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ZEROS OF FUNCTIONS IN BERGMAN-TYPE HILBERT SPACES OF DIRICHLET SERIES

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Abstract

For a real number α the Hilbert space $\mathscr{D}_\alpha$ consists of those Dirichlet series $\sum_{n=1}^\infty a_n/n^s$ for which $\sum_{n=1}^\infty |a_n|^2/[d(n)]^\alpha < \infty$, where $d(n)$ denotes the number of divisors of n. We extend a theorem of Seip on the bounded zero sequences of functions in $\mathscr{D}_\alpha$ to the case $\alpha > 0$. Generalizations to other weighted spaces of Dirichlet series are also discussed, as are partial results on the zeros of functions in the Hardy spaces of Dirichlet series $\mathscr{H}^p$, for $1 \leq p < 2$.

1. Introduction

Let $d(n)$ denote the divisor function let α be a real number. We are interested in the following Hilbert spaces of Dirichlet series:

$$\mathscr{D}_\alpha = \left\{ f(s) = \sum_{n=1}^\infty \frac{a_n}{n^s} : \|f\|_{\mathscr{D}_\alpha}^2 = \sum_{n=1}^\infty \frac{|a_n|^2}{[d(n)]^\alpha} < \infty \right\}.$$

The functions of $\mathscr{D}_\alpha$ are analytic in $\mathbb{C}_{1/2} = \{s = \sigma + it : \sigma > 1/2\}$. Bounded Dirichlet series are almost periodic, and this implies that they have either no zeros or infinitely many zeros, as observed by Olsen and Seip in [10]. This leads us to restrict our investigations to bounded zero sequences for spaces of Dirichlet series. In [13], Seip studied bounded zero sequences for $\mathscr{D}_\alpha$, when $\alpha \leq 0$. This includes the Hardy-type ($\alpha = 0$) and Dirichlet-type ($\alpha < 0$) spaces. The topic of the present work is the Bergman-type spaces ($\alpha > 0$).

Let us therefore introduce the weighted Bergman spaces in the half-plane, A_β. For $\beta > 0$, these spaces consists of functions F which are analytic in $\mathbb{C}_{1/2}$ and satisfy

$$\|F\|_{A_\beta} = \left(\int_{\mathbb{C}_{1/2}} |F(s)|^2 \left(\sigma - \frac{1}{2}\right)^{\beta-1} dm(s) \right)^{\frac{1}{2}} < \infty.$$

It was shown by Olsen in [9] that the local behavior of the spaces $\mathscr{D}_\alpha$ is similar to the spaces A_β, where $\beta = 2^\alpha - 1$. This relationship between α and β will be retained throughout this paper.

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For a class of analytic functions $\mathscr{C}$ on some domain $\Omega \subseteq \mathbb{C}$, we will say that a sequence S of not necessarily distinct numbers in Ω is a zero sequence for $\mathscr{C}$ if there is some non-trivial $F \in \mathscr{C}$ vanishing on S, taking into account multiplicities. We will let $Z(\mathscr{C})$ denote the set of all zero sequences for $\mathscr{C}$.

A result proved by Horowitz in [6] shows that if $\mathscr{C} = A_\beta$ we may assume that F vanishes precisely on $S \in Z(A_\beta)$, i.e. F has no extraneous zeros in $\mathbb{C}_{1/2}$. We will exploit this fact to prove our main result.

THEOREM 1. *Suppose $S = (\sigma_j + it_j)$ is a bounded sequence of points in $\mathbb{C}_{1/2}$ and that $\alpha > 0$. Then there is a non-trivial function in $\mathscr{D}_\alpha$ vanishing on S if and only if $S \in Z(A_\beta)$.*

The "only if" part follows from the local embedding of $\mathscr{D}_\alpha$ into A_β of Theorem 1 and Example 4 from [9]. To prove the "if" part, we will adapt the methods of [13], where an analogous result for $\alpha \leq 0$ was obtained.

The "if" part can essentially be split into two steps. The first step is a discretization lemma, which depends on the properties of $\mathscr{D}_\alpha$ – or rather the weights $[d(n)]^\alpha$. The second step is an iterative scheme, where the properties of A_β become more prominent.

Comparing this with [13], the first step is somewhat harder, since we require very precise estimates on the weights as α grows to infinity. The second step is considerably easier, mainly due to the fact that the norms of A_β are easier to work with than those of the Dirichlet spaces used in [13].

We will use the notation $f(x) \ll g(x)$ to indicate that there is some constant $C > 0$ so that $|f(x)| \leq Cg(x)$. Sometimes the constant C may depend on certain parameters, and this will be specified in the text. Moreover, we write $f(x) \asymp g(x)$ if both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold.

2. Proof of Theorem 1

We begin with the Paley-Wiener representation of functions $F \in A_\beta$, and seek to construct a Dirichlet series $f \in \mathscr{D}_\alpha$ which approximates F.

LEMMA 2 (Paley-Wiener Representation). *A_β is isometrically isomorphic to*

$$L^2_\beta = \left\{\phi \text{ measurable on } [0,\infty) : \|\phi\|^2_{L^2_\beta} = \frac{2\pi\,\Gamma(\beta)}{2^\beta}\int_0^\infty |\phi(\xi)|^2\,\frac{d\xi}{\xi^\beta} < \infty\right\},$$

under the Laplace transformation

$$F(s) = \int_0^\infty \phi(\xi)e^{-(s-1/2)\xi}\,d\xi.$$

PROOF. A proof can be found in [2].

The other ingredient needed for the discretization lemma is estimates on the growth of $[d(n)]^\alpha$. We will partition the integers into blocks and use an average order type estimate. To prove this estimate, we will need the precise form of a formula stated by Ramanujan [11] and proved by Wilson [15]: for any real number α and any integer $\nu > 2^\alpha - 2$, we have

$$(1)\quad D_\alpha(x) = \sum_{n \le x} [d(n)]^\alpha = x(\log x)^{2^\alpha - 1}\left(\sum_{\lambda=0}^{\nu} \frac{A_\lambda}{(\log x)^\lambda} + \mathcal{O}\left(\frac{1}{(\log x)^{\nu+1}}\right)\right).$$

Wilson's proof of (1) can be considered at special case of Selberg-Delange method. For more about the Selberg-Delange method, we refer to Chapter II.5 of [14]. However, we mention that the coefficients A_λ depend on the coefficients of the Dirichlet series ϕ_α, which we implicitly define through the relation

$$(2)\quad \zeta_\alpha(s) = \sum_{n=1}^{\infty} [d(n)]^\alpha n^{-s} = \prod_{j=1}^{\infty}\left(1 + \sum_{k=1}^{\infty}(k+1)^\alpha p_j^{-sk}\right) = [\zeta(s)]^{2^\alpha}\phi_\alpha(s).$$

The partial sums of the coefficients of ζ_α are estimated through Perron's formula and the residue theorem. While (2) is only valid for $\mathrm{Re}(s) > 1$, a simple computation using Euler products shows that ϕ_α converges for $\mathrm{Re}(s) > 1/2$, and thus Theorem 5 of [14] may be applied. In particular, the coefficients A_λ depend on the coefficients of ϕ_α, and since the coefficients of ϕ_α depend continuously on α, so does A_λ in (1).

LEMMA 3. *Let α be a real number and $0 < \gamma < 1$. Then*

$$(3)\qquad \sum_{j^\gamma \le \log n \le (j+1)^\gamma} \frac{[d(n)]^\alpha}{n} \asymp j^{\gamma 2^\alpha - 1},$$

as $j \to \infty$. The implied constants may depend on α and γ.

PROOF. We will first assume that 2^α is not an integer. Fix ν such that $\nu > 2^\alpha - 1$ and $\nu > 1/\gamma - 1$. We use Abel summation to rewrite

$$(4)\qquad \sum_{y < n \le x} \frac{[d(n)]^\alpha}{n} = \frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} + \int_y^x \frac{D_\alpha(z)}{z^2}\,dz.$$

By using (1) and the fact that $2^\alpha - 1 - \nu < 0$ we perform some standard calculations to estimate

$$\frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} = \sum_{\lambda=0}^{\nu} A_\lambda\big((\log x)^{2^\alpha-1-\lambda} - (\log y)^{2^\alpha-1-\lambda}\big) + \mathcal{O}\big((\log y)^{2^\alpha-2-\nu}\big),$$

$$\int_y^x \frac{D_\alpha(z)}{z^2}\,dz = \sum_{\lambda=0}^{\nu} \frac{A_\lambda}{2^\alpha-\lambda}\big((\log x)^{2^\alpha-\lambda} - (\log y)^{2^\alpha-\lambda}\big) + \mathcal{O}\big((\log y)^{2^\alpha-1-\nu}\big).$$

Let us now take $x = \exp((j+1)^\gamma)$ and $y = \exp(j^\gamma)$. For any exponent η it is clear that

$$(\log x)^\eta - (\log y)^\eta = \gamma\eta j^{\gamma\eta-1}\left(1 + \mathcal{O}\left(\frac{1}{j}\right)\right).$$

Hence we have

$$\frac{D_\alpha(x)}{x} - \frac{D_\alpha(y)}{y} \asymp \sum_{\lambda=0}^{\nu} A_\lambda(\gamma(2^\alpha-1-\lambda))j^{\gamma(2^\alpha-1-\lambda)-1} + \mathcal{O}\big(j^{\gamma(2^\alpha-2-\nu)}\big),$$

$$\int_y^x \frac{D_\alpha(z)}{z^2}\,dz \asymp \sum_{\lambda=0}^{\nu} A_\lambda j^{\gamma(2^\alpha-\lambda)-1} + \mathcal{O}\big(j^{\gamma(2^\alpha-1-\nu)}\big).$$

We combine these estimates with (4) to obtain

$$\sum_{j^\gamma\le\log n\le(j+1)^\gamma} \frac{[d(n)]^\alpha}{n} \asymp j^{\gamma 2^\alpha-1}\left(A_0 + \sum_{\lambda=1}^{\nu}\frac{B_\lambda}{j^{\gamma\lambda}} + \mathcal{O}\left(\frac{1}{j^{\gamma 2^\alpha-1-\gamma(2^\alpha-1-\nu)}}\right)\right), \tag{5}$$

where $B_\lambda = A_\lambda + A_{\lambda-1}\gamma(2^\alpha-\lambda)$. This proves (3) since $\nu > 1/\gamma - 1$. By continuity on both sides of (5), the assumption that 2^α is not an integer may be dropped.

The parameter $0 < \gamma < 1$ will be used to control the "block size" in our partition of the integers. It will become apparent that as α grows to infinity, we must be able to let γ tend to 0. In [13] it was sufficient to have a similar estimate only for $1/2 < \gamma < 1$.

LEMMA 4 (Discretization Lemma). *Let $\alpha > 0$ and let N be a sufficiently large positive integer. Then there exists positive constants A and B* (*depending* *on* α, *but not* N) *such that the following holds: for every function* $\phi \in L^2_\beta$ *supported on* $[\log N, \infty)$, *there is a function of the form*

$$f(s) = \sum_{n=N}^{\infty} \frac{a_n}{n^s}$$

in $\mathscr{D}_\alpha$ *such that* $\|f\|_{\mathscr{D}_\alpha} \le A\|\phi\|_{L^2_\beta}$. *Moreover,* f *may be chosen so that*

$$\Phi(s) = \int_{\log N}^{\infty} \phi(\xi) e^{-(s-1/2)\xi}\, d\xi - f(s)$$

enjoys the estimate

$$|\Phi(s)| \le B|s - 1/2| N^{-\sigma+1/2} (\log N)^{-1} \|\phi\|_{L^2_\beta},$$

in $\mathbb{C}_{1/2}$.

Proof. Let $\gamma = 2/(4 + 2^\alpha)$ and let J be the largest integer smaller than $(\log(N))^{1/\gamma}$. For $j \ge J$, let n_j be the smallest integer n such that $e^{j^\gamma} \le n$. When γ is small it is possible that $n_j = n_{j+1}$. This can be avoided by taking N sufficiently large. Set $\xi_{n_j} = j^\gamma$ and for $n_j < n \le n_{j+1}$ iteratively choose ξ_n such that

$$\frac{\xi_{n+1}^{\beta+1} - \xi_n^{\beta+1}}{\beta + 1} = A_j \frac{[d(n)]^\alpha}{n}, \tag{6}$$

where A_j is chosen so that $\xi_{n_{j+1}} = (j + 1)^\gamma$. Clearly, Lemma 3 implies that A_j is bounded as $j \to \infty$. Let us set

$$a_n = \sqrt{n} \int_{\xi_n}^{\xi_{n+1}} \phi(\xi)\, d\xi.$$

A simple computation using the Cauchy-Schwarz inequality shows that

$$|a_n|^2 = n \left| \int_{\xi_n}^{\xi_{n+1}} \phi(\xi)\, d\xi \right|^2 \le n \cdot \frac{\xi_{n+1}^{\beta+1} - \xi_n^{\beta+1}}{\beta + 1} \int_{\xi_n}^{\xi_{n+1}} |\phi(\xi)|^2 \frac{d\xi}{\xi^\beta}.$$

In view of (6) it is clear that $\|f\|_{\mathscr{D}_\alpha} \le A\|\phi\|_{L^2_\beta}$. Now, if $n_j \le n \le n_{j+1}$ and $\xi \in [\xi_{n_j}, \xi_{n_{j+1}}]$ we see that

$$\left| e^{-(s-1/2)} - n^{-(s-1/2)} \right| \le N^{-\sigma+1/2} |s - 1/2| j^{\gamma-1}. \tag{7}$$

Then, by (7) and the Cauchy-Schwarz inequality

$$|\Phi(s)| \leq N^{-\sigma+1/2}|s-1/2|\sum_{j=J}^{\infty} j^{\gamma-1}\sum_{n=n_j}^{n_{j+1}-1}\left(\frac{\xi_{n+1}^{\beta}-\xi_n^{\beta}}{\beta}\right)^{\frac{1}{2}}\left(\int_{\xi_n}^{\xi_{n+1}}|\phi(\xi)|^2\,\frac{d\xi}{\xi^{\beta}}\right)^{\frac{1}{2}}.$$

By using the Cauchy-Schwarz inequality again with (6) we get

$$|\Phi(s)| \ll N^{-\sigma+1/2}|s-1/2|\sum_{j=J}^{\infty} j^{\gamma-1}\left(\sum_{n=n_j}^{n_{j+1}-1}\frac{[d(n)]^{\alpha}}{n}\right)^{\frac{1}{2}}\left(\int_{\xi_{n_j}}^{\xi_{n_{j+1}}}|\phi(\xi)|^2\,\frac{d\xi}{\xi^{\beta}}\right)^{\frac{1}{2}}.$$

Now Lemma 3 and the Cauchy-Schwarz inequality yield

$$|\Phi(s)| \ll N^{-\sigma+1/2}|s-1/2|\left(\sum_{j=J}^{\infty} j^{(2+2^{\alpha})\gamma-3}\right)^{\frac{1}{2}}\left(\int_{\log N}^{\infty}|\phi(\xi)|^2\,\frac{d\xi}{\xi^{\beta}}\right)^{\frac{1}{2}}.$$

The series converges since $\gamma < 2/(2+2^{\alpha})$. The proof is completed by a standard estimate of the convergent series,

$$\left(\sum_{j=J}^{\infty} j^{(2+2^{\alpha})\gamma-3}\right)^{\frac{1}{2}} \ll (\log N)^{((2+2^{\alpha})\gamma-2)/(2\gamma)} = (\log N)^{-1},$$

where we used that $J \asymp (\log N)^{1/\gamma}$.

The final result needed for the iterative scheme is the following simple lemma on the $\overline{\partial}$-equation. We omit the proof, which is obvious.

LEMMA 5. *Suppose g is a continuous function on $\mathbb{C}_{1/2}$, supported on*

$$\Omega(R,\tau) = \left\{s = \sigma + it : 1/2 \leq \sigma \leq 1/2+\tau,\ -R \leq t \leq R\right\},$$

for some positive real numbers τ and R. Then

$$u(s) = \frac{1}{\pi}\int_{\Omega}\frac{g(w)}{s-w}\,dm(w)$$

solves $\overline{\partial}u = g$ in $\mathbb{C}_{1/2}$ and satisfies $\|u\|_{\infty} \leq C_{\Omega}\|g\|_{\infty}$.

We have now collected all our preliminary results and are ready to begin the proof of Theorem 1. For any positive integer N we set $E_N(s) = N^{-s+1/2}$ and consider the space $E_N A_{\beta}$. By a substitution it is evident that any $F \in E_N A_{\beta}$ can be represented as

$$F(s) = \int_{\log N}^{\infty}\phi(\xi)e^{-(s-1/2)\xi}\,d\xi$$

for some $\phi \in L^2_\beta[\log N, \infty)$, in view of Lemma 2.

FINAL STEP IN THE PROOF OF THEOREM 1. Let us fix $\alpha > 0$ and a bounded sequence $S = (\sigma_j + it_j) \in Z(A_\beta)$. From this point all constants may depend on α and S. Since S is bounded we may assume $S \subset \Omega(R-2, \tau-2)$ for some $R, \tau > 2$. Let Θ be some smooth function defined on $\overline{\mathbb{C}_{1/2}}$ with the following properties:

- Θ is supported on $\Omega(R, \tau)$,
- $\Theta(s) = 1$ for $s \in \Omega(R-1, \tau-1)$,
- $|\overline{\partial}\Theta(s)| \leq 2$.

Let $G \in A_\beta$ vanish precisely on S and assume furthermore that $\|G\|_{A_\beta} = 1$. Now, suppose that $F \in E_N A_\beta$, and let $f \in \mathscr{D}_\alpha$ be the function obtained by applying Lemma 4 to F, and $\Phi = F - f$. Moreover, let u denote the solution to the equation

$$\overline{\partial} u = \frac{\overline{\partial}(\Theta\Phi)}{G E_N}. \tag{8}$$

The right hand side of (8) is a smooth function compactly supported on $\Omega(R, \tau)$ since $|G(s)|$ is bounded from below where $\overline{\partial}\Theta(s) \neq 0$. We can use Lemma 5 and Lemma 2 to estimate

$$\|u\|_\infty \ll \left\| \frac{\overline{\partial}(\Theta\Phi)}{G E_N} \right\|_\infty \ll (\log N)^{-1} \|\phi\|_{L^2_\beta} = (\log N)^{-1} \|F\|_{A_\beta}. \tag{9}$$

We set $T_N F = \Theta\Phi - G E_N u$. The function $T_N F$ has the following properties:

- $T_N F(s) = \Phi(s)$ for $s \in S$,
- $T_N F$ is analytic in $\mathbb{C}_{1/2}$ since $\overline{\partial} T_N F(s) = 0$ for $s \in \mathbb{C}_{1/2}$,
- $T_N F \in E_N A_\beta$, by the compact support of Θ and the estimate (9).

Hence T_N defines an operator on $E_N A_\beta$. By the triangle inequality, Lemma 4 and the fact that Θ has compact support, it is clear that

$$\|T_N F\|_{A_\beta} \leq \|\Theta\Phi\|_{A_\beta} + \|G E_N u\|_{A_\beta} \ll (\log N)^{-1} \|\phi\|_{L^2_\beta} + \|u\|_\infty \|G\|_{A_\beta}.$$

Since $\|G\|_{A_\beta} = 1$ and $\|\phi\|_{L^2_\beta} = \|F\|_{A_\beta}$ we have $\|T_N\| \ll (\log N)^{-1}$ in view of (9). Let N be large, but arbitrary, and define $F_0(s) = E_N(s) G(s)$. Then $F_0 \in E_N A_\beta$ and its norm in this space is ≤ 1. Set

$$F_j = T_N^j F_0.$$

Let f_j be the Dirichlet series of Lemma 4 obtained from F_j. Then $f_0 + F_1$ vanishes on S, since

$$f_0(s) + F_1(s) = f_0(s) + T_N F_0(s) = f_0(s) + F_0(s) - f_0(s) = F_0(s) = 0,$$

for $s \in S$, by the fact that $T_N F(s) = \Phi(s)$ for $s \in S$. Iteratively, the function $f_0 + f_1 + \cdots + f_j + F_{j+1}$ also vanishes on S. Define

$$f(s) = \sum_{j=0}^{\infty} f_j(s)$$

and choose N so large that $\|T_N\| < 1$ so that $\|F_j\|_{A_\beta} \to 0$ and, say

$$|f(1)| > \sum_{j=1}^{\infty} |f_j(1)|,$$

so that f is non-trivial in $\mathscr{D}_\alpha$ and vanishing on S.

By again following [13], we can modify the iterative scheme in the following way: let $F \in A_\beta$ be arbitrary, and set $F_0 = F$. Using the algorithm in the same manner as above, we see that $F_1(s) + f_0(s) = F_0(s)$ for $s \in S$. Moreover,

$$F_{j+1}(s) + f_j(s) + f_{j-1}(s) + \cdots + f_0(s) = F(s),$$

for $s \in S$. Continuing as above, we obtain the following result:

Corollary 6. *Suppose $S = (\sigma_j + it_j) \in Z(A_\beta)$ is bounded. For every function $F \in A_\beta$ there is some $f \in \mathscr{D}_\alpha$ such that $f(s) = F(s)$ on S.*

We can extend Theorem 1 and Corollary 6 by considering different weights. Let $w = (w_1, w_2, \dots)$ be a non-negative weight. Define the Hilbert space of Dirichlet series $\mathscr{D}_w$ in the same manner as above, with the added convention that the basis vector n^{-s} is excluded if $w_n = 0$. Theorem 1 in [9] states that $\mathscr{D}_w$ embeds locally into A_β if and only if

$$\sum_{n \le x} w_n \ll x(\log x)^\beta, \tag{10}$$

where $\beta > 0$. By modifying the proof of our Theorem 1, we can obtain a similar result for $\mathscr{D}_w$ with respect to A_β provided we additionally have

$$\sum_{j^\gamma \le \log n \le (j+1)^\gamma} \frac{w_n}{n} \asymp j^{\gamma(\beta+1)-1}, \tag{11}$$

as $j \to \infty$, for some $0 < \gamma < 2/(3+\beta)$. Several of the weights considered in [9] are possible, but we only mention the case $w_n = (\log n)^\beta$ for $\beta > 0$. These spaces were introduced by McCarthy in [8]. It is easy to show that these weights satisfy (10) and (11) for any $0 < \gamma < 1$, and similar results with respect to A_β are obtained.

REMARK. The embeddings of [9] extend to any $\beta \leq 0$, in view of (10), and we get the Hardy space ($\beta = 0$) and Dirichlet spaces ($\beta < 0$) in the half-plane. We can extend the results in [13] in a similar manner as above. However, this is only possible for $-1 \leq \beta < 0$. The method of [13] breaks down for $\beta < -1$ due to the fact that the norms of the corresponding Dirichlet spaces in the half-plane uses higher order derivatives and different estimates are needed.

3. Blaschke-type conditions for $\mathscr{D}_\alpha$ and $\mathscr{H}^p$

Now that we have identified the bounded zero sequences of $\mathscr{D}_\alpha$ as those of A_β, let us consider necessary and sufficient conditions for bounded zero sequences of A_β. The zero sequences of Bergman spaces in the unit disc $\mathbb{D}$ have attracted considerable attention. We refer to the monograph [3]. For $\beta > 0$, these are the spaces

$$A_\beta(\mathbb{D}) = \left\{ F \in H(\mathbb{D}) : \|F\| = \int_{\mathbb{D}} |F(z)|^2 (1-|z|)^{\beta-1}\, dm(z) < \infty \right\}.$$

Results pertaining to zero sequences of $A_\beta(\mathbb{D})$ are relevant to our case since

$$\phi(s) = \frac{s-3/2}{s+1/2}$$

is a conformal mapping from $\mathbb{C}_{1/2}$ to $\mathbb{D}$, and

$$F \mapsto (s+1/2)^{-2(\beta+1)} F\left(\frac{s-3/2}{s+1/2}\right)$$

defines an isometric isomorphism from $A_\beta(\mathbb{D})$ to A_β. This implies that $S \in Z(A_\beta)$ if and only if $\phi(S) \in Z(A_\beta(\mathbb{D}))$. Since the Hardy space $H^2(\mathbb{D})$ is included in $A_\beta(\mathbb{D})$ for every $\beta > 0$, it is clear that the Blaschke condition

$$\sum_j (\sigma_j - 1/2) < \infty \tag{12}$$

is sufficient for bounded zero sequences of A_β. Moreover, Theorem 4.1 of [3] shows that the Blaschke condition (12) is both necessary and sufficient provided the bounded sequence S is contained in any cone $|t-t_0| \leq c(\sigma - 1/2)$. Unfortunately, the situation becomes more complicated in the general case and we do not have a precise Blaschke-type condition for bounded zero sequences. In fact, for every $\epsilon > 0$ and every A_β a necessary condition for bounded zero sequences is

$$\sum_j (\sigma_j - 1/2)^{1+\epsilon} < \infty, \tag{13}$$

by Corollary 4.8 of [3]. Clearly, this condition does not offer any insight into what happens as $\beta \to 0^+$. However, using the notion of density introduced by Korenblum in [7] it is possible to provide a generalized condition describing the geometrical information of the zero sequences of $A_\beta(\mathbb{D})$. The most precise results on Korenblum's density are obtained by Seip in [12]. We omit the details, only mentioning that this generalized condition in a certain sense tends to (12) when $\beta \to 0^+$.

The Hardy spaces of Dirichlet series $\mathscr{H}^p$, $1 \leq p < \infty$, can be defined as the closure of the set of all Dirichlet polynomials with respect to the norms

$$\left\| \sum_{n=1}^{N} \frac{a_n}{n^s} \right\|_{\mathscr{H}^p} = \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} \frac{a_n}{n^{it}} \right|^p dt \right)^{\frac{1}{p}}.$$

For the basic properties of these spaces we refer to [4] and [1]. However, we immediately observe that $\mathscr{H}^2 = \mathscr{D}_0$. In [13], the bounded zero sequences of the spaces $\mathscr{H}^p$, for $2 \leq p < \infty$, are studied. In particular, for $\mathscr{H}^2$ the Blaschke condition (12) is shown to be both necessary and sufficient. Results for $2 < p < \infty$ are obtained through embeddings $\mathscr{D}_\alpha \subset \mathscr{H}^p \subset \mathscr{H}^2$, where $\alpha < 0$ depends on p. The embedding of $\mathscr{H}^p$ into $\mathscr{H}^2$ implies that the Blaschke condition (12) is necessary for $\mathscr{H}^p$.

The sufficient conditions are obtained through a similar result as Theorem 1: for $\alpha < 0$, the spaces $\mathscr{D}_\alpha$ have the same bounded zero sequences as certain weighted Dirichlet spaces in $\mathbb{C}_{1/2}$. In particular, for $2 < p < \infty$ there is some $0 < \gamma < 1$ such that a sufficient condition for bounded zero sequences of $\mathscr{H}^p$ is

$$\sum_j (\sigma_j - 1/2)^{1-\gamma} < \infty, \tag{14}$$

and moreover $\gamma \to 0$ as $p \to 2^-$. We omit the details, which can be found in [13].

We will now consider the case $1 \leq p < 2$. That $\mathscr{H}^2 \subset \mathscr{H}^p \subseteq \mathscr{H}^1$ for $1 \leq p < 2$ is trivial, and this shows that (12) is a sufficient condition for bounded zero sequences of $\mathscr{H}^p$. In [5], Helson proved the beautiful inequality

$$\|f\|_{\mathscr{D}_1} = \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathscr{H}^1}, \tag{15}$$

which implies that $\mathscr{H}^p \subset \mathscr{D}_1$. This shows that the Blaschke-type condition (13) is necessary for bounded zero sequences of $\mathscr{H}^p$, for every $\epsilon > 0$. Regrettably, this means we are unable to specify how the situation changes as $p \to 2^-$, in a manner similar to (14). However, if we again restrict S to the cone $|t - t_0| \leq c(\sigma - 1/2)$, the Blaschke condition (12) is both necessary and sufficient for bounded zero sequences of $\mathscr{H}^p$.

REMARK. The Blaschke condition (12) is well-known to be necessary and sufficient for bounded zero sequences of the Hardy spaces $H^p(\mathbb{C}_{1/2})$. By a theorem in [4], $\mathscr{H}^2$ embeds locally into $H^2(\mathbb{C}_{1/2})$. This trivially extends to even integers p. Whether the local embedding extends to every $p \geq 1$ is an open question. Observe that if (12) is not the optimal necessary condition for bounded zero sequences of $\mathscr{H}^p$, when $1 \leq p < 2$, then the local embedding would be impossible for these p. However, since (14) is a sufficient condition for bounded zero sequences of $\mathscr{H}^p$ when $p \geq 2$, its optimality would not contradict the local embedding for these p.

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