g\}$ to be non-simple is that it is an exceptional value of the parameter of a family with modality. Considering Lemma 4.6, since $A_{F,\phi}(f)$ is simple, the modal family would be $\{A_{F,\phi'}(f), g\}$ with $\tau(\phi') = \tau(\phi) - 1$ and clearly this is not the case. Therefore, $\{A_{F,\phi}(f), g\}$ is simple.

Now suppose that $\{A_{F,\phi}(f), g\}$ is simple. Its normal form is

$$\begin{cases} (f_{\phi(z)}(x), z) \\ \big(X, \sum_{j=p+1}^{n+1} v_j^2\big) \end{cases} \tag{2}$$

If we take the 1-parameter stable unfolding of the augmentation $\widetilde{F}(x,z,\lambda) = (f_{\phi(z)+\lambda}(x), z, \lambda)$, it turns out by part ii) of Theorem 4.4 that

$$\begin{cases} (f_{\phi(z)+\lambda}(x), z, \lambda) \\ \big(X, \sum_{j=p+1}^{n+1} v_j^2, \lambda\big) \end{cases} \tag{3}$$

is a 1-parameter stable unfolding of $\{A_{F,\phi}(f), g\}$. Therefore, if we consider the deformation $\big\{(f_{\phi(z)+\lambda}(x), z), \big(X, \sum_{j=p+1}^{n+1} v_j^2\big)\big\}$, it only has stable singularities. Since g has no contact with the strata of $A_{F,\phi}(f)$, g is the best possible with respect to $A_{F,\phi}(f)$ and $\widetilde{F}$ and so

$$\begin{aligned} \mathscr{A}\text{-}\mathrm{cod}(\{A_{F,\phi}(f), g\}) &= \mathscr{A}\text{-}\mathrm{cod}(A_{F,\phi}(f)) + \mathscr{A}\text{-}\mathrm{cod}(g) \\ &= \mathscr{A}\text{-}\mathrm{cod}(A_{F,\phi}(f)) + n - p + 1. \end{aligned}$$

Wilson's formula yields

$$\begin{aligned}
&\mathscr{A}_e\text{-cod}(\{A_{F,\phi}(f), g\}) \\
&\quad = \mathscr{A}\text{-cod}(\{A_{F,\phi}(f), g\}) + (r+1)(p-n) - p \\
&\quad = \mathscr{A}\text{-cod}(A_{F,\phi}(f)) + n - p + 1 + (r+1)(p-n) - p \\
&\quad = \mathscr{A}_e\text{-cod}(A_{F,\phi}(f)) + 1 \\
&\quad = \mathscr{A}_e\text{-cod}(f)\tau(\phi) + 1.
\end{aligned}$$

On the other hand, since $\mathscr{A}_e\text{-cod}(\{A_{F,\phi}(f), g\}) = \mathscr{A}_e\text{-cod}(f)(\tau(\phi)+1)$, we have $\mathscr{A}_e\text{-cod}(f) = 1$.

EXAMPLE 4.8. i) Let $f(y) = (y^2, y^3)$ and consider the augmentations and concatenations

$$\begin{cases} (y^2, y^3 + x^{k+1}y, x) \\ (y, x, 0) \end{cases} \tag{4}$$

These bigerms are called A_0S_k ($k \geq 1$) in [7] and [25] and are simple.

ii) Let $f(y) = (y^2, y^5)$ and consider the augmentation and concatenation

$$\begin{cases} (y^2, y^5 + x^2y, x) \\ (y, x, 0) \end{cases} \tag{5}$$

The bigerm A_0B_2 is not simple since $\mathscr{A}_e\text{-cod}(f) = 2$ and the immersion is transverse to the strata of B_2. Therefore, the bigerms A_0B_k are not simple for $k > 1$.

iii) Consider the codimension 1, n-germ from $\mathbb{K}^{n-1}$ to $\mathbb{K}^{n-1}$

$$\begin{cases} (x_1^2, x_2, \ldots, x_{n-1}) \\ \ldots \\ (x_1, x_2, \ldots, x_{n-1}^2) \\ (x_1^2 + x_2 + \cdots + x_{n-1}, x_2, \ldots, x_{n-1}) \end{cases} \tag{6}$$

and augment and concatenate to obtain the $n+1$-germ from $\mathbb{K}^n \to \mathbb{K}^n$

$$\begin{cases} (x_1^2, x_2, \ldots, x_{n-1}, z) \\ \ldots \\ (x_1, x_2, \ldots, x_{n-1}^2, z) \\ (x_1^2 + x_2 + \cdots + x_{n-1} + \phi(z), x_2, \ldots, x_{n-1}, z) \\ (x_1, x_2, \ldots, x_{n-1}, z^2) \end{cases} \tag{7}$$

If ϕ is quasihomogeneous, $\phi(z) = z^k$ and we obtain a simple multigerm of codimension k. This means that there are infinitely many simple multigerms with $n+1$ fold branches. However, as we will see later, there is no simple multigerm with $n+2$ branches. We remark here that by [17, Corollary 3.9], any multigerm with $n+1$ fold branches is an augmentation and concatenation. These examples also hold for the case $(n, n+1)$ considering immersions instead of folds.

iv) Consider the codimension 1, $n-1$-germ from $\mathbb{K}^{n-2}$ to $\mathbb{K}^{n-2}$ and augment and concatenate it twice. We obtain infinitely many non-simple multigerms from $\mathbb{K}^n$ to $\mathbb{K}^n$ with $n+1$ fold branches of codimension $(\tau(\phi_1)+1)(\tau(\phi_2)+1)$. The last fold is transverse to the strata of the previous n-germ.

$$(8) \qquad \begin{cases} (x_1^2, x_2, \dots, x_{n-2}, y, z) \\ \dots \\ (x_1, x_2, \dots, x_{n-2}^2, y, z) \\ (x_1^2 + x_2 + \dots + x_{n-2} + \phi_1(y) + \phi_2(z), x_2, \dots, x_{n-2}, y, z) \\ (x_1, x_2, \dots, x_{n-2}, y^2, z) \\ (x_1, x_2, \dots, x_{n-2}, y, z^2) \end{cases}$$

v) The extra hypothesis for the converse of Theorem 4.7 to be true is necessary. If we simultaneously augment and concatenate the codimension 2 bigerm $\{(x^2, y), (x^2+y^3, y)\}$ we obtain the codimension 4 simple trigerm ([25])

$$(9) \qquad \begin{cases} (x^2, y, z) \\ (x^2 + y^3 + z^2, y, z) \\ (x, y, z^2) \end{cases}$$

Notice that the double point curve for $\{(x^2, y, z), (x^2+y^3+z^2, y, z)\}$ describes a cusp which is tangent in the limit to g.

4.2. Generalised concatenations

Now we study the simplicity of multigerms admitting a decomposition $h = \{f, g\}$ where f and g are stable germs. We prove in Proposition 4.11 that if $\widetilde{\tau}(f) = \{0\}$ and $\dim_{\mathbb{K}} \widetilde{\tau}(g) = p-2$, then h is not simple. From this we deduce in Corollary 4.14 that generalised concatenations where $\widetilde{\tau}(f) = \{0\}$ are non-simple. Furthermore, we discuss simplicity of h when $1 \leq \dim \widetilde{\tau}(f)$, $\dim \widetilde{\tau}(g) < p-1$, which may or may not be generalised concatenations.

DEFINITION 4.9 ([17]). Let $f\colon (\mathbb{K}^{n-s}, S) \to (\mathbb{K}^{p-s}, 0)$, $s < p$, be a germ of finite $\mathscr{A}_e$-codimension and let $F\colon (\mathbb{K}^n, S \times \{0\}) \to (\mathbb{K}^p, 0)$ be a s-parameter stable unfolding of f with

$$F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_{p-s}(x_1, \dots, x_n), x_{n-s+1}, \dots, x_n),$$

where $F_i(x_1, \dots, x_{n-s}, 0, \dots, 0) = f_i(x_1, \dots, x_{n-s})$. Let $\overline{g}\colon (\mathbb{K}^{n-p+s}, T) \to (\mathbb{K}^s, 0)$ be stable. Then the multigerm $h = \{F, g\}$ is a generalised concatenation of f with g, where $g = Id_{\mathbb{K}^{p-s}} \times \overline{g}$.

Observe that with this definition, $\dim \widetilde{\tau}(g) \geq p-s \geq 1$. If g is a monogerm and $\dim \widetilde{\tau}(g) = p - s$, it is of the form

$$g(x_1, \dots, x_n) = (x_1, \dots, x_{p-s}, g_{p-s+1}(x_{p-s+1}, \dots, x_n), \dots, g_p(x_{p-s+1}, \dots, x_n)).$$

When $s = 1$ and $g_p(x_p, \dots, x_n) = \sum_{i=p}^n x_i^2$ (or $g_p = 0$ when $n = p - 1$), h is called a monic concatenation. When h is of the form

$$\begin{cases} (X, y, u) \mapsto (f_u(y), u, X) \\ (x, Y, u) \mapsto (Y, u, g_u(x)) \end{cases} \tag{10}$$

where $(f_u(y), u)$ and $(u, g_u(x))$ are 1-parameter stable unfoldings of a certain f and g respectively, h is called a binary concatenation.

In [15], Nishimura proved the following Theorem:

THEOREM 4.10. *Let $f = \{f_1, \dots, f_r\}\colon (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ with $n \leq p$ be a multigerm with minimal corank. If $np \neq 1$ and f is $\mathscr{A}$-simple, then the following inequality holds*

$$m_0(f) \leq \frac{p^2 + (n-1)r}{n(p-n) + n - 1}.$$

From this result we obtain

PROPOSITION 4.11. *Let $h = \{f, g\}$ be a multigerm with f, g stable and $n = p \neq 1, 2$ or $n = p - 1$. Suppose that $\widetilde{\tau}(f) = \{0\}$ and $\dim_{\mathbb{K}} \widetilde{\tau}(g) = p - 2$, then h is not $\mathscr{A}$-simple.*

PROOF. Suppose that h is simple.

1) First take the case $n = p$. From Nishimura's result we have that $m_0(h) \leq (n^2 + (n-1)r)/(n-1)$. Since f is stable and $\widetilde{\tau}(f) = \{0\}$, it must be an $A_{k_1,\dots,k_s}$-singularity with $\sum_{i=1}^s k_i = n$. On the other hand $\dim_{\mathbb{K}} \widetilde{\tau}(g) = n - 2$ implies that g is either an A_2-singularity or an A_1^2-singularity. We have that

$$m_0(h) = m_0(f) + m_0(g) = \sum_{i=1}^s (k_i + 1) + m_0(g) = n + s + m_0(g),$$

where $m_0(g) = 3$ or 4 depending on whether g is an A_2 or an A_1^2, respectively.

For the A_2 case we have that $n+s+3 \leq (n^2+(n-1)(s+1))/(n-1)$ where $s+1=r$. This implies that $n-2 \leq 0$ and therefore $n=1,2$. For example the bigerms $\{x^2, x^3\}$ when $n=1$ and $\{(x^3+xy, y), (x, y^3+xy)\}$ when $n=2$ are simple ([17]).

In the A_1^2 case $n+s+4 \leq (n^2+(n-1)(s+2))/(n-1)$ where $s+2=r$. Again this implies that $n=1,2$. For example the trigerms $\{x^2, x^2, x^2\}$ when $n=1$ and $\{(x^3+xy, x), (x, y^2), (x, y^2+x)\}$ when $n=2$ are simple ([17]).

2) For the case $n=p-1$, Nishimura yields $m_0(h) \leq \frac{p^2+(p-2)s}{2p-3}$. Here $\dim_{\mathbb{K}} \widetilde{\tau}(g) = p-2$ implies that g is a transversal intersection of two immersions. We distinguish between the cases where n is even or odd.

If n is even, $\widetilde{\tau}(f) = \{0\}$ implies that f is a monogerm with $m_0(f) = (n+2)/2$ or it is a p-tuple point with $m_0(f)=p$. If f is a monogerm we have that $\frac{n+2}{2}+2 = \frac{p+5}{2} \leq \frac{p^2+3(p-2)}{2p-3}$, which implies $p \leq 3$, however when $p=3$ a cross-cap together with two immersions is not simple ([25]) so h is not simple. If f is a p-tuple point we have that $p+2 \leq (p^2+(p-2)(p+2))/(2p-3)$ and so $p \leq 2$, which is a contradiction.

If n is odd, $\widetilde{\tau}(f) = \{0\}$ implies that f is either a bigerm $\{f_1, f_2\}$ with $m_0(f_1) = (n-1+2)/2$ and $m_0(f_2)=1$ or it is a p-tuple point. The case where h is a $p+2$-tuple point is the same as in the case that n is even and yields $p \leq 2$, however, the cross-ratio shows that a quadruple point when $p=2$ is not simple. When f is a bigerm we get the inequality $\frac{n+1}{2}+1+2 = \frac{p+6}{2} \leq \frac{p^2+4(p-2)}{2p-3}$ which again implies $p \leq 2$.

Example 4.12. i) In the equidimensional case, the bigerm A_2A_n from $\mathbb{K}^n$ to $\mathbb{K}^n$ given by

$$\begin{cases} (x_1^{n+1}+x_2x_1+\cdots+x_{n-2}x_1^{n-3} \\ \qquad +yx_1^{n-2}+zx_1^{n-1}, x_2, \ldots, x_{n-2}, y, z) \\ (x_1, \ldots, x_{n-2}, y, z^3+yz) \end{cases} \tag{11}$$

is not simple when $n > 2$. It has $\mathscr{A}_e$-codimension n ([17]) but the stratum codimension is always 2.

ii) A $p+2$-tuple point for any (n,p) with $n \geq p-1$ is not simple.

Corollary 4.13. *Let $h = \{f, g\}$ be a multigerm with f, g stable and $\widetilde{\tau}(f) = \{0\}$. If h is simple, then g is a prism on a Morse function or an immersion.*

Corollary 4.14. *Let $h = \{f, g\}$ be a non-monic generalised concatenation (i.e. g is not a prism on a Morse function or an immersion) and suppose that $\widetilde{\tau}(f) = \{0\}$, then h is not simple.*

The case $h = \{f, g\}$ with f and g stable and $1 \le \dim \widetilde{\tau}(f), \dim \widetilde{\tau}(g) < p - 1$ is not included in the above results. Suppose h is of type $A_{k_1 \dots k_r}$ from $\mathbb{K}^n$ to $\mathbb{K}^n$ where $\sum_{i=1}^{r} k_i = n + 1$. This implies that $m_0(h) = \sum_{i=1}^{r} k_i + r = n + r + 1 < n + r + 1 + 1/(n-1) = (n^2 + r(n-1))/(n-1)$, which means that the multiplicity of such a multigerm is the maximum possible below Nishimura's bound. We have the following

PROPOSITION 4.15. *There exists a simple* $h \colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, 0)$ *of type* $A_{k_1 \dots k_r}$ *with* $\sum_{i=1}^{r} k_i = n + 1$.

PROOF. We can decompose h in two stable germs $A_{k_{i_1} \dots k_{i_s}}$ and $A_{k_{j_1} \dots k_{j_{r-s}}}$ such that $k_{i_1} + \cdots + k_{i_s} = l$ and $k_{j_1} + \cdots + k_{j_{r-s}} = n + 1 - l$. There exist germs of type $A_{k_{i_1} \dots k_{i_s}}$ and $A_{k_{j_1} \dots k_{j_{r-s}}}$ which have codimension 1 as germs in $\mathbb{K}^{l-1}$ and $\mathbb{K}^{n-l}$ ([5]). With them we can construct a codimension 1 binary concatenation which is of type $A_{k_1 \dots k_r}$ in $\mathbb{K}^n$ and is therefore simple.

A similar study can be done for the case of multigerms $h \colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ where the multiplicity is the maximum possible below Nishimura's bound $N = ((n+1)^2 + r(n-1))/(2n-1)$.

Suppose $n = 2l + 1$. Since $r \le n + 2 = 2l + 3$ then for $(l, r) \ne (1, 2)$, $N = ((2l+2)^2 + 2lr)/(4l+1) \le l + 1 + r/2$ when r is even or $N \le l + 1 + (r+1)/2$ when r is odd. In fact $N = l + 2 + (r/2) - (2l - 4 + r)/2(4l+1)$ and $0 < (2l - 4 + r)2(4l+1) \le 1/2$. If $l = 1$ and $r = 2$ then $N = 4$ and from [4] there is no simple bigerm $h = \{f, g\}$ with f, g stable of multiplicity 4.

Now suppose $n = 2l$. Then for $l \ne 1$ and $(l, r) \ne (2, 1)$, we have $N = ((2l+1)^2 + (2l-1)r)/(4l-1) \le [l + 1 + r/2]$. In fact $N = l + 1 + (r/2) + (l+2)/(4l-1) - r/2(4l+1)$ and $(l+2)/(4l-1) - r/2(4l+1) < 1/2$. When $l = 2$ and $r = 1$, there is not a stable monogerm of multiplicity 4. Suppose $l = 1$. If $r = 3$ then $N = 4$ and from [25] the only simple trigerms are those composed by 3 immersions and therefore have multiplicity 3. If $r = 2$ then $N = 3$ and there are simple bigerms whose branches are a cross-cap and an immersion ([25]). We have the following

PROPOSITION 4.16. *There exists a simple multigerm* $h \colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ *with* $m_0(h) = l + 1 + r/2$ *when* $n = 2l$ *or* $n = 2l + 1$ *and* r *is even, or* $m_0(h) = l + 1 + (r+1)/2$ *when* $n = 2l + 1$ *and* r *is odd.*

PROOF. First suppose that $n = 2l + 1$ and r is even. We can write $h = \{f, g\}$ such that $m_0(f) = l + 1$ and is stable. Notice that $1 \le \dim \widetilde{\tau}(f) \le l + 1$. Consider g the multigerm of $r/2$ immersions with $m_0(g) = r/2$ and take f with $r/2$ branches. Then h has the desired multiplicity and number of branches and is stable since the analytic strata have regular intersection. In fact, $\operatorname{cod} \widetilde{\tau}(f) =$

$2l+2-\dim\widetilde{\tau}(f)=2l+2-r/2$ and $\operatorname{cod}\widetilde{\tau}(g)=\operatorname{cod}\widetilde{\tau}(A_0^{r/2})=r/2$, so we can always choose them in a way that they have regular intersection. Obviously, any stable germ in the nice dimensions is simple.

If $n=2l$ and r is even then there exists a codimension 1 germ whose versal unfolding is the germ h constructed above and therefore is simple.

Now suppose $n=2l+2$ and r is odd. Then $[(l+1)+1+r/2]=l+1+(r+1)/2$. We can write $h=\{f,g\}$ such that $m_0(f)=l+1$ and is stable. This means that $2\leq\dim\widetilde{\tau}(f)\leq l+2$. Similarly to the previous case, consider g the multigerm of $(r+1)/2$ immersions with $m_0(g)=(r+1)/2$ and take f with $(r-1)/2$ branches, then h has the desired multiplicity and number of branches and is stable since the analytic strata have regular intersection and therefore simple.

If $n=2l+1$ and r is odd then there exists a codimension 1 germ whose versal unfolding is the germ h constructed above and is therefore simple.

However, there are examples of multigerms with the highest possible multiplicity below Nishimura's bound that are not simple:

EXAMPLE 4.17. i) Consider the codimension 2 trigerm of folds given by $\{(x^2,y),(x,y^2),(x,y^2+x^2)\}$. If we augment and concatenate it we obtain the codimension 4 quadrigerm

$$\begin{cases}(x^2,y,z)\\(x,y^2,z)\\(x,y^2+x^2+z^2,z)\\(x,y,z^2)\end{cases}\tag{12}$$

If we take the first two branches as f and the last two as g we have that $1=\dim\widetilde{\tau}(f)=\dim\widetilde{\tau}(g)$ but this multigerm is not simple by Theorem 4.7. The same example is valid for $(n,p)=(2,3)$ considering immersions instead of folds.

ii) Suppose we have a germ of type $A_{k_1\dots k_r}$ from $\mathbb{K}^n$ to $\mathbb{K}^n$ such that $\sum_{i=1}^r k_i=n+1$ and $k_{r-1}=k_r=1$. Since $\sum_{i=1}^{r-2}k_i=n-1$, there exists a germ of type $A_{k_1\dots k_{r-2}}$ which has codimension 1 in $\mathbb{K}^{n-2}$. We can augment and concatenate it with an augmenting function ϕ such that $\tau(\phi)=t>1$ to obtain a germ in $\mathbb{K}^{n-1}$ of codimension $t+1>2$. If we augment and concatenate this germ again we obtain a non-simple germ of type $A_{k_1\dots k_r}$, provided the last fold $A_{k_r}=A_1$ is transverse to the strata of the germ of type $A_{k_1\dots k_{r-1}}$.

iii) By Theorem 4.10, two A_{n-1} singularities in $\mathbb{K}^n$ are simple only when $n\leq 3$ (two cuspidal edges, for example).

It follows by [15] that if $f\colon(\mathbb{K}^n,S)\to(\mathbb{K}^p,0)$ $(n\leq p)$ is simple, then the number of branches r is bounded by $p^2/n(p-n)$. In the equidimensional case this is not an upper bound. However, if we consider only non-submersive branches we can prove the following.

PROPOSITION 4.18. *Let* $f = \{f_1, \dots, f_r\}\colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, 0)$ *be a germ of type* $A_{k_1,\dots,k_r}$ *with* $|S| = r > 1$ *and* $n \geq k_i \geq k_{i+1}$ $\forall i = 1, \dots, r-1$. *If* f *is simple, then* $r \leq n - k_1 + 2 = n - m_0(f_1) + 1$.

PROOF. If $k_1 = 1$, then all the other branches are also fold singularities. From Example 4.12, a simple multigerm with only fold singularities can have at most $n+1$ branches.

If $k_1 = 2$, from Proposition 4.11 since f is simple then $\dim \widetilde{\tau}(f') > 0$ where $f' = \{f_2, \dots, f_r\}$. Therefore f' has at most $n-1$ branches, and so $r \leq n$. In fact the best multigerm that has analytic stratum zero is the n-tuple transversal point.

If $k_1 = k \leq n$, then $\dim \widetilde{\tau}(f_1) = n-k$. In the best of the cases, the remaining branches are folds. Suppose we take $n-k$ transversal folds whose intersection has dimension k. Then $\widetilde{\tau}(\{f_1, A_1^{n-k}\}) = \{0\}$, and so, by Corollary 4.13, there is just one more branch which is a prism on a Morse function. Therefore $r \leq n - k + 1 + 1 = n - k + 2$.

4.3. *Multigerms with a non-stable branch*

We study here germs $h = \{f, g\}$ where f is a non-stable primitive germ. We classify all simple germs where g is a prism on a Morse function or an immersion and give some results for the general case.

COROLLARY 4.19. *Let* $f = \{f_1, \dots, f_r\}\colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, 0)$ *be a primitive* $\mathscr{A}_e$*-codimension* 1 *germ,* $n > 2$. *Then the multigerm* $h = \{f, A_1\}$ *is not simple.*

PROOF. If f is a multigerm, from [5] f_i is stable for all $i = 1, \dots, r$, so $h = A_{k_1,\dots,k_r,1}$ and $m_0(h) = n+1+r+2$. If f is a monogerm, $m_0(f) = n+2$ ([21]) and $m_0(h) = n+2+2 = n+1+r+2$. Suppose that h is simple. By Nishimura's result $n+r+3 \leq (n^2 + (n-1)(r+1))/(n-1)$ and so $n \leq 2$.

COROLLARY 4.20. *Let* $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ *be a primitive* $\mathscr{A}_e$*-codimension* 1 *germ,* $n > 3$. *Then the multigerm* $h = \{f, A_0\}$ *is not simple.*

PROOF. From [5] we know that $m_0(f) = (n+3)/2$ and that n is odd, since there are no primitive $\mathscr{A}_e$-codimension 1 when n is even. Suppose that h is simple. By Nishimura's result $1 + \frac{n+3}{2} \leq \frac{(n+1)^2 + 2(n-1)}{2n-1}$ and so $n \leq 3$.

These results can be deduced from the proof of [17, Propostion 5.9] which states that if $h = \{f, g\}$ is a multigerm with f a primitive monogerm of $\mathscr{A}_e$-codimension 1 and g a prism on a Morse function or an immersion, then h has codimension greater than or equal to p when $n \geq p$ and greater than or equal to $\frac{p}{2}$ when $n = p - 1$.

EXAMPLE 4.21. i) When $p = 1$, the bigerm of a Morse function and an A_2-singularity and the trigerm of 3 Morse functions have codimension 2 and are simple.

ii) If $n = 1$, $p = 2$, there is the simple codimension 2 bigerm $\{(x^2,x^3),\ (0,x)\}$, and if $n = p = 2$ there are the simple codimension 2 bigerm

$$\begin{cases} (x^4 + yx, y) \\ (x, y^2 + x) \end{cases} \tag{13}$$

and the trigerm

$$\begin{cases} (x^3 + xy, x) \\ (x, y^2) \\ (x, y^2 + x) \end{cases} \tag{14}$$

iii) In the equidimensional case, given the bigerm

$$\begin{cases} (x_1^{n+2} + x_2x_1 + \cdots + x_nx_1^{n-1}, x_2, \ldots, x_n) \\ (x_1, \ldots, x_{n-1}, x_n^2 + x_{n-1}) \end{cases} \tag{15}$$

the codimension is exactly n (except when $n = 1$, see case 1) above) and is non-simple when $n > 2$.

iv) When $(n, p) = (3, 4)$, the bigerm

$$\begin{cases} (u, v, x^3 + ux, x^4 + vx) \\ (u, u, v, x) \end{cases} \tag{16}$$

has codimension 2 and is simple ([4]). There are no primitive codimension 1 multigerms in these dimensions.

v) When $(n, p) = (2, 3)$, a cross-cap and two immersions or a quintuple point are not simple ([7], [25]).

THEOREM 4.22. *Let $h = \{f, g\}$ is a multigerm with f a non-stable germ and g a prism on a Morse function or an immersion and suppose that g is transverse to the limits of the tangent spaces of f. Then h is simple if and only if either f is an augmentation of an $\mathscr{A}_e$-codimension* 1 *germ (i.e. $h = \{A_{P,\phi}(p), g\}$ with* $\mathscr{A}_e$-cod$(p) = 1$) *or h is one of examples* i), ii) *or* iv) *above.*

PROOF. Follows directly from Theorem 4.7 and Corollaries 4.19 and 4.20.

Example 4.21 shows that simple multigerms $h = \{f, g\}$ where f is a primitive monogerm and g is a prism on a Morse function or an immersion are exceptional. We expect that if g is a more degenerate stable singularity, h will not be simple. In what follows we discuss the case where f is an augmentation and g is more degenerate than prism on a Morse function or an immersion.

COROLLARY 4.23. *Let* $A_{F,\phi}(f)\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ *be an augmentation and* g *be a cuspidal edge or two transversal folds* (*when* $n \geq p$) *or two transversal immersions* (*when* $n = p-1$). *If* $m_0(A_{F,\phi}(f)) > (n^2 - n + 1)/(n - 1)$ (*when* $n \geq p$) *or* $m_0(A_{F,\phi}(f)) > (n^2 + n)/(2n - 1)$ (*when* $n = p - 1$) *then the multigerm* $\{A_{F,\phi}(f), g\}$ *is not simple.*

PROOF. Suppose $h = \{A_{F,\phi}(f), g\}$ is simple. First, if $n = p$, by Nishimura's result $m_0(h) \leq (n^2 + r(n - 1))/(n - 1)$. If g is a cuspidal edge we have

$$m_0(A_{F,\phi}(f)) + m_0(g) = m_0(A_{F,\phi}(f)) + 3 = m_0(h) \leq \frac{n^2 + 2(n-1)}{n-1},$$

which implies $m_0(A_{F,\phi}(f)) \leq (n^2 + r(n - 1))/(n - 1)$. The case where g is two transversal folds follows similarly by using $m_0(g) = 4$ and $r = 3$.

If $n = p - 1$, then $r = 3$ and $m_0(g) = 2$, and the result follows similarly.

EXAMPLE 4.24. i) The bigerms

$$\begin{cases} (x^3 + (y^2 + z^l)x, y, z) \\ (x, y, z^3 + yz) \end{cases} \tag{17}$$

have codimension $l + 1$ and are simple. The versal unfolding can be obtained similarly to the proof of Theorem 4.12 in [17].

ii) The trigerms

$$\begin{cases} (x, y, z^2) \\ (x, y, z^2 + y^2 + x^l) \\ (x^3 + yx, y, z) \end{cases} \tag{18}$$

have codimension $l + 1$ and are simple, by the same argument as above.

iii) The trigerms

$$\begin{cases} (x^3 + (y^2 + z^l)x, y, z) \\ (x, y^2, z) \\ (x, y, z^2) \end{cases} \tag{19}$$

are augmentation and concatenation of the codimension 2 bigerm $\{(x^3 + y^2x, y), (x, y^2)\}$ and so have codimension $2l$ and are non-simple.

iv) The bigerms

$$\begin{cases} (x^4 + yx + z^l x, y, z) \\ (x, y, z^3 + yz) \end{cases} \tag{20}$$

are not simple.

5. Simple multigerms from $\mathbb{C}^3$ to $\mathbb{C}^3$

In this section we obtain a list which includes all simple multigerms from $\mathbb{C}^3$ to $\mathbb{C}^3$ using our results and some simple calculations.

5.1. Monogerms

Table 1, obtained by W. L. Marar and F. Tari in [12] and earlier by V. Goryunov in [6], contains a list of normal forms for simple corank 1 monogerms of maps from $\mathbb{R}^3$ to $\mathbb{R}^3$.

TABLE 1.

Name	Normal form	$\mathscr{A}_e$-codimension
A_1	(x, y, z^2)	0
$3_{\mu(P)}$	$(x, y, z^3 + P(x, y)z)$	$\mu(P)$
4_1^k	$(x, y, z^4 + xz \pm y^k z^2), k \geq 1$	$k - 1$
4_2^k	$(x, y, z^4 + (y^2 \pm x^k)z + xz^2), k \geq 2$	k
5_1	$(x, y, z^5 + xz + yz^2)$	1
5_2	$(x, y, z^5 + xz + y^2z^2 + yz^3)$	2

Here $P(x, y)$ are simple functions in two variables and $\mu(P)$ denotes the Milnor number of P. We use the standard notation A_2 for the cuspidal edge 3_0 and A_3 for the swallowtail 4_1^1.

5.2. Bigerms

We consider bigerms $h = \{f, g\}$.

We study first the case where f is non-stable. Suppose f is an augmentation and g is a fold singularity A_1. When h is an augmentation and concatenation and from Theorem 4.7 we know that if f is an augmentation of a codimension 1 germ then h is simple. So augmenting the codimension 1 germs $(x^3 + y^2x, y)$ and $(x^4 + yx, y)$ we obtain the families of simple germs $3_\mu A_1$ with P an A_μ singularity and $4_1^k A_1$:

$$\begin{cases} (x^3 + (y^2 + z^{\mu+1})x, y, z) \\ (x, y, z^2) \end{cases} \quad \text{and} \quad \begin{cases} (x^4 + yx + z^k x^2, y, z) \\ (x, y, z^2) \end{cases} \tag{21}$$

If we augment and concatenate the codimension μ germ $(x^3 + z^{\mu+1}x, z)$, since (x, y^2, z) is not transversal to the limits of the strata of the augmentations, we must consider $\{(x^3 + (y^2 + z^{\mu+1})x, y, z), (x, y^2, z)\}$.

The 3_μ cases where P is a D_k or E_i singularity with $k \geq 4$ and $i = 6, 7, 8$ can be seen as augmentations of the codimension 2 germ $(x^3 + y^3x, y)$ and the 4_2^k cases can be seen as augmentations of the codimension 2 germ $(x^4 + y^2x + yx^2, y)$. In all these cases, (x, y, z^2) is transverse to the corresponding strata, so the corresponding bigerm is not simple.

There are no simple germs in this case when h is not an augmentation and concatenation.

Suppose g is not a fold singularity. Since $m_0(f) \geq 3$, from Nishimura's bound we have that $m_0(g) \leq 3$ so the only possibilities are $3_\mu A_2$ singularities. Following the calculations in Example 4.24 i) and the fact that $3_\mu A_1$ is not simple if P is not an A_μ singularity, these bigerms are only simple when the function P in 3_μ has an A_μ singularity.

If f is primitive, from Corollary 4.19, there are no simple bigerms.

Now suppose that f and g are stable. First suppose that both are A_1 singularities. From [17, Proposition 3.7 and Corollary 3.8], h must be an augmentation. It is well known that a bigerm with two fold singularities is simple if and only if they are transversal (A_1^2) or they have a simple contact (the contact function is simple). The only possibilities are

$$(22) \qquad \begin{cases} (x, y, z^2) \\ (x, y, z^2 + h(x, y)) \end{cases}$$

where $h(x, y)$ is a simple function singularity.

We need the following Lemma to proceed which is an equidimensional version of a Theorem in [25].

LEMMA 5.1. *Let $h = \{f, g\}$ and $h' = \{f', g\}$ be finitely determined germs. Consider ${}_V\mathscr{K}$ the subgroup of the group $\mathscr{K}$ whose diffeomorphism in the source preserves V, where V is the discriminant of g. If h and h' are $\mathscr{A}$-equivalent then λ is ${}_V\mathscr{K}$-equivalent to λ', where $\lambda, \lambda' \in \mathscr{O}_3$ are reduced defining equations for the discriminants of f and f' respectively.*

PROOF. Since h and h' are $\mathscr{A}$-equivalent there exist germs of diffeomorphisms such that $\psi \circ f \circ \varphi = f'$ and $\psi \circ g \circ \phi = g$. Let $D(f)$ denote the discriminant of f. Then ψ preserves $D(g)$ and takes $D(f)$ into $D(f')$. So $(\lambda' \circ \psi)^{-1}(0) = \lambda^{-1}(0)$ as they are reduced equations. Therefore $\lambda' \circ \psi$ is $\mathscr{C}$-equivalent to λ and as ψ preserves $V = D(g)$, they are ${}_V\mathscr{K}$-equivalent.

From this lemma we deduce that if the function λ defining the discriminant of a fold singularity f is non-simple, then h will be non-simple. We continue our discussion.

If f is an A_1 singularity and g is an A_2 singularity, again all such bigerms are augmentations. Using the classification of simple submersions preserving a cuspidal edge carried out in [18] we obtain a list of all possible simple bigerms with a fold and a cuspidal edge. In fact, these are all obtained by augmenting the codimension 1 and two bigerms $\{(x^3+yx, y), (x, y^2)\}$ and $\{(x^3+yx, y), (x^2, y)\}$. This gives the families

$$\begin{cases} (x^3+yx, y, z) \\ (x, y^2+z^k, z) \end{cases} \quad \text{and} \quad \begin{cases} (x^3+yx, y, z) \\ (x^2+z^k, y, z) \end{cases} \tag{23}$$

The case $k=1$ in both families is the stable germ A_1A_2.

In [3], the authors obtain a classification of submersions under ${}_V\mathscr{R}$-equivalence, where V is the discriminant of the swallowtail. Similarly we can obtain the classification of submersions under ${}_V\mathscr{K}$-equivalence. The possible simple bigerms with an A_1 and an A_3 singularity:

$$\begin{cases} (x^4+yx+zx^2, y, z) \\ (x, y, z^2) \end{cases} \quad \text{and for } k\geq 2 \quad \begin{cases} (x^4+yx+zx^2, y, z) \\ (x, y^2+z^k, z) \end{cases} \tag{24}$$

The first one is a codimension 1 monic concatenation of (x^4+yx, y), and the family is $\mathscr{A}$-equivalent to codimension k monic concatenations of $(x^4+y^kx+yx^2, y)$.

If both f and g are A_2 singularities we have a codimension 1 binary concatenation

$$\begin{cases} (x^3+yx, y, z) \\ (x, y, z^3+yz) \end{cases} \tag{25}$$

We should consider two cuspidal edges with some type of contact. First we study the contact between one of the cuspidal edges and the limiting tangent plane to the other. From [17, Example 4.15 ii)], there is only one $\mathscr{A}$-class for any type of contact and it has codimension 2, a normal form is

$$\begin{cases} (x^3+y^lx+zx, y, z) \\ (x, y, z^3+yz) \end{cases} \tag{26}$$

The next type of contact is between the two limiting tangent planes. Using the complete transversal method we obtain the simple bigerms of codimensions 3 and 4 respectively

$$\begin{cases} (x^3+yx, y, z) \\ (x^3+zx+x^2y, y, z) \end{cases} \quad \text{and} \quad \begin{cases} (x^3+yx, y, z) \\ (x^3+zx, y, z) \end{cases} \tag{27}$$

Based on the previous example, different types of contact between the limiting tangent planes yield the same germ.

The multiplicity of a bigerm of type $A_{2,3}$ overpasses Nishimura's bound for simplicity.

5.3. Trigerms

Due to Nishimura's bound we can only have either 3 folds or 2 folds and a germ of multiplicity 3.

With 3 folds either the trigerm is a stable triple point ($k = 1$ in any of the families below) or it is an augmentation (again by [17, Corollary 3.8]). The germ h must be an augmentation of one of the germs $\{(x^2, y), (x^2 + y^l, y), (x, y^2)\}$ since they are the only trigerms of 3 fold singularities from $\mathbb{C}^2$ to $\mathbb{C}^2$ which admit a 1-parameter stable unfolding. Comparing with the simple trigerms of 3 immersions in [25] a trigerm with 3 fold singularities is simple if it is equivalent to one of the following

$$\begin{cases} (x^2, y, z) \\ (x^2 + y + z^k, y, z) \\ (x, y^2, z) \end{cases} \quad \begin{cases} (x^2, y, z) \\ (x^2 + y^l + z^2, y, z) \\ (x, y^2, z) \end{cases} \tag{28}$$

$$\begin{cases} (x^2, y, z) \\ (x^2 + yz + z^k, y, z) \\ (x, y^2, z) \end{cases} \quad \text{and} \quad \begin{cases} (x^2, y, z) \\ (x^2 + y^2 + z^3, y, z) \\ (x, y^2, z) \end{cases} \tag{29}$$

The last case corresponds to Example 4.8 v). Notice that the second family is also a simultaneous augmentation and concatenation of a codimension 1 germ.

If we have two fold singularities and a cuspidal edge we must consider two cases. Firstly, the two A_1 singularities must be an augmentation so we study what kind of augmentations together with a cuspidal edge give simple germs. We use [17, Theorem 4.12] about the codimension of a cuspidal concatenation. The only simple germs here are those in Example 4.24 ii) of codimension $l+1$

$$\begin{cases} (x, y, z^2) \\ (x, y, z^2 + y^2 + x^l) \\ (x^3 + yx, y, z) \end{cases} \tag{30}$$

with $l \geq 1$. For $l = 1$ we get a codimension 2 germ which can be seen as a monic concatenation.

Secondly, a fold and a cuspidal edge are also an augmentation, so together with another fold, h might be a simultaneous augmentation and concatenation.

TABLE 2.

$\mathscr{K}$-orbit	Normal form	$\mathscr{A}_e$-cod
A_1A_1	$\{(x, y, z^2); (x, y, z^2 + h(x, y))\}$	$\mu(h)$
A_1A_2	$\{(x^3 + yx, y, z); (x, y^2 + z^k, z)\}$	$k - 1$
	$\{(x^3 + yx, y, z); (x^2 + z^k, y, z)\}$	$2(k - 1)$
A_1A_3	$\{(x^4 + yx + zx^2, y, z); (x, y^2 + z^k, z)\}$	k
A_2A_2	$\{(x^3 + zx, y, z); (x, y, z^3 + yz)\}$	1
	$\{(x^3 + y^2x + zx, y, z); (x, y, z^3 + yz)\}$	2
	$\{(x^3 + yx, y, z); (x^3 + zx + x^2y, y, z)\}$	3
	$\{(x^3 + yx, y, z); (x^3 + zx, y, z)\}$	4
$3_\mu A_1$	$\{(x^3 + (y^2 + z^{\mu+1})x, y, z); (x, y, z^2)\}$	$\mu + 1$
	$\{(x^3 + (y^2 + z^{\mu+1})x, y, z); (x, y^2, z)\}$	2μ
$4_1^k A_1$	$\{(x^4 + yx + z^kx^2, y, z); (x, y, z^2)\}$	k
$3_\mu A_2$	$\{(x^3 + (y^2 + z^{\mu+1})x, y, z); (x, y, z^3 + yz)\}$	$\mu + 2$
$A_1A_1A_1$	$\{(x^2, y, z); (x^2 + y + z^k, y, z); (x, y^2, z)\}$	$k - 1$
	$\{(x^2, y, z); (x^2 + y^k + z^2, y, z); (x, y^2, z)\}$	k
	$\{(x^2, y, z); (x^2 + yz + z^k, y, z); (x, y^2, z)\}, k \geq 2$	k
	$\{(x^2, y, z); (x^2 + y^2 + z^3, y, z); (x, y^2, z)\}$	4
$A_1A_1A_2$	$\{(x, y, z^2); (x, y, z^2 + y^2 + x^k); (x^3 + yx, y, z)\}$	$k + 1$
	$\{(x, y, z^2); (x, y^2 + z^k, z); (x^3 + yx, y, z)\}$	k
$3_\mu A_1A_1$	$\{(x^3 + (y^2 + z^{\mu+1})x, y, z); (x, y, z^2); (x, y, z^2 + y)\}$	$\mu + 2$
$A_1A_1A_1A_1$	$\{(x^2, y, z); (x, y^2, z); (x^2 + y + z^k, y, z); (x, y, z^2)\}$	k

In this case, if the augmentation comes from a codimension 1 germ, then h is simple so we get the simple trigerms of the family

$$\begin{cases} (x, y, z^2) \\ (x, y^2 + z^k, z) \\ (x^3 + yx, y, z) \end{cases} \tag{31}$$

for $k \geq 1$, which come from the only codimension 1 germ from $\mathbb{C}^2$ to $\mathbb{C}^2$ with a fold and a cusp. For $k = 1$ we get a codimension 1 monic concatenation. If we consider the codimension 2 germ $f = \{(x^3 + yx, y), (x^2, y)\}$, since the germ (x, y, z^2) is transverse to the strata of any augmentation of f, the simultaneous augmentation and concatenation of f will yield non-simple germs.

If h were not a simultaneous augmentation and concatenation, the cuspidal edge with the other fold would be an augmentation too. So we would have normal forms

$$(32) \qquad \begin{cases} (x, y^2 + z^l, z) \\ (x^2 + z^k, y, z) \\ (x^3 + yx, y, z) \end{cases}$$

However, the germ $\{(x, y, z^2), (x^2 + z^k, y, z), (x^3 + yx, y, z)\}$ which is not simple due to the previous example, is in the adjacency of these germs. So, in this case, h is not simple.

Example 4.24 iii) shows a $3_\mu A_1^2$ case which is not simple, however, the first fold is not the best possible with respect to the first branch, so we must consider $\{(x^3 + (y^2 + z^l)x, y, z), (x, y, z^2), (x, y, z^2 + y)\}$.

5.4. *Quadrigerms*

Here all branches must be fold singularities. From Example 4.8 iii) and iv), the only simple quadrigerms are

$$(33) \qquad \begin{cases} (x^2, y, z) \\ (x, y^2, z) \\ (x^2 + y + z^l, y, z) \\ (x, y, z^2) \end{cases}$$

From Example 4.12 ii) we know that there are no simple pentagerms.

Table 2 includes all simple multigerms from $\mathbb{C}^3$ to $\mathbb{C}^3$. Here $h(x, y)$ is a simple function in two variables, $\mu(h)$ stands for the Milnor number of h and $k \geq 1$ unless stated otherwise.

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