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ON OPEN AND CLOSED STRINGS

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Abstract

Cobordism categories are highly complicated structures that can be analyzed by way of their classifying spaces. In the case of surfaces, meaning 2-dimensional cobordisms, this has led to many important results in recent years. This paper studies the subcategory of open strings of the category of open and closed strings as introduced by Baas, Cohen and Ramírez and identifies the homotopy type of its classifying spaces.

1. Introduction

Dating back over 50 years through the works of Thom and others, cobordism theory is an important subject in geometry and topology. In the late 1980s cobordisms and cobordism categories appeared in mathematical physics, where they are instrumental in topological and conformal field theories as defined by Atiyah [1] and Segal [13], respectively. These field theories are monoidal functors from cobordism categories to suitable categories of vector spaces. Such field theories are very interesting from a topological standpoint as they produce topological invariants of manifolds.

Cobordism categories are highly complicated structures and require advanced machinery to be analyzed and understood. In the last 10 years one fruitful way to understand them has been by studying their classifying spaces. The case of surfaces, meaning 2-dimensional cobordisms, has led to important results, in particular the proof of the Mumford conjecture by Madsen and Weiss [11]. See also [10] for a survey of the Madsen-Weiss theorem and its proof. Moreover, the case of categories of surfaces is also of special interest because of their relevance to string theory.

In string theory, a closed string is a circle and an open string is a closed (unit) interval. The closed and open strings can be taken to be objects of a category S^{oc}, whose morphisms are surfaces with boundary consisting of source and

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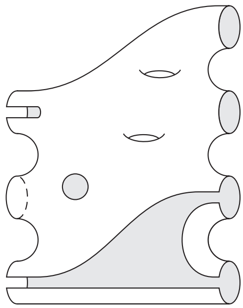


FIGURE 1. An open-closed cobordism

target strings, and a third part, the so-called free boundary. An example of such an open-closed cobordism, i.e. a morphism in S^{oc}, is depicted in Figure 1.

The category S^{oc} is an example of a surface category (a 2-dimensional cobordism category) that is enriched by the diffeomorphism groups (or the mapping class groups) of the surfaces. Hence, S^{oc} can be thought of as a (strict) symmetric monoidal 2-category with disjoint union as the monoidal product.

For the simplest of such surface categories, S, in which only closed strings appear and morphisms have no free boundary, the classifying space is known by work of Tillmann [16]. Baas, Cohen and Ramírez [2] determined the classifying space for the largest such surface category, the open-closed category $\mathsf{S}^{\mathrm{oc}}_{\mathscr{D}}$ where the D-branes, i.e. boundary conditions for the open strings, are labelled by an indexing set $\mathscr{D}$. In particular, they prove that

$$\Omega B\mathsf{S}^{\mathrm{oc}}_{\mathscr{D}} \simeq \Omega^{\infty} MTSO(2) \times \prod_{d \in \mathscr{D}} Q(BS^1_+),$$

where $MTSO(2)$ denotes the Thom spectrum of minus the canonical line bundle $\gamma_1 \to \mathbb{C}\mathrm{P}^{\infty}$, cf. [6], and $Q(BS^1_+) = \mathrm{colim}_{n\to\infty} \Omega^n \Sigma^n BS^1_+$, [2, Theorem 1]. For $\mathscr{D} = \{*\}$, this simplifies to

$$\Omega B\mathsf{S}^{\mathrm{oc}} \simeq \Omega^{\infty} MTSO(2) \times Q(BS^1_+). \tag{1}$$

In comparison, the main theorem in [16] can be stated, as

$$\Omega B\mathsf{S} \simeq \mathbb{Z} \times B\Gamma^+_{\infty}, \tag{2}$$

where $B\Gamma^+_{\infty}$ denotes the Quillen construction applied to the classifying space of the stable mapping class group. By the Madsen-Weiss theorem this implies that $\Omega B\mathsf{S} \simeq \Omega^{\infty} MTSO(2)$.

Main result

The main result presented here extends the results of [2] to the subcategory of open strings, $\mathsf{S}^{\mathsf{open}}$, of S^{oc} ($\mathscr{D} = \{*\}$), in which only open strings are permitted as objects. Furthermore, the description given in [2] is not suitable for the category of open strings $\mathsf{S}^{\mathsf{open}}$ as described below. Tillmann introduced the concept of atomic surfaces in [17] as a way of defining the category of closed strings S. Her atomic surfaces are ideally suited to describe $\mathsf{S}^{\mathsf{open}}$ in order to compute its classifying space.

The main objective is to prove the following theorem.

THEOREM 1. *The classifying space of* $\mathsf{S}^{\mathsf{open}}$ *has the homotopy type of an infinite loop space with*

$$\Omega B\mathsf{S}^{\mathsf{open}} \simeq \Omega^\infty MTSO(2) \times Q(BS^1_+).$$

Outline

The cobordism category $\mathsf{S}^{\mathsf{open}}$ is described in Section 2. Its classifying space is computed in Section 3, thus proving Theorem 1. The techniques used to compute the classifying space rely heavily on the ones introduced in [16] and later expanded to cover open and closed strings in [2].

2. Atomic surfaces

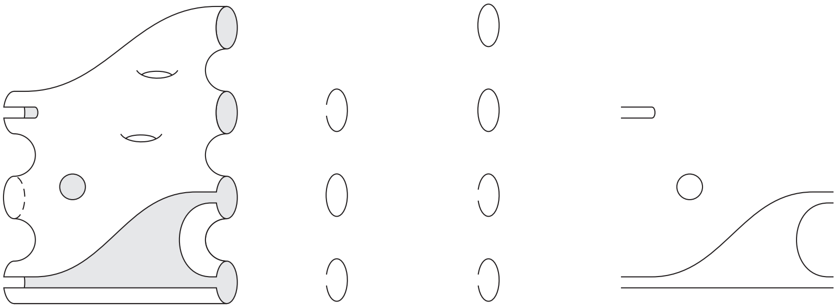
Baas, Cohen and Ramírez introduced in [2] the notion of an open-closed cobordism (as illustrated in Figure 1). In the following manifolds are always assumed to be smooth and compact.

DEFINITION 2. Let $\vec{n} = (n_1, n_2, \dots, n_k)$ where $n_i \in \{0, 1\}$ and $n_i = 0$ corresponds to an oriented copy of the circle S^1 and $n_i = 1$ corresponds to an oriented copy of the closed unit interval $I = [0, 1]$. Furthermore, let $M_{\vec{n}}$ be the disjoint union of oriented copies of S^1 and I with $\pi_0 M_{\vec{n}}$ equipped with an ordering described by $\vec{n}$. Similarly, let $-M_{\vec{n}}$ be the same manifold but with opposite orientation. An oriented *open-closed cobordism* $\Sigma\colon \vec{n} \to \vec{m}$ is an oriented 2-dimensional manifold Σ together with a choice of subsets $\partial_{\text{in}}\Sigma$, $\partial_{\text{out}}\Sigma$ and $\partial_{\text{free}}\Sigma$ of its boundary $\partial\Sigma$ such that:

(1) $\partial\Sigma = (\partial_{\text{in}}\Sigma \sqcup \partial_{\text{out}}\Sigma) \cup \partial_{\text{free}}\Sigma$;

(2) There are orientation-preserving diffeomorphisms $\partial_{\text{in}}\Sigma \cong M_{\vec{n}}$ and $\partial_{\text{out}}\Sigma \cong -M_{\vec{m}}$;

(3) The free boundary $\partial_{\text{free}}\Sigma$ is a 1-dimensional oriented (ordinary) cobordism from $\partial(\partial_{\text{in}}\Sigma)$ to $\partial(\partial_{\text{out}}\Sigma)$, i.e. $\partial_{\text{free}}\Sigma \cap (\partial_{\text{in}}\Sigma \sqcup \partial_{\text{out}}\Sigma) = \partial(\partial_{\text{free}}\Sigma) = \partial(\partial_{\text{in}}\Sigma \sqcup \partial_{\text{out}}\Sigma)$.



(a) $\Sigma\colon \vec{n} \to \vec{m}$ (b) $\partial_{\text{in}}\Sigma$ (c) $\partial_{\text{out}}\Sigma$ (d) $\partial_{\text{free}}\Sigma\colon \partial(\partial_{\text{in}}) \to \partial(\partial_{\text{out}})$

FIGURE 2. An open-closed cobordism $\Sigma\colon \vec{n} \to \vec{m}$

The sets $\partial_{\text{in}}\Sigma$, $\partial_{\text{out}}\Sigma$ and $\partial_{\text{free}}\Sigma$ are referred to as the *incoming*, *outgoing* and *free* boundaries, respectively.

Figure 1 illustrates an open-closed cobordism $\Sigma\colon \vec{n} \to \vec{m}$ for $\vec{n} = (1, 0, 1)$ and $\vec{m} = (0, 0, 1, 1)$. Its various boundaries are shown in Figure 2.

DEFINITION 3. Two open-closed cobordisms $\Sigma, \Sigma'\colon \vec{n} \to \vec{m}$ are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism $\varphi\colon \Sigma \Rightarrow \Sigma'$ with $\varphi(\partial_{\text{in}}\Sigma) = \partial_{\text{in}}\Sigma'$ and $\varphi(\partial_{\text{out}}\Sigma) = \partial_{\text{out}}\Sigma'$.

The proof of (1) relies on homological stability of the decorated stable mapping class group, [3], and a generalized group completion argument, cf. [12]. As mentioned in the introduction, an open disk removed from the surface Σ, meaning a closed component of the free boundary, will be referred to as a *window*. It is an artifact of the proof that it is necessary to have open-closed cobordisms, $\Sigma\colon \vec{n} \to S^1$, completely determined up to isomorphism by their genus and number of windows. In [2] the authors solve this by introducing something they call *open boundary permutations* but these are not suitable when dealing with the case of only open strings. Another way of dealing with this problem is introduced by Hanbury [7] using a construction involving Quillen over-categories. In the case of open strings this problem of fixing the free boundary will be solved using an atomic surface description of the category.

2.1. The category of open strings

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the natural numbers.

DEFINITION 4. The *category of open strings* $\mathsf{S}^{\mathsf{open}}$ is the monoidal 2-category consisting of the following data.

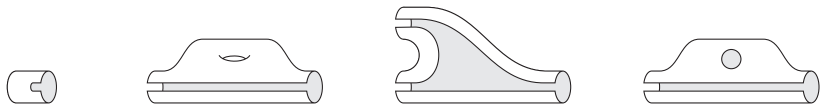
(i) An object is a natural number n corresponding to n oriented copies of $I = [0, 1]$.

(ii) For $m, n \in \mathbb{N}$, a 1-morphism is an oriented open-closed cobordism $\Sigma\colon n \to m$ where Σ has at least one outgoing boundary component, i.e. $m \geq 1$. Any such Σ is constructed by combining
- a fixed disk $\Sigma_D\colon 0 \to 1$
- a fixed pair of pants $\Sigma_P\colon 2 \to 1$
- a fixed genus increasing operator $\Sigma_G\colon 1 \to 1$ which has genus 1
- a fixed windows increasing operator $\Sigma_W\colon 1 \to 1$ which has one window

each of which comes equipped with a fixed collar, meaning for each of its boundary intervals a diffeomorphism $I \times [0, 1) \to U$ where U is a neighborhood of the boundary interval in question. These building blocks, called *atomic surfaces*, are combined by way of composition, i.e. gluing along the outgoing boundary of one with the incoming boundary of the other using the parametrization induced by the collars, or by taking disjoint union.

(iii) Given two 1-morphisms $\Sigma, \Sigma'\colon n \to m$, a 2-morphism is an orientation-preserving diffeomorphism $\varphi\colon \Sigma \Rightarrow \Sigma'$ that leave the collar of incoming and outgoing boundaries pointwise fixed. Let $\mathrm{Diff}^+_{\mathrm{oc}}(\Sigma, \Sigma'; \partial)$ denote the set of such isomorphisms.

Note that if there is a 2-morphism between two 1-morphisms then these are isomorphic in the sense of Definition 3. The identity 1-morphism, $n \to n$, is thought of as a cylinder of length zero. Any 1-morphism is by definition some (finite) combination of composition and disjoint union of the atomic surfaces Σ_D, Σ_P, Σ_G and Σ_W. The atomic surfaces are illustrated in Figure 3.



(a) $\Sigma_D\colon 0 \to 1$ (b) $\Sigma_G\colon 1 \to 1$ (c) $\Sigma_P\colon 2 \to 1$ (d) $\Sigma_W\colon 1 \to 1$

FIGURE 3. Atomic surfaces of $\mathsf{S}^{\mathrm{open}}$

Composition of 1-morphisms $\Sigma\colon n \to m$ and $\Sigma'\colon m \to l$ is denoted by $\Sigma' \circ \Sigma\colon n \to l$. Disjoint union of objects n and m in $\mathsf{S}^{\mathrm{open}}$ gives an object $n \sqcup m = n + m$ and so taking the disjoint union of $\Sigma_1 \in \mathsf{S}^{\mathrm{open}}(n_1, m_1)$ and $\Sigma_2 \in \mathsf{S}^{\mathrm{open}}(n_2, m_2)$ gives an object $\Sigma_1 \sqcup \Sigma_2 \in \mathsf{S}^{\mathrm{open}}(n_1 + n_2, m_1 + m_2)$, i.e. an open-closed cobordism $\Sigma_1 \sqcup \Sigma_2\colon (n_1 + n_2) \to (m_1 + m_2)$.

To explain vertical and horizontal composition of 2-morphisms, consider the following two diagrams

$$\begin{tikzcd} n \arrow[r, bend left=60, "\Sigma_1"{name=U}] \arrow[r, "\Sigma_2"{name=M}] \arrow[r, bend right=60, "\Sigma_3"'{name=D}] & m \arrow[Rightarrow, from=U, to=M, "\varphi_1"] \arrow[Rightarrow, from=M, to=D, "\varphi_2"] \end{tikzcd} \qquad \begin{tikzcd} n \arrow[r, bend left=60, "\Sigma_1"{name=A}] \arrow[r, bend right=60, "\Sigma_2"'{name=B}] & m \arrow[r, bend left=60, "\Sigma_3"{name=C}] \arrow[r, bend right=60, "\Sigma_4"'{name=D}] & l \arrow[Rightarrow, from=A, to=B, "\psi_1"] \arrow[Rightarrow, from=C, to=D, "\psi_2"] \end{tikzcd}$$

where the diagram on the left is the case of vertical composition of 2-morphisms and the diagram on the right is the case of horizontal composition of 2-morphisms.

Vertical composition written $\varphi_2\varphi_1\colon \Sigma_1 \Rightarrow \Sigma_3$ is the composition of diffeomorphisms φ_1 with φ_2. Horizontal composition written $\psi_2 \circ \psi_1\colon \Sigma_3 \circ \Sigma_1 \Rightarrow \Sigma_4 \circ \Sigma_2$ is the composition of glued open-closed cobordisms. This is well-defined because the collars are fixed pointwise, so the diffeomorphisms ψ_1 and ψ_2 define $\psi_2 \circ \psi_1\colon \Sigma_3 \circ \Sigma_1 \Rightarrow \Sigma_4 \circ \Sigma_2$. Finally, by disjoint union there is a monoidal structure $\psi_1 \sqcup \psi_2\colon \Sigma_1 \sqcup \Sigma_3 \Rightarrow \Sigma_2 \sqcup \Sigma_4$. Both vertical and horizontal composition of 2-morphisms are strictly associative.

The incoming boundary intervals and the outgoing boundary interval of objects $\Sigma \in \mathsf{S}^{\mathsf{open}}(n, 1)$ are all contained in a single boundary component of Σ which provides a canonical labelling of the boundary components due to the fact that for such an open-closed cobordism there is an orientation-preserving diffeomorphism to the disk with boundary with $n + 1$ marked intervals. See Figure 4. The canonical labelling is crucial for the proof of Theorem 1.

Note that objects in $\mathsf{S}^{\mathsf{open}}(n, m)$ have m connected components with each having exactly one outgoing boundary interval. In particular, $\mathsf{S}^{\mathsf{open}}(n, 0)$ is empty.

REMARK 5. If $\mathsf{S}^{\mathsf{open}}$ had been given the structure of a symmetric monoidal 2-category as opposed to a monoidal 2-category such a canonical labelling would not exist[1]. As the canonical labelling fixes the non-closed free boundary of $\Sigma \in \mathsf{S}^{\mathsf{open}}(n, 1)$ such a Σ is completely determined up to isomorphism by its genus and number of windows.

The category of open strings is topologized as follows. Both the set of objects and the set of 1-morphisms are given the discrete topology. Each set of iso-

[1] There are no orientation-preserving diffeomorphisms from the circle with $(n + 1)$-labelled parts that interchanges two labelled parts with each other and fixes the remaining $(n - 1)$-labelled parts for $n \geq 2$.

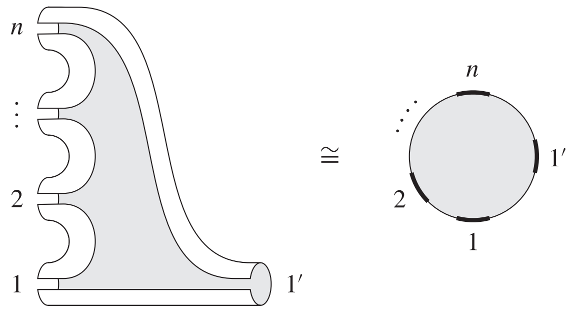


FIGURE 4. The canonical labelling of boundary components of objects $\Sigma \in \mathsf{S}^{\mathsf{open}}(n, 1)$

morphisms between two isomorphic 1-morphisms Σ and Σ', $\mathrm{Diff}^+_{\mathrm{oc}}(\Sigma, \Sigma'; \partial)$, is given the compact-open topology. This gives a topology on the set of 2-morphisms. As composition of 1-morphisms, vertical and horizontal composition of 2-morphisms, source, target, identity and disjoint union are all continuous maps with respect to these topologies, $\mathsf{S}^{\mathsf{open}}$ is a topological monoidal 2-category.

3. Determining the classifying space of $\mathsf{S}^{\mathsf{open}}$

3.1. The decorated mapping class group

Fix a compact, oriented, smooth surface $F^{(w)}_{g,n}$ with genus g, w marked points and n boundary components. See Figure 5.

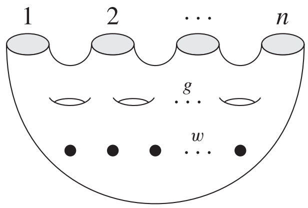


FIGURE 5. The surface $F^{(w)}_{g,n}$

Let $\mathrm{Diff}^+(F^{(w)}_{g,n}; \partial)$ denote the group of orientation-preserving diffeomorphisms of $F^{(w)}_{g,n}$ equipped with the compact-open topology that fix the boundary pointwise and the marked points setwise. The *decorated mapping class group* of $F^{(w)}_{g,n}$ is defined as

$$\Gamma^{(w)}_{g,n} := \pi_0\big(\mathrm{Diff}^+(F^{(w)}_{g,n}; \partial)\big).$$

By attaching a surface with two boundary components, one genus and one marked point to $F^{(w)}_{g,n}$, say onto the boundary component of $F^{(w)}_{g,n}$ to the far right, and extending diffeomorphisms by the identity, there is a map $\Gamma^{(w)}_{g,n} \to \Gamma^{(w+1)}_{g+1,n}$. The *stable decorated mapping class group* is defined to be the colimit taken over those maps,

$$\Gamma^{(\infty)}_{\infty,n} := \operatorname*{colim}_{g,w\to\infty} \Gamma^{(w)}_{g,n}.$$

3.2. The classifying space of the category of open strings

Let $\mathsf{BS}^{\mathsf{open}}$ denote the associated topological category to $\mathsf{S}^{\mathsf{open}}$. In other words, the space of objects of $\mathsf{BS}^{\mathsf{open}}$ is the same as the space of objects of $\mathsf{S}^{\mathsf{open}}$ and each morphism category $\mathsf{S}^{\mathsf{open}}(n,m)$ is replaced by its classifying space, i.e. $\mathsf{BS}^{\mathsf{open}}(n,m) := B\mathsf{S}^{\mathsf{open}}(n,m)$. The classifying space of the category of open strings $\mathsf{S}^{\mathsf{open}}$ is defined as the classifying space of the associated topological category $B(\mathsf{BS}^{\mathsf{open}})$, i.e. $B\mathsf{S}^{\mathsf{open}} := B(\mathsf{BS}^{\mathsf{open}})$.

By Remark 5, there is a homotopy equivalence

$$\mathsf{BS}^{\mathsf{open}}(n,1) \simeq \coprod_{\Sigma} B\operatorname{Diff}^{+}_{\mathrm{oc}}(\Sigma;\partial)$$

where the disjoint union is over isomorphism types of Σ.

In [4] and [5] the authors proved that projecting the group of orientation-preserving diffeomorphisms to its set of connected components induces a homotopy equivalence. In other words, $\operatorname{Diff}^{+}_{\mathrm{oc}}(\Sigma;\partial) \simeq \pi_0\big(\operatorname{Diff}^{+}_{\mathrm{oc}}(\Sigma;\partial)\big)$ for all $\Sigma \in \mathsf{S}^{\mathsf{open}}(n,m)$. Hence, there is a homotopy equivalence $B\operatorname{Diff}^{+}_{\mathrm{oc}}(\Sigma;\partial) \simeq B\big(\pi_0\big(\operatorname{Diff}^{+}_{\mathrm{oc}}(\Sigma;\partial)\big)\big)$.

Note that diffeomorphisms between two open-closed cobordisms from n to m fixes the boundary components pointwise and the windows setwise. Let $\Sigma \in \mathsf{S}^{\mathsf{open}}(n,1)$. An orientation-preserving diffeomorphism, $\varphi\colon \Sigma \Rightarrow \Sigma$, with $\varphi(\partial_{\mathrm{in}}\Sigma) = \partial_{\mathrm{in}}\Sigma$ and $\varphi(\partial_{\mathrm{out}}\Sigma) = \partial_{\mathrm{out}}\Sigma$, restricts to a self-diffeomorphism on the free boundary. Furthermore, as φ fixes every boundary interval, it is isotopic to a diffeomorphism that fixes the free boundary pointwise. Viewing Σ as a surface with genus g, w number of windows and $n+1$ boundary intervals, where the free boundary connects all of them, there is an orientation-preserving diffeomorphism to the same surface with one boundary circle with $(n+1)$-labelled parts. See Figure 6.

This proves the following result.

LEMMA 6. *Let n and 1 be objects in $\mathsf{S}^{\mathsf{open}}$. Then there is a homotopy equivalence*

$$\mathsf{BS}^{\mathsf{open}}(n,1) \simeq \coprod_{g,w\geq 0} B\Gamma^{(w)}_{g,1_{n+1}}$$

where 1_{n+1} denotes one boundary component with $(n+1)$-labelled parts of the decorated mapping class group.

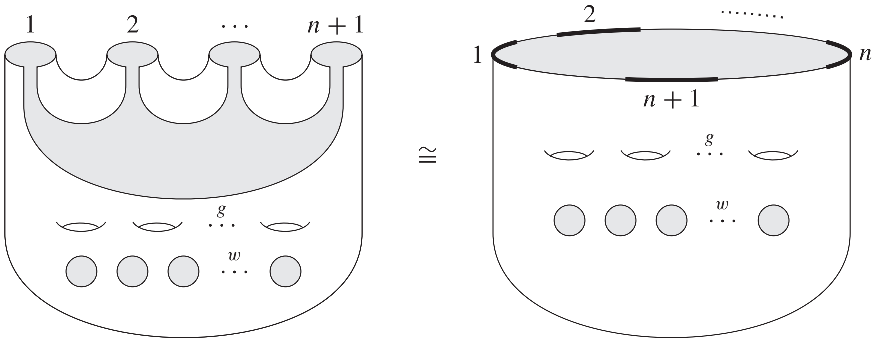


FIGURE 6. An object $\Sigma \in \mathsf{S}^{\mathsf{open}}(n, 1)$ viewed as a surface with one boundary component with $(n+1)$-labelled parts

Note that here windows are seen as marked points. A marked point on a surface is the same as a puncture, i.e. that a point has been removed from the surface. Compare this with a window. As an orientation-preserving diffeomorphism, $\varphi \in \mathrm{Diff}^{+}_{\mathrm{oc}}(\Sigma; \partial)$, fixes the windows of Σ only setwise, it follows that there is an isotopy $\Phi\colon \varphi \simeq \varphi_p$ where φ_p is an orientation-preserving diffeomorphism on the corresponding surface with punctures playing the part of windows. Note that φ_p can permute the marked points.

Let $\Sigma_T\colon 1 \to 1$ be the composition of $\Sigma_G\colon 1 \to 1$ (see Figure 3(b)) and $\Sigma_W\colon 1 \to 1$ (see Figure 3(d)), i.e. $\Sigma_T = \Sigma_W \circ \Sigma_G$. See Figure 7.

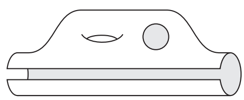


FIGURE 7. The 1-morphism $\Sigma_T\colon 1 \to 1$ in $\mathsf{S}^{\mathsf{open}}$

It follows that $\Sigma_T \in \mathsf{S}^{\mathsf{open}}(1, 1)$ induces a map $t\colon \mathsf{BS}^{\mathsf{open}}(n, 1) \to \mathsf{BS}^{\mathsf{open}}(n, 1)$ by gluing on Σ_T to the outgoing boundary of an object in $\mathsf{S}^{\mathsf{open}}(n, 1)$. The map t increases both the genus and the number of windows by one. Let $\mathsf{BS}^{\mathsf{open}}_{\infty}(n)$ be the homotopy colimit of the system

$$\mathsf{BS}^{\mathsf{open}}(n, 1) \xrightarrow{t} \mathsf{BS}^{\mathsf{open}}(n, 1) \xrightarrow{t} \mathsf{BS}^{\mathsf{open}}(n, 1) \xrightarrow{t} \cdots .$$

By applying Lemma 6, the following result holds.

LEMMA 7. *Let n be an object in $\mathsf{S}^{\mathsf{open}}$. Then there is a homotopy equivalence*

$$\mathsf{BS}^{\mathsf{open}}_{\infty}(n) \simeq \mathbb{Z} \times \mathbb{Z} \times B\Gamma^{(\infty)}_{\infty, 1_{n+1}}.$$

Note that the two copies of $\mathbb{Z}$ on the right hand side reflects that the decorated mapping class group has been stabilized with respect to both genus and windows. Precomposition with $\Sigma \in \mathsf{BS}^{\mathsf{open}}(n, m)$ induces a map,

$$\Sigma^*\colon \mathsf{BS}^{\mathsf{open}}_{\infty}(m) \to \mathsf{BS}^{\mathsf{open}}_{\infty}(n),$$

which in light of Lemma 7 can be thought of as a map,

$$\Sigma^*\colon \mathbb{Z} \times \mathbb{Z} \times B\Gamma^{(\infty)}_{\infty,1_{m+1}} \to \mathbb{Z} \times \mathbb{Z} \times B\Gamma^{(\infty)}_{\infty,1_{n+1}}.$$

Hence, there is a contravariant functor

$$\mathsf{BS}^{\mathsf{open}}_{\infty}\colon \mathsf{BS}^{\mathsf{open}} \to \mathsf{Spaces}$$

where Spaces is the category of topological spaces and continuous maps. This functor sends objects $n \in \mathsf{BS}^{\mathsf{open}}$ to the space $\mathsf{BS}^{\mathsf{open}}_{\infty}(n)$ and morphisms $\Sigma \in \mathsf{BS}^{\mathsf{open}}(n, m)$ to the continuous map $\Sigma^*\colon \mathsf{BS}^{\mathsf{open}}_{\infty}(m) \to \mathsf{BS}^{\mathsf{open}}_{\infty}(n)$.

Let $\mathsf{BS}^{\mathsf{open}}_{1}(n) := \mathsf{BS}^{\mathsf{open}}(n, 1)$. Precomposition with $\Sigma \in \mathsf{BS}^{\mathsf{open}}(n, m)$ induces a map

$$\Sigma^*\colon \mathsf{BS}^{\mathsf{open}}_{1}(m) \to \mathsf{BS}^{\mathsf{open}}_{1}(n).$$

Hence, there is a contravariant functor

$$\mathsf{BS}^{\mathsf{open}}_{1}\colon \mathsf{BS}^{\mathsf{open}} \to \mathsf{Spaces}$$

that sends objects $n \in \mathsf{BS}^{\mathsf{open}}$ to the space $\mathsf{BS}^{\mathsf{open}}_{1}(n)$ and morphisms $\Sigma \in \mathsf{BS}^{\mathsf{open}}(n, m)$ to the continuous map $\Sigma^*\colon \mathsf{BS}^{\mathsf{open}}_{1}(m) \to \mathsf{BS}^{\mathsf{open}}_{1}(n)$. It follows by the construction of the functors $\mathsf{BS}^{\mathsf{open}}_{\infty}$ and $\mathsf{BS}^{\mathsf{open}}_{1}$, that for any object $n \in \mathsf{BS}^{\mathsf{open}}$,

$$\mathsf{BS}^{\mathsf{open}}_{\infty}(n) = \operatorname{hocolim}\big(\mathsf{BS}^{\mathsf{open}}_{1}(n) \xrightarrow{t} \mathsf{BS}^{\mathsf{open}}_{1}(n) \xrightarrow{t} \mathsf{BS}^{\mathsf{open}}_{1}(n) \xrightarrow{t} \cdots\big).$$

Let C be a topological category and let $F\colon \mathsf{C} \to \mathsf{Spaces}$ be a contravariant functor. The category $\mathsf{C} \wr F$ is the category consisting of the following data.

(i) An object is a pair (c, x) where c is an object in C and $x \in F(c)$.

(ii) A morphism is a pair $(m, x')\colon \big(c, F(m)(x')\big) \to (c', x')$ where $m \in \mathsf{C}(c, c')$ and $x' \in F(c')$.

Let $m \in \mathsf{C}(c, c')$ and $m' \in \mathsf{C}(c', c'')$. The composition of $(m, x')\colon \big(c, F(m)(x')\big) \to (c', x')$ and $(m', x'')\colon \big(c', F(m')(x'')\big) \to (c'', x'')$ in $\mathsf{C} \wr F$ is given by

$$(m', x'') \circ (m, x') = (m' \circ m, x'')\colon \big(c, F(m' \circ m)(x'')\big) \to (c'', x'').$$

The category $\mathsf{C} \wr F$ is often referred to as the *Grothendieck construction* of F.

Projecting onto the first factor for both the objects and the morphisms gives a natural functor $\pi\colon \mathsf{C} \wr F \to \mathsf{C}$. The homotopy colimit of the functor F is known as the classifying space of the Grothendieck construction $\mathsf{C} \wr F$. Hence, there is a map hocolim $F = B(\mathsf{C} \wr F) \xrightarrow{B\pi} B\mathsf{C}$.

LEMMA 8. *Let* $\mathsf{BS}^{\mathsf{open}}_{\infty} \colon \mathsf{BS}^{\mathsf{open}} \to \mathsf{Spaces}$ *be the contravariant functor as described above. Then* hocolim $\mathsf{BS}^{\mathsf{open}}_{\infty} = B(\mathsf{BS}^{\mathsf{open}} \wr \mathsf{BS}^{\mathsf{open}}_{\infty})$ *is contractible.*

PROOF. As $\mathsf{BS}^{\mathsf{open}}_{\infty}(n) = \operatorname{hocolim}\big(\mathsf{BS}^{\mathsf{open}}_{1}(n) \xrightarrow{t} \mathsf{BS}^{\mathsf{open}}_{1}(n) \xrightarrow{t} \mathsf{BS}^{\mathsf{open}}_{1}(n) \xrightarrow{t} \cdots\big)$,

$$\operatorname{hocolim} \mathsf{BS}^{\mathsf{open}}_{\infty} = B(\mathsf{BS}^{\mathsf{open}} \wr \mathsf{BS}^{\mathsf{open}}_{\infty}) \simeq \operatorname{hocolim} B(\mathsf{BS}^{\mathsf{open}} \wr \mathsf{BS}^{\mathsf{open}}_{1}).$$

So the result follows by showing that $B(\mathsf{BS}^{\mathsf{open}} \wr \mathsf{BS}^{\mathsf{open}}_{1})$ is contractible. This fact follows by the observation that $(1, \mathrm{id}_1)$ is a terminal object in $\mathsf{BS}^{\mathsf{open}} \wr \mathsf{BS}^{\mathsf{open}}_{1}$.

A map between topological spaces $f \colon X \to Y$ is a *homology equivalence* if the induced map in homology with integer coefficients is an isomorphism, i.e. $f_* \colon H_*(X) \to H_*(Y)$ is an isomorphism.

LEMMA 9. *The map*

$$\Sigma^* \colon \mathbb{Z} \times \mathbb{Z} \times B\Gamma^{(\infty)}_{\infty, 1_{m+1}} \to \mathbb{Z} \times \mathbb{Z} \times B\Gamma^{(\infty)}_{\infty, 1_{n+1}}$$

is a homology equivalence for all $\Sigma \in \mathsf{BS}^{\mathsf{open}}(n, m)$.

This result follows by applying the Harer stability theorem [8], as in [3, Proposition 3.4].

A map of topological spaces $f \colon X \to Y$, is a *homology fibration* if, for every $y \in Y$, the inclusion of the geometric fiber over y to the homotopy fiber over y is a homology equivalence.

By combining Lemma 7, Lemma 9 and the generalized group-completion theorem ([12], [16]) the following result holds.

LEMMA 10. *The projection map*

$$p \colon \operatorname{hocolim} \mathsf{BS}^{\mathsf{open}}_{\infty} \to B\mathsf{S}^{\mathsf{open}}$$

is a homology fibration with geometric fiber $p^{-1}(n) = \mathsf{BS}^{\mathsf{open}}_{\infty}(n) \simeq \mathbb{Z} \times \mathbb{Z} \times B\Gamma^{(\infty)}_{\infty, 1_{n+1}}$ *for any natural number n.*

It is a standard fact from algebraic topology that if $F \to E \xrightarrow{p} B$ is a fibration or fiber bundle with E contractible then there is a weak homotopy equivalence between the fiber $F = p^{-1}(b_0)$ and the loop space $\Omega_{b_0} B$ based at b_0, see e.g. [9, Proposition 4.66]. By combining this fact with Lemma 8 and Lemma 10 the following holds.

PROPOSITION 11. *Let $\Omega_n B\mathsf{S}^{\mathsf{open}}$ denote the space of loops in $B\mathsf{S}^{\mathsf{open}}$ based at $n \in \mathsf{S}^{\mathsf{open}}$. Then the map*

$$\delta\colon \mathbb{Z} \times \mathbb{Z} \times B\Gamma^{(\infty)}_{\infty,1_{n+1}} \to \Omega_n B\mathsf{S}^{\mathsf{open}}$$

is a homology equivalence for all objects n in $\mathsf{S}^{\mathsf{open}}$.

This proposition is a major step towards proving Theorem 1. By [2, Corollary 12], there is a homotopy equivalence $\mathbb{Z} \times (B\Gamma^{(\infty)}_{\infty,n})^+ \simeq B\Gamma^+_\infty \times Q(BS^1_+)$ for all natural numbers n. Together with the Madsen-Weiss theorem, [11], this implies that

$$\mathbb{Z} \times \mathbb{Z} \times (B\Gamma^{(\infty)}_{\infty,n})^+ \simeq \Omega^\infty MTSO(2) \times Q(BS^1_+).$$

PROOF OF THEOREM 1. By Proposition 11, there is a homology equivalence $\delta\colon \mathbb{Z} \times \mathbb{Z} \times B\Gamma^{(\infty)}_{\infty,1_{n+1}} \to \Omega_n B\mathsf{S}^{\mathsf{open}}$ for all $n \in \mathsf{S}^{\mathsf{open}}$. Applying the Quillen plus construction then gives that there is a homology equivalence $\delta^+\colon \mathbb{Z} \times \mathbb{Z} \times (B\Gamma^{(\infty)}_{\infty,1_{n+1}})^+ \to \Omega_n B\mathsf{S}^{\mathsf{open}}$, i.e. the homology equivalence δ can be factorized through δ^+ as described by the following commutative diagram.

$$\begin{tikzcd}
\mathbb{Z} \times \mathbb{Z} \times B\Gamma^{(\infty)}_{\infty,1_{n+1}} \arrow[r, "\delta"] \arrow[d, "q"'] & \Omega_n B\mathsf{S}^{\mathsf{open}} \\
\mathbb{Z} \times \mathbb{Z} \times (B\Gamma^{(\infty)}_{\infty,1_{n+1}})^+ \arrow[ur, "\delta^+"'] &
\end{tikzcd}$$

The fact that δ^+ is a homology equivalence implies that it is a bijection of the set of connected components. Furthermore, restricting δ^+ to a connected component is also a homology equivalence.

The source of δ^+ has the homotopy type of an infinite loop space by [2, Corollary 12] (with one boundary component with $(n+1)$-labelled parts) and (2) while the target of δ^+ is a loop space. Every loop space is an H-group, see e.g. [14, p. 38]. The identity component of an H-group is also an H-group. Furthermore,

$$\pi_0\big(\mathbb{Z} \times \mathbb{Z} \times (B\Gamma^{(\infty)}_{\infty,1_{n+1}})^+\big)$$

is a group, and hence all the path components of $\mathbb{Z}\times\mathbb{Z}\times(B\Gamma^{(\infty)}_{\infty,1_{n+1}})^+$ are homotopy equivalent. Specifically, they are homotopy equivalent with the identity (with respect to the H-group structure) component of $\mathbb{Z} \times \mathbb{Z} \times (B\Gamma^{(\infty)}_{\infty,1_{n+1}})^+$. Every path-connected H-space is simple, see e.g. [14, Theorem 7.3.9].

Hence, by the Whitehead theorem, δ^+, when restricted to a connected component, is a homotopy equivalence. This gives a homotopy equivalence

$$\mathbb{Z} \times \mathbb{Z} \times (B\Gamma^{(\infty)}_{\infty,1_{n+1}})^+ \simeq \Omega_n B\mathsf{S}^{\mathsf{open}}.$$

By [2, Corollary 12] (with one boundary component with $(n+1)$-labelled parts) and the Madsen-Weiss theorem, there is a homotopy equivalence $\mathbb{Z} \times \mathbb{Z} \times (B\Gamma^{(\infty)}_{\infty,1_{n+1}})^+ \simeq \Omega^\infty MTSO(2) \times Q(BS^1_+)$.

As $\mathsf{BS}^{\mathsf{open}}$ is a connected category, it follows that $B\mathsf{S}^{\mathsf{open}}$ is path-connected. Thus $\Omega_n B\mathsf{S}^{\mathsf{open}} \simeq \Omega_m B\mathsf{S}^{\mathsf{open}}$ for all $n, m \in \mathsf{S}^{\mathsf{open}}$. Hence, it does not matter where the loops are based in $B\mathsf{S}^{\mathsf{open}}$. This finishes the proof.

In [15], the author proves, following the arguments above and that of [16], that homotopy type of the classifying spaces of the various subcategories of the category of open and closed strings depends on whether "windows" are allowed or not, and not whether the strings are closed or open or both.

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