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A PALEY-WIENER THEOREM FOR THE SPHERICAL TRANSFORM ASSOCIATED WITH THE GENERALIZED GELFAND PAIR $(U(p,q), H_n)$, $p+q=n$

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Abstract

In this work we prove a Paley-Wiener theorem for the spherical transform associated to the generalized Gelfand pair $(H_n \ltimes U(p,q), H_n)$, where H_n is the $2n+1$-dimensional Heisenberg group.

In particular, by using the identification of the spectrum of $(U(p,q), H_n)$ with a subset Σ of $\mathbb{R}^2$, we prove that the restrictions of the spherical transforms of functions in $C_0^\infty(H_n)$ to appropriated subsets of Σ, can be extended to holomorphic functions on $\mathbb{C}^2$. Also, we obtain a real variable characterizations of such transforms.

1. Introduction

In the last years the spherical analysis on the Heisenberg group H_n has been a subject of considerable interest, that is, the study of the spherical functions and the Gelfand transform related to a Gelfand pair $(K \ltimes H_n, K)$, where K is a compact subgroup of automorphisms of H_n (see [5], [2] and [3]).

We recall that $(K \ltimes H_n, K)$, also denoted by (K, H_n), is called a Gelfand pair if any of the following equivalent conditions hold:

(i) $L^1_K(H_n) = \{f \in L^1(H_n) : f(kx) = f(x), \forall x \in H_n, k \in K\}$ is a commutative convolution algebra,

(ii) the algebra $\mathcal{U}_K(H_n)$ of left-invariant and K-invariant differential operators on H_n is commutative,

(iii) for any irreducible unitary representation of the semidirect product $K \ltimes H_n$, the space of vectors fixed by K is at most one dimensional.

In this case, we denote by $\Delta(K, H_n)$ the Gelfand spectrum of $L^1_K(H_n)$, which can be identified with the set of bounded spherical functions.

For $f \in L^1_K(H_n)$, the Gelfand transform $\widehat{f}$ is given by

$$\widehat{f}(\varphi) = \int_{H_n} f\overline{\varphi}, \quad \varphi \in \Delta(K, H_n).$$

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When K is no longer assumed to be compact, $L^1_K(H_n)$ is trivial and (K, H_n) is called a *generalized Gelfand pair* if condition (iii) holds. Moreover, in this case, the algebra of K-invariant, left invariant differential operators is commutative.

Here we consider the Heisenberg group as the real manifold $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z, t)(w, s) = \big(z + w, t + s - \tfrac{1}{2}\operatorname{Im} B(z, w)\big),$$

where

$$B(z, w) = \sum_{j=1}^{p} z_j \overline{w}_j - \sum_{j=p+1}^{n} z_j \overline{w}_j .$$

Thus, $K = U(p, q) = \{g \in Gl(n, \mathbb{C}) : B(gz, gw) = B(z, w)\ \forall (z, w) \in \mathbb{C}^n\}$ acts by automorphisms on H_n via

$$g.(z, t) = (gz, t), \quad \text{for } (z, t) \in H_n,$$

and $(U(p, q) \ltimes H_n, H_n)$ is a generalized Gelfand pair (see [10]). When $p = n$ and $q = 0$, $U(n, 0)$ is the unitary group denoted by $U(n)$.

Let $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}$ be the standard basis of the Heisenberg Lie algebra, with $[X_i, Y_j] = \delta_{i,j} T$ and all other brackets equal to zero. It was proved in [7] and [6] that $\Delta(U(n), H_n)$ is homeomorphic to the subset

$$\Gamma = \{(\lambda, (2k + n)|\lambda|) : \lambda \neq 0, k \in \mathbb{N}\} \cup \{(0, \sigma) : \sigma \geq 0\}$$

of $\mathbb{R}^2$ equipped with the relative topology, where λ and $(2k + n)|\lambda|$ are the eigenvalues of $T = \partial/\partial t$ and $L = \sum_{j=0}^{n} X_j^2 + Y_j^2$ respectively.

In this case the characterization of the image of the space $S_{U(n)}(H_n)$ of the Schwartz functions on H_n which are $U(n)$-invariant under the Gelfand transform, was given as the space of the restrictions to Γ of Schwartz functions on $\mathbb{R}^2$ (see [18] and [3]).

In analogy to the compact case, the spectrum $\Delta(U(p, q), H_n)$ associated to the generalized Gelfand pair $(U(p, q), H_n)$ was introduced in [14] as the extremal points of the $U(p, q)$-invariant positive type distributions on H_n, also called spherical distributions by Molcanov in [15]. It is known that these are joint eigendistributions of $\mathcal{U}_{U(p,q)}(H_n)$ (see [11]).

The algebra $\mathcal{U}_{U(p,q)}(H_n)$ is generated by

$$\mathcal{D} = \sum_{j=0}^{p} X_j^2 + Y_j^2 - \sum_{j=p+1}^{n} X_j^2 + Y_j^2 \quad \text{and} \quad T = \frac{\partial}{\partial t}.$$

Let us consider the subset of $\mathbb{R}^2$ given by

$$\Sigma = \{(\lambda, (2k + p - q)|\lambda|) : \lambda \neq 0,\ k \in \mathbb{Z}\} \cup \{(0, \sigma) : \sigma \in \mathbb{R}\}.$$

Then $\Delta\,(U(p,q), H_n)$ can be identified with Σ via the map which sends an spherical distribution to its corresponding pair of eingenvalues. We denote by $S_{\lambda,k}$ and S_σ the spherical distributions satisfying

$$\begin{aligned} iT(S_{\lambda,k}) &= \lambda S_{\lambda,k}, \quad & -\mathscr{D}(S_{\lambda,k}) &= |\lambda|(2k + p - q)S_{\lambda,k},\\ iT(S_\sigma) &= 0, & -\mathscr{D}(S_\sigma) &= \sigma S_\sigma. \end{aligned}$$

Explicit formulas for $S_{\lambda,k}$ and S_σ were obtained in [10], [12] and [14].

The normalized spherical transform associated with the generalized Gelfand pair $(U(p,q), H_n)$ was introduced in [8]. If f a *Schwartz function* on H_n, its normalized spherical transform $\mathscr{F}(f)$ is defined by

$$\mathscr{F}(f)(\lambda, (2k + p - q)|\lambda|) = \begin{cases} |\lambda|^{n-1}\langle S_{\lambda,k}, f\rangle, & k \geq 0,\\ (-1)^n|\lambda|^{n-1}\langle S_{\lambda,k}, f\rangle, & k < 0, \end{cases}$$

and by

$$\mathscr{F}(f)(0,\sigma) = \langle S_\sigma, f\rangle, \quad \sigma \in \mathbb{R}.$$

The characterization of the image of the spherical transform has been generalized in [8] in the following way. Let Σ^+ and Σ^- be the subsets of Σ defined by

$$\Sigma^+ = \{(\lambda, (2k + p - q)|\lambda|) : \lambda \neq 0,\ k \geq -p + 1\} \cup \{(0, \sigma) : \sigma \geq 0\}$$

and

$$\Sigma^- = \{(\lambda, (2k + p - q)|\lambda|) : \lambda \neq 0, k \leq q - 1\} \cup \{(0, \sigma) : \sigma \leq 0\}.$$

Then $\mathscr{F}(f)|_{\Sigma^+}$ and $\mathscr{F}(f)|_{\Sigma^-}$ can be separately extended to Schwartz functions on $\mathbb{R}^2$. Moreover, the spherical transform $\mathscr{F}(f)$ of a Schwartz function f on H_n can be extended to a Schwartz function on $\mathbb{R}^2$ if and only if $s \mapsto \mathscr{F}(f)(0,s)$ is a Schwartz function.

Since $U(p,q)$ acts trivially on the center of H_n, the description of the space $\mathscr{S}'(H_n)^{U(p,q)}$ of the tempered and $U(p,q)$-invariant distribution on H_n, reduces to that of $\mathscr{S}'(C^n)^{U(p,q)}$. Moreover, as $U(p,q)$ and $SO(2p,2q)$ have the same orbits on $\mathbb{R}^{2n}$, one can adapt the results in [17], in order to define a linear map N from $\mathscr{S}(H_n)$ onto $\mathscr{H}^\#$, the space of functions φ on $\mathbb{R}^2$ of the form

$$\varphi(\tau, t) = \varphi_1(\tau, t) + \tau^{n-1}\varphi_2(\tau, t)H(\tau), \quad \varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R}^2),$$

where H is the Heaviside function, in such a way that its adjoint $N'\colon H^{\#} \to \mathcal{S}'(H_n)^{U(p,q)}$ is a homeomorphism.

Roughly speaking, N is given by integration on the $U(p,q)$-orbits $B(z,z) = \tau$ with respect to the orbital measures $d\mu(\tau)$, and it satisfies, for $f \in \mathcal{S}(H_n)$,

$$\int_{H_n} f(z,t)\,dz\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Nf(\tau,t)\,d\tau\,dt.$$

Moreover, since $\mathrm{Ker}(N) = \mathrm{Ker}(\mathcal{F})$ (see [13]), we may consider the spherical transform defined on $\mathcal{H}^{\#}$.

The aim of this work is to obtain a Paley-Wiener theorem for the spherical transform of the functions on $\mathcal{H}_n^{\#}$ with compact support (see Theorem 3.8). This result was motivated by [1] and [4].

In [1], the following Paley-Wiener theorem was proved: a function f on $\mathbb{R}^n$ is the Fourier transform of a C^∞ function with compact support if and only if it is a Schwartz function and, for some $p \in [1,\infty]$,

$$\limsup_{k\to\infty} \left\| \Delta^k f \right\|_p^{1/k} < \infty,$$

where Δ is the Laplace operator on $\mathbb{R}^n$. In this case, the left-hand side is finite for every p, the "lim sup" is a limit and, for every $p \in [1,\infty]$,

$$\lim_{k\to\infty} \left\| \Delta^k f \right\|_p^{1/k} = \max_{x\in \operatorname{supp} \mathcal{F}^{-1} f} |x|^2.$$

In [4], an analogous result was proved for the spherical transform associated to the Gelfand pair $(U(n), H_n)$ which relies on the choice of the differential operators $M_\pm$ introduced by Benson, Jenkins and Ratcliff in [5] on Γ and the Korányi norm on H_n.

We are interested in finding an inversion formula recovering Nf from the spherical transform $\mathcal{F}(f)$ of a Schwartz function f on H_n. With this purpose, in the second section, we define measures on Σ^+ and Σ^- and we introduce the operators $M^\pm$ on Σ^+ and $\mathcal{M}^\pm$ on Σ^-, proving some relevant properties.

In the third section we prove the main result of this work.

In the last section we show that the restrictions to Σ^+ and Σ^- of the spherical transform of functions on $\mathcal{H}^{\#}$ with compact support can be extended to holomorphic functions on $\mathbb{C}^2$ (see Theorem 4.2).

2. Preliminaries

We recall some known facts in order to present explicitly the spherical distributions $S_{\lambda,k}$ for $\lambda \neq 0$, $k \in \mathbb{Z}$ and S_σ for $\sigma \in \mathbb{R}$. Let H be the Heaviside function (that is, $H = \chi_{(0,\infty)}$) and let

$$\mathscr{H} = \big\{\varphi\colon \mathbb{R} \mapsto \mathbb{C} : \varphi(\tau) = \varphi_1(\tau) + \tau^{n-1}\varphi_2(\tau)H(\tau),\ \varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R})\big\}.$$

It was proved in [17] that $\mathscr{H}$, equipped with a natural topology, is a Fréchet space. Also, for $p, q \in \mathbb{N}$, $p + q = n$, a linear, continuous and surjective map $N\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{H}$ was given in such a way that its adjoint $N'\colon \mathscr{H}' \to \mathscr{S}'(\mathbb{R}^n)^{O(p,q)}$ is a linear homeomorphism onto the space of the $O(p, q)$-invariant, tempered distributions on $\mathbb{R}^n$. As $U(p, q)$ and $SO(2p, 2q)$ have the same orbits and orbital measures on $\mathbb{R}^{2n}$, this construction can also be used to describe the space $\mathscr{S}'(\mathbb{C}^n)^{U(p,q)}$, that is, there exists a linear, continuous and surjective map, still denoted by $N\colon \mathscr{S}(\mathbb{C}^n) \to \mathscr{H}$, whose adjoint map $N'\colon \mathscr{H}' \to \mathscr{S}'(\mathbb{C}^n)^{U(p,q)}$ is a homeomorphism.

In order to give N explicitly, we introduce new coordinates in $\mathbb{C}^n$. Given $u = (u_1, \dots, u_p, u_{p+1}, \dots, u_n) \in \mathbb{C}^n$, we write $\rho = |u_1|^2 + \cdots + |u_n|^2$ and $\tau = (|u_1|^2 + \cdots + |u_p|^2) - (|u_{p+1}|^2 + \cdots + |u_n|^2)$. It is clear that

$$\big\|(u_1, \dots, u_p)\big\| = \Big(\frac{\rho+\tau}{2}\Big)^{1/2}, \quad \big\|(u_{p+1}, \dots, u_n)\big\| = \Big(\frac{\rho-\tau}{2}\Big)^{1/2},$$

and let

$$w_1 = \Big(\frac{\rho+\tau}{2}\Big)^{-1/2}(u_1, \dots, u_p) \in S^{2p-1},$$

$$w_2 = \Big(\frac{\rho-\tau}{2}\Big)^{-1/2}(u_{p+1}, \dots, u_n) \in S^{2q-1}.$$

A straightforward adaptation of the Tengstrand map in [17] showed that the map $N\colon \mathscr{S}(H_n) \to \mathscr{H}^{\#}$ defined by

$$Nf(\tau, t) = \int_{\rho>|\tau|}\int_{S^{2p-1}\times S^{2q-1}} f\bigg(\Big(\frac{\rho+\tau}{2}\Big)^{1/2}\omega_u, \Big(\frac{\rho-\tau}{2}\Big)^{1/2}\omega_v, t\bigg)\, d\omega_u\, d\omega_v \times (\rho+\tau)^{p-1}(\rho-\tau)^{q-1}\, d\rho,$$

is linear, continuous and surjective, and its adjoint map $N'\colon (\mathscr{H}^{\#})' \to \mathscr{S}'(H_n)^{U(p,q)}$ is a homeomorphism.

On the other hand, we recall the definition of the Laguerre polynomials

$$L_m^{(0)}(\tau) = \sum_{j=0}^{m}\binom{m}{j}(-1)^j\frac{\tau^j}{j!}, \quad L_{m-1}^{(\alpha+1)}(\tau) = -\frac{d}{d\tau}L_m^{(\alpha)}(\tau),$$

for $m, \alpha \in \mathbb{N}_0$, according to [16]. Therefore, $L_m^{(\alpha)}(0) = \binom{\alpha+m}{m}$.

Then, the distributions $S_{\lambda,k}$ calculated in [12] are given by

$$S_{\lambda,k} = F_{\lambda,k} \otimes e^{-i\lambda t},$$

with $F_{\lambda,k} \in \mathscr{S}'(\mathbb{C}^n)^{U(p,q)}$ defined by

$$\langle F_{\lambda,k}, f(\cdot,t)\rangle = \big\langle (L^{(0)}_{k-q+n-1}H)^{n-1}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, t)\big\rangle,$$

for $k \geq 0$, $\lambda \neq 0$, and by

$$\langle F_{\lambda,k}, f(\cdot,t)\rangle = \big\langle (L^{(0)}_{-k-p+n-1}H)^{n-1}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(-2|\lambda|^{-1}\tau, t)\big\rangle,$$

for $k < 0$, $\lambda \neq 0$. Therefore,

$$\langle S_{\lambda,k}, f\rangle = \sum_{j=0}^{n-1}\binom{n-1}{j}(-4)^j|\lambda|^{n-1-j} \times \int_0^\infty\int_{-\infty}^\infty L^{(0)}_{k-q+n-1}\left(\frac{s|\lambda|}{2}\right)\frac{\partial^j Nf}{\partial s^j}(s,t)e^{-i\lambda t-\frac{s|\lambda|}{4}}\,dt\,ds,$$

for $k \geq 0$, $\lambda \neq 0$ and

$$\langle S_{\lambda,k}, f\rangle = (-1)^n\sum_{j=0}^{n-1}\binom{n-1}{j}4^j|\lambda|^{n-1-j} \times \int_0^\infty\int_{-\infty}^\infty L^{(0)}_{-k-p+n-1}\left(\frac{s|\lambda|}{2}\right)\frac{\partial^j Nf}{\partial s^j}(-s,t)e^{-i\lambda t-\frac{s|\lambda|}{4}}\,dt\,ds,$$

for $k < 0$, $\lambda \neq 0$, where $Nf(\tau,\hat{\lambda})$ denotes the Fourier transform of $Nf(\tau,\cdot)$ in λ.

Moreover, the distributions S_σ calculated in [13] are given by

$$\langle S_\sigma, f\rangle = (-1)^{n-1}\int_{\mathbb{R}}\int_0^\infty J_0((\sigma\tau)^{1/2})\frac{\partial^{n-1}Nf}{\partial\tau^{n-1}}(\tau,t)\,d\tau\,dt$$

for $\sigma \geq 0$ and by

$$\langle S_\sigma, f\rangle = (-1)^{n-2}\int_{\mathbb{R}}\int_0^\infty J_0((-\sigma\tau)^{1/2})\frac{\partial^{n-1}Nf}{\partial\tau^{n-1}}(-\tau,t)\,d\tau\,dt$$

for $\sigma < 0$, where $J_0(\tau) = \sum_{k=0}^\infty \frac{(-1)^k}{k!k!}(\frac{\tau}{2})^{2k}$ is the Bessel function of order 0 of the first kind.

DEFINITION 2.1. For $k > -p$ and $\lambda \neq 0$ we define the distribution $\Lambda^+_{\lambda,k}$ on $\mathscr{S}(H_n)$ by

$$\langle \Lambda^+_{\lambda,k}, f\rangle = \big\langle (L^{(0)}_{k-q+n-1}H)^{n-1}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(2|\lambda|^{-1}\tau, \hat{\lambda})\big\rangle$$

and for $k < q$ and $\lambda \neq 0$ let $\Lambda^-_{\lambda,k}$ given by

$$\langle \Lambda^-_{\lambda,k}, f\rangle = (-1)^n \big\langle (L^{(0)}_{-k-p+n-1}H)^{n-1}, \tau \mapsto 2|\lambda|^{-1}e^{-\tau/2}Nf(-2|\lambda|^{-1}\tau, \hat{\lambda})\big\rangle$$

So, due to Corollary 2.3 in [12], for $f \in \mathscr{S}(H_n)$ we have that

$$\mathscr{F}(f)|_{\Sigma^+}(\lambda, (2k+p-q)|\lambda|) = |\lambda|^{n-1}\langle \Lambda^+_{\lambda,k}, f\rangle,$$

and

$$\mathscr{F}(f)|_{\Sigma^-}(\lambda, (2k+p-q)|\lambda|) = |\lambda|^{n-1}\langle \Lambda^-_{\lambda,k}, f\rangle.$$

It is known that

$$\mathscr{D} = \sum_{j=1}^{p}(X_j^2 + Y_j^2) - \sum_{j=p+1}^{n}(X_j^2 + Y_j^2) = B(z)\frac{\partial^2}{\partial t^2} + \Box + R,$$

where $\Box = \sum_{j=1}^{p}\big(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\big) - \sum_{j=p+1}^{n}\big(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\big)$ and $R = \frac{\partial}{\partial t}\sum_{j=1}^{n}\big(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\big)$.

As $S_{\lambda,k}$ is $U(p,q)$-invariant, we have $\langle S_{\lambda,k}, Rf\rangle = 0$ and there exists $T_{\lambda,k}$ in $(\mathscr{H}^\#)'$ such that $\langle S_{\lambda,k}, f\rangle = \langle T_{\lambda,k}, Nf\rangle$, for all $f \in \mathscr{S}(H_n)$. Thus, we have that

$$\begin{aligned}\langle S_{\lambda,k}, \mathscr{D}f\rangle &= \left\langle S_{\lambda,k}, \left(B(z)\frac{\partial^2}{\partial t^2} + \Box\right)f\right\rangle \\ &= \left\langle T_{\lambda,k}, (\tau,t) \mapsto \left(\tau\frac{\partial^2}{\partial t^2} + D\right)Nf(\tau,t)\right\rangle,\end{aligned} \tag{1}$$

where the differential operator D satisfy $N(\Box f) = DNf$ and is defined by $D = 4(\tau\frac{\partial^2}{\partial\tau^2} + (2-n)\frac{\partial}{\partial\tau})$ (see [17]).

Moreover,

$$\xi\mathscr{F}(f)|_{\Sigma^\pm} = \mathscr{F}(\mathscr{D}f)|_{\Sigma^\pm} \tag{2}$$

since $\mathscr{D}S_{\lambda,k} = (2k+p-q)|\lambda|S_{\lambda,k}$.

2.1. The spaces $L^2(\Sigma^+, \mu^+)$ and $L^2(\Sigma^-, \mu^-)$

We observe that in the compact case (see [4]), by using the inversion formula for the Gelfand transform, the following holds for $U(n)$-invariant Schwartz functions f, g:

$$\langle f, g\rangle_{L^2(H_n)} = \langle \widehat{f}, \widehat{g}\rangle_{L^2(\Gamma)},$$

where $L^2(\Gamma) = \{\varphi\colon \Gamma \to \mathbb{C} : \int_\Gamma |\varphi|^2 d\mu < \infty\}$ and μ is the Plancherel measure defined by

$$\int_\Gamma \varphi\, d\mu = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \sum_{j=0}^{\infty} \binom{j+n-1}{j} \varphi(\lambda, (2j+n)|\lambda|)|\lambda|^n\, d\lambda,$$

$\forall \varphi \in C_c(\Gamma)$. In our case, for $f \in \mathscr{S}(H_n)$ we will introduce a function G_f defined on $\mathbb{R}^2$, and we will define measures μ^+ and μ^- on Σ^+ and Σ^- respectively, such that

$$\begin{aligned} &\langle \mathscr{F}(f), \mathscr{F}(g)\rangle_{\Sigma^+} = \langle G_f, G_g\rangle_{\mathbb{R}_{\geq 0}\times\mathbb{R}} \\ \text{and} \qquad &\langle \mathscr{F}(f), \mathscr{F}(g)\rangle_{\Sigma^-} = \langle G_f, G_g\rangle_{\mathbb{R}_{<0}\times\mathbb{R}}, \end{aligned}$$

for all f and g in $\mathscr{S}(H_n)$.

In fact, we shall write $\mathscr{F}(f)(\lambda, k)$ instead of $\mathscr{F}(f)(\lambda, (2k+p-q)|\lambda|)$ for all $\lambda \neq 0$ and $k \in \mathbb{Z}$ and we make the following remark: by the definition of $\mathscr{F}(f)$ we have

$$\begin{aligned} \mathscr{F}(f)(\lambda, k) &= \sum_{j=0}^{n-1} \binom{n-1}{j} (-4)^j |\lambda|^{n-1-j} \\ &\quad \times \int_0^\infty L^{(0)}_{k-q+n-1}\left(\frac{s|\lambda|}{2}\right) \frac{\partial^j Nf}{\partial s^j}(s, \hat{\lambda}) e^{-\frac{s|\lambda|}{4}}\, ds \\ &= \sum_{j=0}^{n-1} c_j \int_0^\infty L^{(0)}_{k-q+n-1}\left(\frac{s|\lambda|}{2}\right) \\ &\quad \times \int_{-\infty}^\infty \lambda^{n-1-j} \frac{\partial^j Nf}{\partial s^j}(s, t) e^{-i\lambda t}\, dt\, e^{-\frac{s|\lambda|}{4}}\, ds, \end{aligned}$$

for $\lambda \neq 0$ and $k > -p$, where $c_j = \binom{n-1}{j}(-4)^j \operatorname{sgn}(\lambda)^{n-1-j}$. In a similar way, it follows that

$$\begin{aligned} \mathscr{F}(f)(\lambda, k) &= (-1)^n \sum_{j=0}^{n-1} \binom{n-1}{j} 4^j |\lambda|^{n-1-j} \\ &\quad \times \int_0^\infty L^{(0)}_{-k-p+n-1}\left(\frac{s|\lambda|}{2}\right) \frac{\partial^j Nf}{\partial s^j}(-s, \hat{\lambda}) e^{-s|\lambda|/4}\, ds \\ &= \sum_{j=0}^{n-1} d_j \int_0^\infty L^{(0)}_{-k-p+n-1}\left(\frac{s|\lambda|}{2}\right) \\ &\quad \times \int_{-\infty}^\infty \lambda^{n-1-j} \frac{\partial^j Nf}{\partial s^j}(-s, t) e^{-i\lambda t}\, dt\, e^{-s|\lambda|/4}\, ds, \end{aligned}$$

for $\lambda \neq 0$ and $k < q$, where $d_j = (-1)^n \binom{n-1}{j} 4^j \operatorname{sgn}(\lambda)^{n-1-j}$.

So, for f in $\mathscr{S}(H_n)$ we define the function G_f on $\mathbb{R}^2$ by

$$
(3) \qquad G_f(\tau,\lambda) = \begin{cases} \displaystyle\sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{n-1} 4^j i^{n-1-j} \\ \qquad \times \operatorname{sgn}(\lambda)^{n-1-j} (i\lambda)^{n-1-j} \frac{\partial^j Nf}{\partial \tau^j}(\tau,\hat{\lambda}), \quad \text{if } \tau \geq 0, \\ \displaystyle\sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{j+1} 4^j i^{n-1-j} \\ \qquad \times \operatorname{sgn}(\lambda)^{n-1-j} (i\lambda)^{n-1-j} \frac{\partial^j Nf}{\partial \tau^j}(\tau,\hat{\lambda}), \quad \text{if } \tau < 0, \end{cases}
$$

where $\frac{\partial^j Nf}{\partial \tau^j}(\tau,\hat{\lambda})$ denotes the Fourier transform of $\frac{\partial^j Nf}{\partial \tau^j}(\tau,\cdot)$ in λ.

As $\{L^{(0)}_{k-q+n-1}(s)e^{-\frac{s}{2}}\}_{k>-p}$ and $\{L^{(0)}_{-k-p+n-1}(s)e^{-\frac{s}{2}}\}_{k<q}$ are orthogonal bases of $L^2(0,\infty)$ and

$$
\left\| s \mapsto L^{(0)}_{k-q+n-1}\left(\frac{s|\lambda|}{2}\right) e^{-\frac{s|\lambda|}{4}} \right\|^2_{L^2(0,\infty)} = \frac{2}{|\lambda|} = \left\| s \mapsto L^{(0)}_{-k-p+n-1}\left(\frac{s|\lambda|}{2}\right) e^{-\frac{s|\lambda|}{4}} \right\|^2_{L^2(0,\infty)},
$$

we have that

$$
\left\{ \varphi_k(s) = \left(\frac{2}{|\lambda|}\right)^{-1/2} L^{(0)}_{k-q+n-1}\left(\frac{s|\lambda|}{2}\right) e^{-\frac{s|\lambda|}{4}} \right\}_{k>-p}
$$

and

$$
\left\{ \phi_k(s) = \left(\frac{2}{|\lambda|}\right)^{-1/2} L^{(0)}_{-k-p+n-1}\left(\frac{s|\lambda|}{2}\right) e^{-\frac{s|\lambda|}{4}} \right\}_{k<q}
$$

are orthonormal bases of $L^2(0,\infty)$. It follows from the remark above that

$$
(4) \qquad \mathscr{F}(f)(\lambda,k) = \begin{cases} \left(\dfrac{2}{|\lambda|}\right)^{1/2} \langle G_f(\cdot,\lambda), \varphi_k \rangle_{L^2(\tau)}, & \text{if } k > -p, \\ \left(\dfrac{2}{|\lambda|}\right)^{1/2} \langle G_f(\cdot,\lambda)^{\vee}, \phi_k \rangle_{L^2(\tau)}, & \text{if } k < q, \end{cases}
$$

where $\langle\cdot,\cdot\rangle_{L^2(\tau)}$ denotes the inner product in $L^2(0,\infty)$. Then

$$G_f(\tau,\lambda) = \begin{cases} \displaystyle\sum_{k>-p} \left(\frac{|\lambda|}{2}\right)^{1/2} \mathscr{F}(f)(\lambda,k)\varphi_k(\tau), & \text{if } \tau \geq 0, \\ \displaystyle\sum_{k<q} \left(\frac{|\lambda|}{2}\right)^{1/2} \mathscr{F}(f)(\lambda,k)\phi_k(\tau), & \text{if } \tau < 0, \end{cases}$$

and

$$\int_{\mathbb{R}} \sum_{k>-p} |\mathscr{F}(f)(\lambda,k)|^2 \frac{|\lambda|}{2}\, d\lambda = \int_{\mathbb{R}} \int_0^\infty |G_f(\tau,\lambda)|^2\, d\tau\, d\lambda,$$

$$\int_{\mathbb{R}} \sum_{k<q} |\mathscr{F}(f)(\lambda,k)|^2 \frac{|\lambda|}{2}\, d\lambda = \int_{\mathbb{R}} \int_0^\infty |G_f(-\tau,\lambda)|^2\, d\tau\, d\lambda.$$

This suggests to define the measures μ^+, μ^- by

$$\int_{\Sigma^+} \varphi\, d\mu^+ = \int_{\mathbb{R}} \sum_{k\geq -p+1} \varphi(\lambda,k) \frac{|\lambda|}{2}\, d\lambda, \quad \forall \varphi \in C_c(\Sigma^+),$$

$$\int_{\Sigma^-} \varphi\, d\mu^- = \int_{\mathbb{R}} \sum_{k\leq q-1} \varphi(\lambda,k) \frac{|\lambda|}{2}\, d\lambda, \quad \forall \varphi \in C_c(\Sigma^-).$$

Moreover, given f and g Schwartz functions on H_n we have that

$$\langle \mathscr{F}(f), \mathscr{F}(g)\rangle_{\Sigma^+} = \langle G_f, G_g\rangle_{\mathbb{R}_{\geq 0}\times\mathbb{R}}, \tag{5}$$

$$\langle \mathscr{F}(f), \mathscr{F}(g)\rangle_{\Sigma^-} = \langle G_f, G_g\rangle_{\mathbb{R}_{<0}\times\mathbb{R}}. \tag{6}$$

2.2. The operators $M^\pm$ and $\mathscr{M}^\pm$

Given $f \in \mathscr{S}(H_n)$, by the definition of the spherical transform of f, we have that

$$\mathscr{F}(f)(\lambda,k) = I(\lambda,k) + \sum_{j=0}^{n-2} c_{j,k} |\lambda|^{n-j-2} \langle \delta^{(j)}, Nf(\cdot,\hat{\lambda})\rangle,$$

where

$$I(\lambda,k)=\begin{cases}(-|\lambda|)^{n-1}\displaystyle\int_{s>0}e^{-\frac{s|\lambda|}{4}}L_{k-q}^{(n-1)}\left(\frac{s|\lambda|}{2}\right)Nf(s,\hat{\lambda})\,ds, & \text{if } k\geq q,\\[2ex](-|\lambda|)^{n-1}\displaystyle\int_{s>0}e^{-\frac{s|\lambda|}{4}}L_{-k-p}^{(n-1)}\left(\frac{s|\lambda|}{2}\right)Nf(-s,\hat{\lambda})\,ds, & \text{if } k\leq -p,\\[2ex]0, & \text{if } -p<k<q,\end{cases}$$

and

$$c_{j,k}=\begin{cases}4^j\displaystyle\sum_{i=j}^{n-2}\frac{1}{2^i}\binom{i}{j}(L_{k-q+n-1}^{(0)})^{(n-i-2)}(0), & \text{for } k\geq 0,\\[2ex](-1)^{n-2-j}4^j\displaystyle\sum_{i=j}^{n-2}\frac{1}{2^i}\binom{i}{j}(L_{-k-p+n-1}^{(0)})^{(n-i-2)}(0), & \text{for } k<0.\end{cases}$$

Let $\mathscr{A}$ be the function defined on H_n by $\mathscr{A}(z,t)=-it+\frac{1}{4}B(z,z)$ and let $\mathscr{A}f(z,t)=(-it+\frac{1}{4}B(z,z))f(z,t)$.

The following results were proved in [9].

Proposition 2.2. *Let $f\in\mathscr{S}(H_n)$. Then, for $k>-p$ we have that*

$$\frac{\partial\mathscr{F}(f)}{\partial\lambda}(\lambda,k)=\begin{cases}\dfrac{(k-q+n-1)}{\lambda}[\mathscr{F}(f)(\lambda,k)\\ \quad-\mathscr{F}(f)(\lambda,k-1)]-\mathscr{F}(\overline{\mathscr{A}}f)(\lambda,k), & \text{if } \lambda>0,\\[2ex]\dfrac{(k-q+n-1)}{\lambda}[\mathscr{F}(f)(\lambda,k)\\ \quad-\mathscr{F}(f)(\lambda,k-1)]+\mathscr{F}(\mathscr{A}f)(\lambda,k), & \text{if } \lambda<0.\end{cases}$$

and for $k<q$ we have that

$$\frac{\partial\mathscr{F}(f)}{\partial\lambda}(\lambda,k)=\begin{cases}\dfrac{(-k-p+n-1)}{\lambda}[\mathscr{F}(f)(\lambda,k)\\ \quad-\mathscr{F}(f)(\lambda,k+1)]-\mathscr{F}(\overline{\mathscr{A}}f)(\lambda,k), & \text{if } \lambda>0,\\[2ex]\dfrac{(-k-p+n-1)}{\lambda}[\mathscr{F}(f)(\lambda,k)\\ \quad-\mathscr{F}(f)(\lambda,k+1)]+\mathscr{F}(\mathscr{A}f)(\lambda,k), & \text{if } \lambda<0.\end{cases}$$

PROPOSITION 2.3. *Let $f \in \mathcal{S}(H_n)$. If $\lambda \neq 0$ and $k > -p$, then*

$$-\frac{(k-q+n-1)}{|\lambda|}[\mathcal{F}(f)(\lambda,k)-\mathcal{F}(f)(\lambda,k-1)]+\mathcal{F}(\mathcal{A}f)(\lambda,k)$$

$$=-\frac{(k-q+1)}{|\lambda|}[\mathcal{F}(f)(\lambda,k+1)-\mathcal{F}(f)(\lambda,k)]-\mathcal{F}(\overline{\mathcal{A}}f)(\lambda,k),$$

and if $\lambda \neq 0$ and $k < q$, then

$$-\frac{(-k-p+n-1)}{|\lambda|}[\mathcal{F}(f)(\lambda,k)-\mathcal{F}(f)(\lambda,k+1)]+\mathcal{F}(\mathcal{A}f)(\lambda,k)$$

$$=-\frac{(-k-p+1)}{|\lambda|}[\mathcal{F}(f)(\lambda,k-1)-\mathcal{F}(f)(\lambda,k)]-\mathcal{F}(\overline{\mathcal{A}}f)(\lambda,k).$$

The proposition above justifies the following definition.

DEFINITION 2.4. Let F be a function defined on Σ^+, we define the functions M^+F and M^-F on $\Sigma^+ \setminus \{(0,s) \in \mathbb{R}^2 : s \geq 0\}$ by

$$M^+F(\lambda,k)=\begin{cases}\dfrac{\partial F}{\partial\lambda}(\lambda,k)-\dfrac{(k-q+1)}{\lambda}\\ \qquad\times[F(\lambda,k+1)-F(\lambda,k)], & \text{if } \lambda>0,\\ \dfrac{\partial F}{\partial\lambda}(\lambda,k)-\dfrac{(k-q+n-1)}{\lambda}\\ \qquad\times[F(\lambda,k)-F(\lambda,k-1)], & \text{if } \lambda<0.\end{cases}$$

$$M^-F(\lambda,k)=\begin{cases}\dfrac{\partial F}{\partial\lambda}(\lambda,k)-\dfrac{(k-q+n-1)}{\lambda}\\ \qquad\times[F(\lambda,k)-F(\lambda,k-1)], & \text{if } \lambda>0,\\ \dfrac{\partial F}{\partial\lambda}(\lambda,k)-\dfrac{(k-q+1)}{\lambda}\\ \qquad\times[F(\lambda,k+1)-F(\lambda,k)], & \text{if } \lambda<0.\end{cases}$$

DEFINITION 2.5. Let F be a function defined on Σ^-, we define the functions

$\mathcal{M}^+F$ and $\mathcal{M}^-F$ on $\Sigma^- \setminus \{(0,s) \in \mathbb{R}^2 : s \leq 0\}$ by

$$\mathcal{M}^+F(\lambda,k) = \begin{cases} \dfrac{\partial F}{\partial\lambda}(\lambda,k) - \dfrac{(-k-p+1)}{\lambda} \\ \qquad \times [F(\lambda,k-1) - F(\lambda,k)], & \text{if } \lambda > 0, \\ \dfrac{\partial F}{\partial\lambda}(\lambda,k) - \dfrac{(-k-p+n-1)}{\lambda} \\ \qquad \times [F(\lambda,k) - F(\lambda,k+1)], & \text{if } \lambda < 0. \end{cases}$$

$$\mathcal{M}^-F(\lambda,k) = \begin{cases} \dfrac{\partial F}{\partial\lambda}(\lambda,k) - \dfrac{(-k-p+n-1)}{\lambda} \\ \qquad \times [F(\lambda,k) - F(\lambda,k+1)], & \text{if } \lambda > 0, \\ \dfrac{\partial F}{\partial\lambda}(\lambda,k) - \dfrac{(-k-p+1)}{\lambda} \\ \qquad \times [F(\lambda,k-1) - F(\lambda,k)], & \text{if } \lambda < 0. \end{cases}$$

From now on, we will deal only with the restriction of the spherical transform to Σ^+, since the other case follows in a similar way.

From the propositions above it follows that the operators $M^\pm$ have the following relevant property,

$$\mathcal{F}(\mathcal{A}f) = M^+(\mathcal{F}(f)) \quad \text{and} \quad \mathcal{F}(\overline{\mathcal{A}}f) = -M^-(\mathcal{F}(f)). \tag{7}$$

Moreover, a single integration by parts shows that, for $F = \mathcal{F}(f)$ and $G = \mathcal{F}(g)$,

$$\begin{aligned} &\int_{\Sigma^+} M^+F(\lambda,k)G(\lambda,k)\,d\mu^+ \\ &= \sum_{k>-p}\int_0^\infty \left\{\frac{\partial F}{\partial\lambda}(\lambda,k) - \frac{k-q+1}{\lambda}[F(\lambda,k+1)-F(\lambda,k)]\right\}\overline{G(\lambda,k)}\frac{\lambda}{2}\,d\lambda \\ &\quad - \sum_{k>-p}\int_{-\infty}^0 \left\{\frac{\partial F}{\partial\lambda}(\lambda,k) - \frac{k-q+n-1}{\lambda} \right. \\ &\qquad\qquad \left. \times [F(\lambda,k)-F(\lambda,k-1)]\right\}\overline{G(\lambda,k)}\frac{\lambda}{2}\,d\lambda \\ &= -\sum_{k>-p}\int_0^\infty \overline{\left\{\frac{\partial G}{\partial\lambda}(\lambda,k) - \frac{(k-q+1)}{\lambda}\right.} \\ &\qquad\qquad \overline{\left.\times[G(\lambda,k+1)-G(\lambda,k)]\right\}}F(\lambda,k)\frac{\lambda}{2}\,d\lambda \end{aligned}$$

$$+\sum_{k>-p}\int_{-\infty}^{0}\overline{\left\{\frac{\partial G}{\partial\lambda}(\lambda,k)-\frac{(k-q+n-1)}{\lambda}\right.}$$
$$\overline{\left.\times[G(\lambda,k)-G(\lambda,k-1)]\right\}}F(\lambda,k)\frac{\lambda}{2}\,d\lambda$$
$$=-\int_{\Sigma^+}F(\lambda,k)M^-G(\lambda,k)\,d\mu^+.$$

Thus,

$$\langle M^+\mathscr{F}(f),\mathscr{F}(g)\rangle_{\Sigma^+}=-\langle\mathscr{F}(f),M^-\mathscr{F}(g)\rangle_{\Sigma^+}. \tag{8}$$

3. A Paley-Wiener theorem

The following lemmas, propositions and theorems were motivated by [4].

LEMMA 3.1. *Let $f\in\mathscr{S}(H_n)$ then*

$$\|\mathscr{F}(f)\|_{L^\infty(\Sigma^+)}\le C\max_{0\le j\le n-1}\int_0^\infty\int_{-\infty}^{\infty}\left|\frac{\partial^{n-1}Nf}{\partial\tau^j\partial t^{n-1-j}}(\tau,t)\right|dt\,d\tau$$

and

$$\|\mathscr{F}(f)\|_{L^\infty(\Sigma^-)}\le C\max_{0\le j\le n-1}\int_0^\infty\int_{-\infty}^{\infty}\left|\frac{\partial^{n-1}Nf}{\partial\tau^j\partial t^{n-1-j}}(-\tau,t)\right|dt\,d\tau$$

PROOF. This follows from (4) and

$$|G_f(\tau,\lambda)|\le\sum_{j=0}^{n-1}c_j\left|\frac{\partial^{n-1}Nf}{\partial\tau^j\partial t^{n-1-j}}(\tau,\hat{\lambda})\right|\le\sum_{j=0}^{n-1}c_j\int_{\mathbb{R}}\left|\frac{\partial^{n-1}Nf}{\partial\tau^j\partial t^{n-1-j}}(\tau,t)\right|dt.$$

LEMMA 3.2. *Let A be the function defined on $\mathbb{R}^2$ by $A(\tau,t)=(\tau/4)-it$ and let U be the operator $\tau\frac{\partial^2}{\partial t^2}+D$. Then*

$$\left|U^i\big(A^j(\tau,t)\big)\right|\le C_i\,j^{2i}\left|A^{j-i}(\tau,t)\right|,\qquad\forall i\in\mathbb{N},\ j\in\mathbb{Z}.$$

PROOF. We will prove the lemma by induction on i. For $i=1$, we have that

$$U\big(A^j(\tau,t)\big)=A^{j-2}(\tau,t)\big(-\tau j(j-1)+\tfrac{1}{4}\tau j(j-1)+\tfrac{1}{4}(2-n)jA(\tau,t)\big).$$

Then

$$\left|U\big(A^j(\tau,t)\big)\right|\le C_1 j^2\left|A^{j-1}(\tau,t)\right|.$$

For all $r < i$, suppose that $\left|U^r\left(A^j(\tau,t)\right)\right| \le C_r j^{2r}\left|A^{j-r}(\tau,t)\right|$. Then,

$$\begin{aligned}
&U^i\left(A^j(\tau,t)\right)\\
&\quad= U^{i-1}\left(U\left(A^j(\tau,t)\right)\right)\\
&\quad= \sum_{\ell=0}^{i-1}\binom{i-1}{\ell}U^{i-1-\ell}\left(A^{j-2}(\tau,t)\right)\\
&\qquad\times U^{\ell}\left(-\tau j(j-1)+\tfrac14\tau j(j-1)+\tfrac14(2-n)jA(\tau,t)\right)\\
&\quad= \binom{i-1}{1}U^{i-2}\left(A^{j-2}(\tau,t)\right)U\left(-\tau j(j-1)+\tfrac14\tau j(j-1)+\tfrac14(2-n)jA(\tau,t)\right)\\
&\qquad+\binom{i-1}{0}U^{i-1}\left(A^{j-2}(\tau,t)\right)\left(-\tau j(j-1)+\tfrac14\tau j(j-1)+\tfrac14(2-n)jA(\tau,t)\right)\\
&\quad= (i-1)U^{i-2}\left(A^{j-2}(\tau,t)\right)4(2-n)\left(-j(j-1)+\tfrac14 j(j-1)+\tfrac{1}{4^2}(2-n)j\right)\\
&\qquad+U^{i-1}\left(A^{j-2}(\tau,t)\right)\left(-\tau j(j-1)+\tfrac14\tau j(j-1)+\tfrac14(2-n)jA(\tau,t)\right).
\end{aligned}$$

So,

$$\begin{aligned}
\left|U^i(A^j(\tau,t))\right| &\le C_{i-2}j^2(j-2)^{2(i-2)}\left|A^{j-i}(\tau,t)\right|\\
&\qquad + C_{i-1}j^2(j-2)^{2(i-1)}\left|A^{j-i}(\tau,t)\right|\\
&\le C_i j^{2i}\left|A^{j-i}(\tau,t)\right|.
\end{aligned}$$

LEMMA 3.3. *Let $m > 2$ a natural number. Then for $1 \le p < \infty$, the map $(\lambda,\xi)\mapsto(i+\xi)^{-m}$ is in $L^p(\Sigma^+)$.*

PROOF. Let $j > 2$ and let $\varphi(\lambda,\xi) = (i+\xi)^{-j}$. Then

$$\begin{aligned}
&\int_{\mathbb{R}}\sum_{k\ge -p+1}|\varphi(\lambda,(2k+p-q)|\lambda|)|\frac{|\lambda|}{2}\,d\lambda\\
&\quad=\int_{\mathbb{R}}\sum_{k\ge -p+1}\frac{1}{(1+(2k+p-q)^2|\lambda|^2)^{j/2}}\frac{|\lambda|}{2}\,d\lambda\\
&\quad=\sum_{k\ge -p+1}\int_0^{1/|2k+p-q|}\frac{\lambda}{(1+(2k+p-q)^2\lambda^2)^{j/2}}\,d\lambda\\
&\qquad+\sum_{k\ge -p+1}\int_{1/|2k+p-q|}^{\infty}\frac{|\lambda|}{(1+(2k+p-q)^2\lambda^2)^{j/2}}\,d\lambda
\end{aligned}$$

$$\leq \sum_{k\geq -p+1}\int_0^{1/|2k+p-q|}\lambda\,d\lambda+\sum_{k\geq -p+1}\int_{1/|2k+p-q|}^{\infty}\frac{\lambda}{(|2k+p-q|\lambda)^j}\,d\lambda$$
$$=\sum_{k\geq -p+1}\frac{1}{|2k+p-q|^2}+\sum_{k\geq -p+1}\frac{1}{j-2}\frac{1}{|2k+p-q|^2}$$
$$<\infty.$$

For $f\in\mathscr{S}(H_n)$, let R_f be defined by

$$R_f=\sup_{\tau\geq 0,\,t\in\mathbb{R}}\{\,|(\tau,t)|:Nf(\tau,t)\neq 0\,\}.$$

PROPOSITION 3.4. *Let f be a Schwartz function on H_n. Then for every $\ell\geq 0$ and every p in* $[1,\infty]$*, we have*

$$\limsup_{j\to\infty}\big\|(i+\xi)^{\ell}(M^+)^j\mathscr{F}(f)\big\|_{L^p(\Sigma^+)}^{1/j}\leq R_f.$$

PROOF. We can suppose that $0<R_f<\infty$, since in the other cases the conclusion is trivial. By using the equalities (2) and (7), we have that

$$\xi^{\ell}(M^+)^j\mathscr{F}(f)(\lambda,\xi)=\mathscr{F}(\mathscr{D}^{\ell}\mathscr{A}^jf)(\lambda,\xi).$$

On the other hand, let $f_{\ell,j}\in\mathscr{S}(H_n)$ be such that $Nf_{\ell,j}=U^{\ell}N(\mathscr{A}^jf)$. Then $\mathscr{F}(\mathscr{D}^{\ell}\mathscr{A}^jf)=\mathscr{F}(f)_{\ell,j}$ by (1). So, by Lemma 3.1 we have

$$\big\|\xi^{\ell}(M^+)^j\mathscr{F}(f)(\lambda,\xi)\big\|_{L^\infty(\Sigma^+)}\leq C\max_{0\leq r\leq n-1}\int_0^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial^{n-1}Nf_{\ell,j}}{\partial\tau^r\partial t^{n-1-r}}(\tau,t)\right|dt\,d\tau.$$

Using $A(\tau,t)=(\tau/4)-it$ and $N(\mathscr{A}f)=ANf$, we then have

$$\left|\frac{\partial^{n-1}Nf_{\ell,j}}{\partial\tau^r\partial t^{n-1-r}}(\tau,t)\right|$$
$$=\left|U^{\ell}\frac{\partial^{n-1}(A^jNf)}{\partial\tau^r\partial t^{n-1-r}}(\tau,t)\right|$$
$$=\left|U^{\ell}\sum_{s=0}^{n-1-r}\binom{n-1-r}{s}\frac{\partial^r}{\partial\tau^r}\left(\frac{\partial^{n-1-r-s}A^j}{\partial t^{n-1-r-s}}\frac{\partial^sNf}{\partial t^s}\right)\right|$$
$$=\left|U^{\ell}\sum_{s=0}^{n-1-r}\binom{n-1-r}{s}\sum_{m=0}^{r}\binom{r}{m}\frac{\partial^{n-1-s-m}A^j}{\partial\tau^{r-m}\partial t^{n-1-r-s}}\frac{\partial^{s+m}Nf}{\partial\tau^m\partial t^s}\right|$$

$$
\begin{aligned}
&= \left| U^\ell \sum_{s=0}^{n-1-r} \sum_{m=0}^{r} c_{r,m,s} A^{j-(n-1-m-s)}(\tau,t) \frac{\partial^{s+m} Nf}{\partial\tau^m \partial t^s}(\tau,t) \right| \\
&= \left| U^\ell \sum_{s=0}^{n-1-r} \sum_{m=0}^{r} c_{r,m,s} A^{j+m+s}(\tau,t) \frac{\partial^{n-1-s-m} Nf}{\partial\tau^m \partial t^s}(\tau,t) \right| \\
&= \left| \sum_{i=0}^{\ell} \sum_{s=0}^{n-1-r} \sum_{m=0}^{r} c_{\ell,i,r,m,s} U^i A^{j+m+s}(\tau,t) U^{\ell-i} \frac{\partial^{n-1-s-m} Nf}{\partial\tau^m \partial t^s}(\tau,t) \right| \\
&\le \sum_{i=0}^{\ell} \sum_{s=0}^{n-1-r} \sum_{m=0}^{r} c_{\ell,i,r,m,s} \left| U^i A^{j+m+s}(\tau,t) U^{\ell-i} \frac{\partial^{n-1-s-m} Nf}{\partial\tau^m \partial t^s}(\tau,t) \right| \\
&\le \sum_{i=0}^{\ell} \sum_{s=0}^{n-1-r} \sum_{m=0}^{r} j^{2i} d_{\ell,j,m,s,i} \left| A^{j+m+s-i}(\tau,t) U^{\ell-i} \frac{\partial^{n-1-s-m} Nf}{\partial\tau^m \partial t^s}(\tau,t) \right| \\
&\le j^{2\ell} \sum_{i=0}^{\ell} \sum_{s=0}^{n-1-r} \sum_{m=0}^{r} d_{\ell,j,m,s,i} R_f^{j+m+s-i} \left| U^{\ell-i} \frac{\partial^{n-1-s-m} Nf}{\partial\tau^m \partial t^s}(\tau,t) \right|.
\end{aligned}
$$

Using that Nf on $\mathbb{R}_{>0} \times \mathbb{R}$ is a rapidly decreasing function, we get that

$$
\left\| \xi^\ell (M^+)^j \mathscr{F}(f)(\lambda,\xi) \right\|_{L^\infty(\Sigma^+)} \le C_{f,\ell}\, j^{2\ell} R_f^j. \tag{9}
$$

We note that for a sufficiently large integer m, the function $(\lambda,\xi) \mapsto (i+\xi)^{-m}$ is in $L^p(\Sigma^+)$, so that

$$
\begin{aligned}
&\left\| (i+\xi)^\ell (M^+)^j \mathscr{F}(f) \right\|_{L^p(\Sigma^+)} \\
&\quad \le C \left\| (i+\xi)^{m+\ell} (M^+)^j \mathscr{F}(f) \right\|_{L^\infty(\Sigma^+)} \\
&\quad = C \left\| \sum_{r=0}^{m+\ell} \binom{m+\ell}{r} i^{m+\ell-r} \xi^r (M^+)^j \mathscr{F}(f) \right\|_{L^\infty(\Sigma^+)} \\
&\quad \le C_{m,h} \sum_{r=0}^{m+\ell} \binom{m+\ell}{\ell} \left\| \xi^r (M^+)^j \mathscr{F}(f) \right\|_{L^\infty(\Sigma^+)} \\
&\quad \le C_{f,m,\ell} \big(1 + j^{2(m+\ell)}\big) R_f^j,
\end{aligned}
$$

where for the last equality we have used (9).

Conversely, we can prove that a function $Nf|_{\mathbb{R}_{\ge 0}\times\mathbb{R}}$ is compactly supported when a certain limit is finite.

The following Lemma is a straightforward adaptation of Lemma 7.2 in [4].

LEMMA 3.5. *Let $R > 0$ and let j be a positive integer. Let $f \in \mathscr{S}(H_n)$ be such that $Nf|_{\mathbb{R}_{\geq 0}\times\mathbb{R}}$ is a function with support in the set $\{ (\tau, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : |(\tau, t)| > R \}$ and let $f_j = \overline{\mathscr{A}}^{-j} f$.*

Then for every N in $\mathbb{N}$, we have

$$\left\| (i+\xi)^N \mathscr{F}(f_j) \right\|_{L^\infty(\Sigma^+)} \leq C_{N,f}\, j^{2N} R^{-j}.$$

PROOF. We recall that $N(\mathscr{A} f) = ANf$. Then, as Nf is supported away from the origin, the function $Nf_j = \overline{A}^{-j} Nf$ is again smooth and compactly supported. Moreover,

$$\begin{aligned}
\left\| (i+\xi)^N \mathscr{F}(f_j) \right\|_{L^\infty(\Sigma^+)} &= \left\| \mathscr{F}((iI+\mathscr{D})^N f_j) \right\|_{L^\infty(\Sigma^+)} \\
&= \left\| \sum_{\ell=0}^{N} i^\ell \mathscr{F}(\mathscr{D}^\ell f_j) \right\|_{L^\infty(\Sigma^+)} \\
&= \left\| \sum_{\ell=0}^{N} i^\ell \mathscr{F}(f_{\ell,j}) \right\|_{L^\infty(\Sigma^+)},
\end{aligned}$$

where $f_{\ell,j}$ is the function such that $Nf_{\ell,j} = U^\ell N f_j$. Then

$$\begin{aligned}
&\left\| (i+\xi)^N \mathscr{F}(f_j) \right\|_{L^\infty(\Sigma^+)} \\
&\quad \leq C_N \max_{\ell \leq N} \max_{0 \leq r \leq n-1} \int_0^\infty \int_{-\infty}^\infty \left| \frac{\partial^{n-1} N(f_{\ell,r})}{\partial \tau^j \partial t^{n-1-r}}(\tau,t) \right| dt\, d\tau \\
&\quad \leq C_N \max_{\ell \leq N} \max_{0 \leq r \leq n-1} \int_0^\infty \int_{-\infty}^\infty \left| \frac{\partial^{n-1} U^\ell N(f_j)}{\partial \tau^r \partial t^{n-1-r}}(\tau,t) \right| dt\, d\tau \\
&\quad \leq C_N \max_{\ell \leq N} \max_{0 \leq r \leq n-1} \int_0^\infty \int_{-\infty}^\infty \left| \frac{\partial^{n-1} U^\ell N(\overline{\mathscr{A}}^{-j} f)}{\partial \tau^r \partial t^{n-1-r}}(\tau,t) \right| dt\, d\tau \\
&\quad \leq C_N \max_{\ell \leq N} \max_{0 \leq r \leq n-1} \int_0^\infty \int_{-\infty}^\infty \left| \frac{\partial^{n-1} U^\ell \overline{A}^{-j} Nf}{\partial \tau^r \partial t^{n-1-r}}(\tau,t) \right| dt\, d\tau.
\end{aligned}$$

So, by using Lemma 3.2 as in Proposition 3.4, we have that

$$\left\| (i+\xi)^N \mathscr{F}(f_j) \right\|_{L^\infty(\Sigma^+)} \leq C_{N,f}\, j^{2N} R^{-j}.$$

LEMMA 3.6. *Let $f \in \mathscr{S}(H_n)$ such that $Nf(\tau, t) = 0$ for all $|(\tau, t)| > R_2$. If $\langle G_f, G_g \rangle_{\mathbb{R}_{\geq 0}\times\mathbb{R}} = 0$ for each $g \in \mathscr{S}(H_n)$ such that $Ng|_{\mathbb{R}_{\geq 0}\times\mathbb{R}}$ has compact support on $C = \{ (\tau, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : R_1 < |(\tau, t)| < R_2 \}$, then $Nf \equiv 0$ on C.*

PROOF. Without loss of generality we can assume that Nf is a real-valued function on $\mathbb{R}_{\geq 0} \times \mathbb{R}$. By (3), we have for $(\tau, \lambda) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$

$$\begin{aligned} G_f(\tau, \lambda) &= (i\, \mathrm{sgn}(\lambda))^{n-1} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j 4^j i^{-j}\, \mathrm{sgn}(\lambda)^{-j} \frac{\partial^{n-1} Nf}{\partial \tau^j \partial t^{n-1-j}}(\tau, \hat{\lambda}) \\ &= (i\, \mathrm{sgn}(\lambda))^{n-1} \big(DNf(\tau, \hat{\lambda}) - i\, \mathrm{sgn}(\lambda) \tilde{D} Nf(\tau, \hat{\lambda}) \big), \end{aligned}$$

where the differential operators D and $\tilde{D}$ are defined by

$$D = \sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1}{2\ell} 4^{2\ell} (-1)^\ell \frac{\partial^{n-1}}{\partial \tau^{2\ell} \partial t^{n-1-2\ell}}$$

and

$$\tilde{D} = \sum_{\ell=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-1}{2\ell+1} 4^{2\ell+1} (-1)^{\ell+1} \frac{\partial^{n-1}}{\partial \tau^{2\ell+1} \partial t^{n-1-2\ell-1}}.$$

Then,

$$\begin{aligned} &\langle G_f, G_g \rangle \\ &= \int_0^\infty \int_{-\infty}^\infty G_f(\tau, \lambda) \overline{G_g(\tau, \lambda)}\, d\lambda\, d\tau \\ &= \int_0^\infty \int_{-\infty}^\infty DNf(\tau, \hat{\lambda}) DNg(\tau, \hat{\lambda}) + \tilde{D}Nf(\tau, \hat{\lambda}) \tilde{D}Ng(\tau, \hat{\lambda})\, d\lambda\, d\tau \\ &\quad + i \int_0^\infty \int_{-\infty}^\infty \mathrm{sgn}(\lambda) \big(DNf(\tau, \hat{\lambda}) \tilde{D}Ng(\tau, \hat{\lambda}) - \tilde{D}Nf(\tau, \hat{\lambda}) DNg(\tau, \hat{\lambda}) \big)\, d\lambda\, d\tau \\ &= \int_0^\infty \int_{-\infty}^\infty DNf(\tau, t) DNg(\tau, t) + \tilde{D}Nf(\tau, t) \tilde{D}Ng(\tau, t)\, dt\, d\tau \\ &\quad + i \int_0^\infty \int_{-\infty}^\infty \mathrm{sgn}(\lambda) \big(DNf(\tau, \hat{\lambda}) \tilde{D}Ng(\tau, \hat{\lambda}) - \tilde{D}Nf(\tau, \hat{\lambda}) DNg(\tau, \hat{\lambda}) \big)\, d\lambda\, d\tau, \end{aligned}$$

where in the last equality we have used the Plancherel identity. So, if $\langle G_f, G_g \rangle = 0$ then $\int_0^\infty \int_{-\infty}^\infty (DNf\, DNg + \tilde{D}Nf\, \tilde{D}Ng)(\tau, t)\, dt\, d\tau = 0$.

Let $\varepsilon > 0$ and $C_\varepsilon = \{ (\tau, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : R_1 + \varepsilon < |(\tau, t)| < R_2 - \varepsilon \} \subset C$. Let φ_ε be the function defined by $\varphi_\varepsilon(\tau, t) = Nf(\tau, t) \chi_\varepsilon(\tau, t)$, where χ_ε is a Schwartz function such that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon \equiv 1$ on C_ε and $\chi_\varepsilon \equiv 0$ on C^c. Clearly, there exists $g_\varepsilon \in \mathcal{S}(H_n)$ such that $Ng_\varepsilon = \varphi_\varepsilon$, since $\varphi_\varepsilon \in \mathcal{H}^\sharp$.

Moreover, $Ng_\varepsilon \to Nf|_C$, when $\varepsilon \to 0$, and $\langle G_f, G_{g_\varepsilon}\rangle = 0$, for all $\varepsilon > 0$. We then have that

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\int_0^\infty\int_{-\infty}^\infty DNf(\tau,t)DNg_\varepsilon(\tau,t) + \tilde{D}Nf(\tau,t)\tilde{D}Ng_\varepsilon(\tau,t)\,dt\,d\tau \\
&= \lim_{\varepsilon\to 0}\int_C DNf(\tau,t)DNg_\varepsilon(\tau,t) + \tilde{D}Nf(\tau,t)\tilde{D}Ng_\varepsilon(\tau,t)\,dt\,d\tau \\
&= \int_C DNf(\tau,t)DNf(\tau,t) + \tilde{D}Nf(\tau,t)\tilde{D}Nf(\tau,t)\,dt\,d\tau \\
&= \int_C \big(DNf(\tau,t)\big)^2 + \big(\tilde{D}Nf(\tau,t)\big)^2\,dt\,d\tau.
\end{aligned}$$

So,

$$\sum_{\ell=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1}{2\ell}4^{2\ell}(-1)^\ell\frac{\partial^{n-1}Nf}{\partial\tau^{2\ell}\partial t^{n-1-2\ell}}(\tau,t) = 0 \tag{10}$$

and

$$\sum_{\ell=0}^{\lfloor (n-2)/2\rfloor}\binom{n-1}{2\ell+1}4^{2\ell+1}(-1)^{\ell+1}\frac{\partial^{n-1}Nf}{\partial\tau^{2\ell+1}\partial t^{n-1-2\ell-1}}(\tau,t) = 0, \tag{11}$$

for all (τ,t) such that $|(\tau,t)| > R_1$.

Let $\Omega = \{\,(\tau,t) : |(4\tau,t)| > R_1\,\}$ and $\varphi(\tau,t) = Nf(4\tau,t)$. Then, by the Leibniz rule, we have

$$\begin{aligned}
&S\varphi(\tau,t) \\
&\quad:= \left(\frac{\partial}{\partial\tau} - i\frac{\partial}{\partial t}\right)^{n-1}\varphi(\tau,t) \\
&\quad= \sum_{j=0}^{n-1}\binom{n-1}{j}(-i)^{n-1-j}\frac{\partial^{n-1}\varphi}{\partial\tau^j\partial t^{n-1-j}}(\tau,t) \\
&\quad= (-i)^{n-1}\sum_{\ell=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1}{2\ell}4^{2\ell}(-1)^\ell\frac{\partial^{n-1}Nf}{\partial\tau^{2\ell}\partial t^{n-1-2\ell}}(4\tau,t) \\
&\qquad - (-i)^{n-1}i\sum_{\ell=0}^{\lfloor (n-2)/2\rfloor}\binom{n-1}{2\ell+1}4^{2\ell+1}(-1)^{\ell+1}\frac{\partial^{n-1}Nf}{\partial\tau^{2\ell+1}\partial t^{n-1-2\ell-1}}(4\tau,t) \\
&\quad= (-i)^{n-1}\big(DNf - i\tilde{D}Nf\big)(4\tau,t).
\end{aligned}$$

By (10) and (11), $S\varphi(\tau,t)=0$ for all $(\tau,t)\in\Omega$.

Now, we will prove by induction that

$$\left(\frac{\partial}{\partial\tau}-i\frac{\partial}{\partial t}\right)^{n-1}\varphi(\tau,t)=0\ \forall(\tau,t)\in\Omega \implies \varphi(\tau,t)=0\ \forall(\tau,t)\in\Omega.$$

In fact, if $n=2$ the proof is trivial. Suppose $n>2$. Let φ_1,φ_2 be functions in $C^2(\Omega)$ such that $\varphi_1+i\varphi_2=\left(\frac{\partial}{\partial\tau}-i\frac{\partial}{\partial t}\right)^{n-2}\varphi$. Then, we have

$$\left(\frac{\partial}{\partial\tau}-i\frac{\partial}{\partial t}\right)(\varphi_1+\varphi_2)(\tau,t)=0,\quad \forall(\tau,t)\in\Omega,$$

hence $\frac{\partial\varphi_1}{\partial\tau}=-\frac{\partial\varphi_2}{\partial t}$ and $\frac{\partial\varphi_2}{\partial\tau}=\frac{\partial\varphi_1}{\partial t}$. So, $(-i)\left(\frac{\partial}{\partial\tau}-i\frac{\partial}{\partial t}\right)^{n-2}\varphi$ is a holomorphic function on Ω and it vanishes for $|(\tau,t)|>R_2$. Therefore,

$$\left(\frac{\partial}{\partial\tau}-i\frac{\partial}{\partial t}\right)^{n-2}\varphi\equiv 0$$

on Ω. By the inductive hypothesis $\varphi\equiv 0$ on Ω, so

$$Nf(\tau,t)=0,\ \forall|(\tau,t)|>R_1.$$

Finally, if Nf is not a real-valued function, then there exist f_1 and f_2 in $\mathscr{S}(H_n)$ such that $Nf=Nf_1+iNf_2$. So, if $\langle G_f,G_g\rangle_{\mathbb{R}_{\geq 0}\times\mathbb{R}}=0$ for all $g\in\mathscr{S}(H_n)$ such that $Ng|_{\mathbb{R}_{\geq 0}\times\mathbb{R}}$ has compact support on C, then let $\overline{f}\in\mathscr{S}(H_n)$ be such that $N\overline{f}=\overline{Nf}$. It follows that

$$\langle G_{\overline{f}},G_g\rangle_{\mathbb{R}_{\geq 0}\times\mathbb{R}}=\langle\overline{G_f},G_g\rangle_{\mathbb{R}_{\geq 0}\times\mathbb{R}}=\overline{\langle G_f,\overline{G_g}\rangle}_{\mathbb{R}_{\geq 0}\times\mathbb{R}}=\overline{\langle G_f,G_{\overline{g}}\rangle}_{\mathbb{R}_{\geq 0}\times\mathbb{R}}=0,$$

for all $g\in\mathscr{S}(H_n)$ such that $Ng|_{\mathbb{R}_{\geq 0}\times\mathbb{R}}$ has compact support on C. So, by linearity of G, we have

$$\langle G_{f_1},G_g\rangle_{\mathbb{R}_{\geq 0}\times\mathbb{R}}=0\quad\text{and}\quad\langle G_{f_2},G_g\rangle_{\mathbb{R}_{\geq 0}\times\mathbb{R}}=0,$$

for all $g\in\mathscr{S}(H_n)$ such that $Ng|_{\mathbb{R}_{\geq 0}\times\mathbb{R}}$ has compact support on C. Therefore, $Nf_1\equiv 0\equiv Nf_2$ on C.

PROPOSITION 3.7. *Let f be in $\mathscr{S}(H_n)$. Then, for every N in $\mathbb{N}$ and every p in $[1,\infty]$, we have*

$$\liminf_{j\to\infty}\left\|(i+\xi)^{-N}(M^+)^j\mathscr{F}(f)\right\|_{L^p(\Sigma^+)}^{1/j}\geq R_f.$$

PROOF. Suppose that $R_f > 0$ and let $0 < \varepsilon < R_f/2$. Then we may find a function $g \in \mathscr{S}(H_n)$ such that the smooth function Ng has compact support in the set

$$\{ (\tau, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : R_f - \varepsilon < |(\tau, t)| < R_f + \varepsilon \}$$

and $\langle G_f, G_g \rangle_{\mathbb{R}_{\geq 0} \times \mathbb{R}} \neq 0$. The function Ng is supported away from the origin and we let $g_j = \overline{\mathscr{A}}^{-j} g$. By (5) and (7), we have

$$\begin{aligned}
\big|\langle G_f, G_g \rangle_{\mathbb{R}_{\geq 0} \times \mathbb{R}}\big| &= \big|\langle G_f, G_{\overline{\mathscr{A}}^j \overline{\mathscr{A}}^{-j} g} \rangle_{\mathbb{R}_{\geq 0} \times \mathbb{R}}\big| \\
&= \big|\langle \mathscr{F}(f), \mathscr{F}(\overline{\mathscr{A}}^j \overline{\mathscr{A}}^{-j} g)\rangle_{\Sigma^+}\big| && \text{by (5)} \\
&= \big|\langle \mathscr{F}(f), (-1)^j (M^-)^j \mathscr{F}(g_j)\rangle_{\Sigma^+}\big| && \text{by (7)} \\
&= \big|\langle (M^+)^j \mathscr{F}(f), \mathscr{F}(g_j)\rangle_{\Sigma^+}\big| && \text{by (8)} \\
&\leq \big\|(i + \xi)^{-N} (M^+)^j \mathscr{F}(f)\big\|_{L^p(\Sigma^+)} \big\|(i + \xi)^N \mathscr{F}(g_j)\big\|_{L^{p'}(\Sigma^+)}.
\end{aligned}$$

In the case where $\|(i + \xi)^{-N} (M^+)^j \mathscr{F}(f)\|_{L^p(\Sigma^+)} = \infty$ for all j, there is nothing to prove. Otherwise, since $|\langle G_f, G_g\rangle| \neq 0$, we have

$$\begin{aligned}
&\liminf_{j\to\infty} \big\|(i + \xi)^{-N} (M^+)^j \mathscr{F}(f)\big\|_{L^p(\Sigma^+)}^{1/j} \\
&\quad \geq \liminf_{j\to\infty} \left(\frac{\big|\langle G_f, G_g\rangle_{\mathbb{R}_{\geq 0}\times\mathbb{R}}\big|}{\|(i + \xi)^N \mathscr{F}(g_j)\|_{L^{p'}(\Sigma^+)}} \right)^{1/j} \\
&\quad = \liminf_{j\to\infty} \Big(\big\|(i + \xi)^N \mathscr{F}(g_j)\big\|_{L^{p'}(\Sigma^+)} \Big)^{-1/j},
\end{aligned}$$

since there exists M in $\mathbb{N}$ such that $\xi \mapsto (i + \xi)^{N-M}$ is in $L^{p'}(\Sigma^+)$. Moreover, by the previous lemma, we have that

$$\begin{aligned}
\big\|(i + \xi)^N \mathscr{F}(g_j)\big\|_{L^{p'}(\Sigma^+)} &\leq \big\|(i + \xi)^{N-M}\big\|_{L^{p'}(\Sigma^+)} \big\|(i + \xi)^M \mathscr{F}(g_j)\big\|_{L^\infty(\Sigma^+)} \\
&\leq C j^{2M} (R_f - \varepsilon)^{-j},
\end{aligned}$$

and the result follows easily.

When $R_f = \infty$, we use the same argument to show that

$$\liminf_{j\to\infty} \big\|(i + \xi)^{-N} (M^+)^j \mathscr{F}(f)\big\|_{L^p(\Sigma^+)} \geq R$$

for every $R > 0$.

THEOREM 3.8. *Let f be in $\mathscr{S}(H_n)$. The following conditions are equivalent:*

(i) *R_f is finite;*

(ii) *for every $\ell \in \mathbb{N}_0$ and every p in $[1, \infty]$, $\limsup_{j\to\infty} \left\| \xi^\ell (M^+)^j \mathscr{F}(f) \right\|_{L^p(\Sigma^+)}^{1/j}$ is finite;*

(iii) *there exists p in $[1, \infty]$ such that $\liminf_{j\to\infty} \left\| (M^+)^j \mathscr{F}(f) \right\|_{L^p(\Sigma^+)}^{1/j}$ is finite.*

Moreover, if any of these conditions is satisfied, then for every $\ell \in \mathbb{N}_0$ and every $p \in [1, \infty]$, we have

$$\lim_{j\to\infty} \left\| (i+\xi)^\ell (M^+)^j \mathscr{F}(f) \right\|_{L^p(\Sigma^+)}^{1/j} = R_f.$$

PROOF. The implication (iii) $\Rightarrow$ (i) follows from Proposition 3.7, the implication (i) $\Rightarrow$ (ii) follows from Proposition 3.4 and (ii) $\Rightarrow$ (iii) is trivial.

COROLLARY 3.9. *Let $f \in C_0^\infty(H_n)$. Let $R_f^+ = \sup_{(\tau,t)\in\mathbb{R}_{\geq 0}\times\mathbb{R}} \{ |(\tau, t)| : Nf(\tau, t) \neq 0 \}$ and $R_f^- = \sup_{(\tau,t)\in\mathbb{R}_{\leq 0}\times\mathbb{R}} \{ |(\tau, t)| : Nf(\tau, t) \neq 0 \}$. Then,*

$$\lim_{j\to\infty} \left\| (i+\xi)^\ell (M^+)^j \mathscr{F}(f) \right\|_{L^p(\Sigma^+)}^{1/j} = R_f^+,$$

and

$$\lim_{j\to\infty} \left\| (i+\xi)^\ell (\mathscr{M}^+)^j \mathscr{F}(f) \right\|_{L^p(\Sigma^+)}^{1/j} = R_f^-.$$

4. Holomorphic extensions of $\mathscr{F}(f)|_{\Sigma^+}$ and $\mathscr{F}(f)|_{\Sigma^-}$

Let $f \in \mathscr{S}(H_n)$. In this section we prove that if $Nf|_{\mathbb{R}_{\geq 0}\times\mathbb{R}}$ has compact support, then there exists a holomorphic function G on $\mathbb{C}^2$ such that $G|_{\Sigma^+} = \mathscr{F}(f)|_{\Sigma^+}$. The same proof will show that there exists a holomorphic function $\tilde{G}$ on $\mathbb{C}^2$ such that $\tilde{G}|_{\Sigma^-} = \mathscr{F}(f)|_{\Sigma^-}$, if $Nf|_{\mathbb{R}_{\leq 0}\times\mathbb{R}}$ has compact support.

As stated above, in the first section, given $f \in \mathscr{S}(H_n)$ we know that for $k \geq -p+1$,

$$\begin{aligned}
&\mathscr{F}(f)(\lambda, (2k+p-q)|\lambda|) \\
&\qquad = (-1)^{n-1} \int_0^\infty |\lambda|^{n-1} L_{k-q}^{(n-1)}(\tau|\lambda|/2) e^{-|\lambda|\tau/4} Nf(\tau, \hat{\lambda})\, d\tau \\
&\qquad\qquad + \sum_{r=0}^{n-2} c_{r,k} |\lambda|^{n-r-2} \langle \delta^{(r)}, Nf(\cdot, \hat{\lambda}) \rangle,
\end{aligned}$$

where $c_{r,k} = 4^r \sum_{j=r}^{n-2} \frac{1}{2^j} \binom{j}{r} \left(L_{k-q+n-1}^0 \right)^{(n-j-2)}(0)$.

From now on, we will denote by $\mathscr{F}_{\mathrm{ns}}(f)$ the first term of the last equality and we will call it "the non-singular part of $\mathscr{F}(f)$" and we will denote by $\mathscr{F}_{\mathrm{s}}(f)$ the second term and we will call it "the singular part of $\mathscr{F}(f)$".

First, we will show that the non-singular part of $\mathscr{F}(f)|_{\Sigma^+}$ can be extended to a holomorphic function on $\mathbb{C}^2$. In fact, if $\mathscr{L}_k^{(n)}(t) = \frac{1}{\binom{n+k}{k}} L_k^{(n)}(t)$ then we note that

$$
\begin{aligned}
&|\lambda|^{(n-1)} L_{k-q}^{(n-1)}(t) \\
&\quad = |\lambda|^{(n-1)} \binom{k-q+n-1}{k-q} \mathscr{L}_{k-q}^{(n-1)}(t) \\
&\quad = \frac{1}{(n-1)!}(k-q+n-1)|\lambda| \\
&\qquad \times (k-q+n-2)|\lambda| \dots (k-q+1)|\lambda| \mathscr{L}_{k-q}^{(n-1)}(t) \\
&\quad = \frac{1}{(n-1)!} \prod_{j=1}^{n-1} \left(\frac{\xi}{2} - \frac{n}{2}|\lambda| + j|\lambda| \right) \mathscr{L}_{k-q}^{(n-1)}(t) \\
&\quad = \frac{1}{(n-1)!} \frac{1}{2^{n-1}} \prod_{j=1}^{n-1} (\xi - n|\lambda| + 2j|\lambda|) \mathscr{L}_{k-q}^{(n-1)}(t),
\end{aligned}
$$

where $\xi = (2k + p - q)|\lambda|$. Moreover,

$$
\begin{aligned}
\prod_{j=1}^{n-1}(\xi - n|\lambda| + 2j|\lambda|) &= \prod_{j=-p+1}^{q-1} (\xi - (2j+p-q)|\lambda|) \\
&= \begin{cases} \xi \displaystyle\prod_{k=0}^{\frac{n-4}{2}} (\xi^2 - (n-2-2k)^2\lambda^2), & \text{if } n \text{ is even,} \\ \displaystyle\prod_{k=0}^{\frac{n-3}{2}} (\xi^2 - (n-2-2k)^2\lambda^2), & \text{if } n \text{ is odd,} \end{cases}
\end{aligned}
$$

So, if

$$
p_n(z, w) = \begin{cases} \dfrac{1}{(n-1)!} \dfrac{1}{2^{n-1}} w \displaystyle\prod_{k=0}^{\frac{n-4}{2}} (w^2 - (n-2-2k)^2 z^2), & \text{if } n \text{ is even,} \\ \dfrac{1}{(n-1)!} \dfrac{1}{2^{n-1}} \displaystyle\prod_{k=0}^{\frac{n-3}{2}} (w^2 - (n-2-2k)^2 z^2), & \text{if } n \text{ is odd,} \end{cases}
$$

then the polynomial $p_n(z, w)$ is holomorphic on $\mathbb{C}^2$ and we get that

$$\mathscr{F}_{\mathrm{ns}}(f)(\lambda, \xi) = p_n(\lambda, \xi) \int_0^\infty \mathscr{L}_{k-q}^{(n-1)}(\tau|\lambda|/2) e^{-|\lambda|\tau/4} Nf(\tau, \hat{\lambda})\, d\tau.$$

Let $\tau \geq 0$. In [4], it was stated that there exists a holomorphic function on $\mathbb{C}^2$,

$$(\lambda, \xi) \mapsto \Psi_{\lambda,\xi}(\tau),$$

such that $\Psi_{\lambda,\xi}(\tau) = e^{-\frac{\lambda\tau}{4}} F_{1,1}\big(\frac{n}{2} - \frac{\xi}{2\lambda}, n; \frac{\lambda\tau}{2}\big)$, for $\lambda \neq 0$. This is based on the following observation: the normalized solution $F_{1,1}(a, c; s)$ of the confluent hypergeometric differential equation with parameters a, c is a holomorphic function in its parameters. Moreover, if $(\lambda, \xi) \in \mathbb{R}^2$, then $\Psi_{\lambda,\xi} = \Psi_{|\lambda|,\xi}$ and if $\xi = (2(k-q)+n)|\lambda|$, then $\Psi_{\lambda,\xi}$ can be written in terms of the Laguerre polynomial $L_{k-q}^{(n-1)}$ as

$$\Psi_{\lambda,\xi}(\tau) = e^{-\frac{\lambda\tau}{4}} L_{k-q}^{(n-1)}\left(\frac{\lambda\tau}{2}\right).$$

Finally, let $t \in \mathbb{R}$ and let $(\lambda, \xi) \mapsto \Phi_{\lambda,\xi}(\tau, t)$ be the holomorphic function on $\mathbb{C}^2$ given by $\Phi_{\lambda,\xi}(\tau, t) = e^{-i\lambda t}\Psi_{|\lambda|,\xi}(\tau)$. This gives

$$\mathscr{F}_{\mathrm{ns}}(f)(\lambda, \xi) = p_n(\lambda, \xi) \int_0^\infty \int_{\mathbb{R}} \Phi_{\lambda,\xi}(\tau, t) Nf(\tau, t)\, dt\, d\tau.$$

Hence, if the function $Nf|_{\mathbb{R}_{\geq 0}\times\mathbb{R}}$ has compact support, then the map

$$(w_1, w_2) \mapsto p_n(w_1, w_2) \int_0^\infty \int_{\mathbb{R}} \Phi_{w_1,w_2}(\tau, t) Nf(\tau, t)\, dt\, d\tau$$

is a holomorphic function on $\mathbb{C}^2$ and it extends the non-singular part of the spherical transform of f restricted to Σ^+.

Our aim now is to extend the singular part of $\mathscr{F}(f)$ restricted to Σ^+. In fact, we note that

$$\begin{aligned}
c_{r,k} &= 4^r \sum_{j=r}^{n-2} \frac{1}{2^j} \binom{j}{r} (L^0_{k-q+n-1})^{(n-j-2)}(0) \\
&= 4^r \sum_{j=r}^{n-2} \frac{1}{2^j} \binom{j}{r} (-1)^{n-j-2} L^{(n-j-2)}_{k-q+j+1}(0) \\
&= 4^r \sum_{j=r}^{n-2} \frac{1}{2^j} \binom{j}{r} (-1)^{n-j-2} \binom{k-q+n-1}{k-q+j+1}.
\end{aligned}$$

Let $n, \ell \in \mathbb{N}$ be such that $2 \leq \ell \leq n$ and let

$$f_{n,\ell}(x) = 2^{n-2(\ell-1)} \sum_{i=2}^{\ell} \binom{n-i}{n-\ell} (-1)^{i-2} 2^{i-2} \binom{\frac{x+n}{2} - 1}{\frac{x+n}{2} - (i-1)}. \tag{12}$$

This satisfies

$$c_{n-\ell,k} = f_{n,\ell}(2(k-q)+n).$$

PROPOSITION 4.1. *Let $n, \ell \in \mathbb{N}$ be such that $2 \leq \ell \leq n$. Then, $f_{n,\ell}(x)$ is a polynomial of degree $\ell - 2$ such that $f_{n,\ell}(x) = (-1)^{\ell} f_{n,\ell}(-x)$, for all $x \in \mathbb{R}$.*

We will prove this proposition later. We first show some consequences first. If we rewrite $\mathscr{F}_{\mathrm{s}}(f)$ in terms of $f_{n,\ell}$ and use the previous proposition, then

$$\begin{aligned}
&\mathscr{F}_{\mathrm{s}}(f)(\lambda, (2k+p-q)|\lambda|) \\
&\quad = \sum_{\ell=2}^{n} f_{n,\ell}(2k+p-q)|\lambda|^{\ell-2} \langle \delta^{(n-\ell)}, Nf(\cdot, \hat{\lambda}) \rangle \\
&\quad = \sum_{\ell \text{ even}} \left(\sum_{i=0}^{\ell-2/2} a_{n,\ell}^{i} (2k+p-q)^{2i} \right) |\lambda|^{\ell-2} \langle \delta^{(n-\ell)}, Nf(\cdot, \hat{\lambda}) \rangle \\
&\qquad + \sum_{\ell \text{ odd}} \left(\sum_{i=0}^{\ell-3/2} a_{n,\ell}^{i} (2k+p-q)^{2i+1} \right) |\lambda|^{\ell-2} \langle \delta^{(n-\ell)}, Nf(\cdot, \hat{\lambda}) \rangle \\
&\quad = \sum_{\ell \text{ even}} \sum_{i=0}^{\ell-2/2} a_{n,\ell}^{i} ((2k+p-q)|\lambda|)^{2i} \lambda^{\ell-2-2i} \langle \delta^{(n-\ell)}, Nf(\cdot, \hat{\lambda}) \rangle \\
&\qquad + \sum_{\ell \text{ odd}} \sum_{i=0}^{\ell-3/2} a_{n,\ell}^{i} ((2k+p-q)|\lambda|)^{2i+1} \lambda^{\ell-3-2i} \langle \delta^{(n-\ell)}, Nf(\cdot, \hat{\lambda}) \rangle.
\end{aligned}$$

Let Q_ℓ be the polynomial defined on $\mathbb{C}^2$ by

$$Q_\ell(z, w) = \begin{cases} \displaystyle\sum_{i=0}^{\ell-2/2} a_{n,\ell}^{i} w^{2i} z^{\ell-2-2i}, & \text{if } \ell \text{ is even}, \\ \displaystyle\sum_{i=0}^{\ell-3/2} a_{n,\ell}^{i} w^{2i+1} z^{\ell-3-2i}, & \text{if } \ell \text{ is odd}. \end{cases}$$

Clearly, by using the Paley-Wiener theorem for the real Fourier transform, we obtain that the singular part of $\mathscr{F}(f)$ is extended by the following holomorphic map

$$(z, w) \mapsto \sum_{\ell=2}^{n} Q_\ell(z, w)(-1)^{n-\ell} \int_{\mathbb{R}} e^{-itz} \frac{\partial^{n-\ell} Nf}{\partial s^{n-\ell}}(0, t)\, dt,$$

when the function Nf has compact support.

In this way, we have proved the following.

THEOREM 4.2. *Let f be in $\mathscr{S}(H_n)$. If $Nf|_{\mathbb{R}_{\geq 0}\times\mathbb{R}}$ has compact support, then there exists a holomorphic function G on $\mathbb{C}^2$ such that $G|_{\Sigma^+} = \mathscr{F}(f)|_{\Sigma^+}$.*

If $Nf|_{\mathbb{R}_{\leq 0}\times\mathbb{R}}$ has compact support, then there exists a holomorphic function $\tilde{G}$ on $\mathbb{C}^2$ such that $\tilde{G}|_{\Sigma^-} = \mathscr{F}(f)|_{\Sigma^-}$.

COROLLARY 4.3. *If $f \in C_0^\infty(H_n)$, then $\mathscr{F}(f)|_{\Sigma^+}$ and $\mathscr{F}(f)|_{\Sigma^-}$ admit holomorphic extensions on $\mathbb{C}^2$.*

In the remainder of this section we prove Proposition 4.1 beginning with a series of auxiliary results.

LEMMA 4.4. *Let $\ell \geq 6$. Then,*

$$f_{\ell+1,\ell+1}(x) = a_1 f_{\ell-1,\ell-1}(x) + a_2(x^3 + a_3 x)p_\ell(x),$$

where a_1, a_2, a_3 are constant and

$$p_\ell(x) = \begin{cases} \displaystyle\prod_{k=1}^{\ell-5}(x^2-k^2), & \text{if } \ell \text{ is even,} \\ \displaystyle x\prod_{k=1}^{\ell-5}(x^2-k^2), & \text{if } \ell \text{ is odd.} \end{cases}$$

Note that $p_\ell(-x) = (-1)^\ell p_\ell(x)$, for all $x \in \mathbb{R}$.

PROOF. By (12) we have

$$\begin{aligned} f_{\ell+1,\ell+1}(x) &= 4^0 \frac{1}{2^{\ell+1-2}} \sum_{i=2}^{\ell+1} \binom{\ell+1-i}{0} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2}(x+\ell+1-2s) \\ &= \frac{1}{2^{\ell-1}} \sum_{i=2}^{\ell-1} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2}(x+\ell+1-2s) \\ &\quad + \frac{1}{2^{\ell-1}} \frac{(-1)^{\ell-2}}{(\ell-2)!} \prod_{s=1}^{\ell-2}(x+\ell+1-2s) \\ &\quad + \frac{1}{2^{\ell-1}} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=1}^{\ell-1}(x+\ell+1-2s) \end{aligned}$$

$$= \frac{1}{2^{\ell-1}} + \frac{1}{4}\frac{1}{2^{\ell-3}} \sum_{i=3}^{\ell-1} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=0}^{i-3}(x+\ell-1-2s)$$
$$+ \frac{1}{2^{\ell-1}} \frac{(-1)^{\ell-2}}{(\ell-2)!}(x+\ell-1)(x+\ell-3)p_\ell(x)$$
$$+ \frac{1}{2^{\ell-1}} \frac{(-1)^{\ell-1}}{(\ell-1)!}(x+\ell-1)(x+\ell-3)(x-(\ell-3))p_\ell(x).$$

Adding and subtracting $\frac{1}{4} f_{\ell-1,\ell-1}(x)$ to the above sum we get

$$f_{\ell+1,\ell+1}(x)$$
$$= \frac{1}{2^{\ell-1}} \sum_{i=3}^{\ell-1} \frac{(-1)^{i-2}}{(i-2)!} \left(\prod_{s=0}^{i-3}(x+\ell-1-2s) - \prod_{s=1}^{i-2}(x+\ell-1-2s) \right)$$
$$+ \frac{1}{4} f_{\ell-1,\ell-1}(x) + \frac{1}{2^{\ell-1}} \frac{(-1)^{\ell-1}}{(\ell-1)!}$$
$$\times (x+\ell-1)(x+\ell-3)(x-2(\ell-2))p_\ell(x)$$
$$= \frac{1}{4} f_{\ell-1,\ell-1}(x) - \frac{1}{2^{\ell-2}} \sum_{i=2}^{\ell-2} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2}(x+\ell-1-2s)$$
$$+ \frac{1}{2^{\ell-1}} \frac{(-1)^{\ell-1}}{(\ell-1)!}(x+\ell-1)(x+\ell-3)(x-2(\ell-2))p_\ell(x).$$

Finally, as

$$\frac{1}{4} f_{\ell-1,\ell-1}(x) - \frac{1}{2^{\ell-2}} \sum_{i=2}^{\ell-2} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2}(x+\ell-1-2s)$$
$$= -\frac{1}{4} f_{\ell-1,\ell-1}(x) + \frac{1}{2^{\ell-2}} \frac{(-1)^{\ell-3}}{(\ell-3)!} \prod_{s=1}^{\ell-3}(x+\ell-1-2s),$$

we obtain

$$f_{\ell+1,\ell+1}(x)$$
$$= -\frac{1}{4} f_{\ell-1,\ell-1}(x) + \frac{1}{2^{\ell-2}} \frac{(-1)^{\ell-2}}{(\ell-3)!}(x+\ell-3)p_\ell(x)$$
$$+ \frac{1}{2^{\ell-1}} \frac{(-1)^{\ell-1}}{(\ell-1)!}(x+\ell-1)(x+\ell-3)(x-2(\ell-2))p_\ell(x)$$

$$= -\frac{1}{4} f_{\ell-1,\ell-1}(x) + \frac{1}{2^{\ell-1}} \frac{(-1)^{\ell-1}}{(\ell-1)!} \big(x^3 + ((\ell-1)(\ell-3) - 2(\ell-2)^2)x\big) p_\ell(x).$$

LEMMA 4.5. *Let $\ell \geq 7$. Then,*

$$f_{\ell+2,\ell+1}(x) = a_1 f_{\ell,\ell-1}(x) + a_2 q_\ell(x)(x^2 - a_3)(x^2 - a_4),$$

where a_1, a_2, a_3, a_4 are constants and

$$q_\ell(x) = \begin{cases} \displaystyle\prod_{k=1}^{\ell-6}(x^2 - k^2), & \text{if } \ell \text{ is odd,} \\ \displaystyle x\prod_{k=1}^{\ell-6}(x^2 - k^2), & \text{if } \ell \text{ is even.} \end{cases}$$

Note that $q_\ell(-x) = (-1)^{\ell+1} q_\ell(x)$, for all $x \in \mathbb{R}$.

PROOF. By (12) we have

$$\begin{aligned}
f_{\ell+2,\ell+1}(x) &= \frac{4}{2^{\ell+2-2}} \sum_{i=2}^{\ell+1} (\ell+2-i) \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2} (x+\ell+2-2s) \\
&= \frac{4}{2^\ell} \sum_{i=2}^{\ell-1} (\ell+2-i) \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=0}^{i-3} (x+\ell-2s) \\
&\quad + \frac{4}{2^{\ell-1}} \frac{(-1)^{\ell-2}}{(\ell-2)!} \prod_{s=0}^{\ell-3} (x+\ell-2s) + \frac{4}{2^\ell} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=0}^{\ell-2} (x+\ell-2s) \\
&= \frac{4}{2^\ell} \sum_{i=2}^{\ell-1} (\ell-i) \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=0}^{i-3} (x+\ell-2s) \\
&\quad + \frac{4}{2^{\ell-1}} \sum_{i=2}^{\ell-1} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=0}^{i-3} (x+\ell-2s) + \frac{4}{2^{\ell-1}} \frac{(-1)^{\ell-2}}{(\ell-2)!} \prod_{s=0}^{\ell-3} (x+\ell-2s) \\
&\quad + \frac{4}{2^\ell} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=0}^{\ell-2} (x+\ell-2s)
\end{aligned}$$

$$= \frac{4}{2^{\ell}} \sum_{i=2}^{\ell-1} (\ell - i) \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=0}^{i-3} (x + \ell - 2s)$$
$$+ \frac{4}{2^{\ell-1}} \sum_{i=2}^{\ell} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=0}^{i-3} (x + \ell - 2s) + \frac{4}{2^{\ell}} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=0}^{\ell-2} (x + \ell - 2s)$$

We add and subtract $\frac{1}{4} f_{\ell,\ell-1}(x)$ and we get

$$f_{\ell+2,\ell+1}(x)$$
$$= \frac{4}{2^{\ell}} \sum_{i=2}^{\ell-1} (\ell - i) \frac{(-1)^{i-2}}{(i-2)!} 2(i-2) \prod_{s=1}^{i-3} (x + \ell - 2s) + \frac{1}{4} f_{\ell,\ell-1}(x)$$
$$+ \frac{4}{2^{\ell-1}} \sum_{i=2}^{\ell} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=0}^{i-3} (x + \ell - 2s) + \frac{4}{2^{\ell}} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=0}^{\ell-2} (x + \ell - 2s)$$
$$= -\frac{4}{2^{\ell-1}} \sum_{i=2}^{\ell-1} (\ell - i - 1) \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2} (x + \ell - 2s) + \frac{1}{4} f_{\ell,\ell-1}(x)$$
$$+ \frac{4}{2^{\ell-1}} \sum_{i=2}^{\ell} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=0}^{i-3} (x + \ell - 2s) + \frac{4}{2^{\ell}} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=0}^{\ell-2} (x + \ell - 2s)$$
$$= -\frac{1}{4} f_{\ell,\ell-1}(x) + \frac{4}{2^{\ell-1}} \sum_{i=2}^{\ell-1} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2} (x + \ell - 2s)$$
$$+ \frac{4}{2^{\ell-1}} \sum_{i=2}^{\ell} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=0}^{i-3} (x + \ell - 2s) + \frac{4}{2^{\ell}} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=0}^{\ell-2} (x + \ell - 2s).$$

So,

$$f_{\ell+2,\ell+1}(x)$$
$$= -\frac{1}{4} f_{\ell,\ell-1}(x) + \frac{4}{2^{\ell-1}} \sum_{i=2}^{\ell-1} \frac{(-1)^{i-2}}{(i-2)!} \left(\prod_{s=1}^{i-2} (x + \ell - 2s) + \prod_{s=0}^{i-3} (x + \ell - 2s) \right)$$
$$+ \frac{4}{2^{\ell-1}} \frac{(-1)^{\ell-2}}{(\ell-2)!} \prod_{s=0}^{\ell-3} (x + \ell - 2s) + \frac{4}{2^{\ell}} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=0}^{\ell-2} (x + \ell - 2s)$$

$$= -\frac{1}{4} f_{\ell,\ell-1}(x) + \frac{4}{2^{\ell-1}} \sum_{i=2}^{\ell-1} \frac{(-1)^{i-2}}{(i-2)!} (2(x+\ell) - 2(i-2)) \prod_{s=1}^{i-3} (x+\ell-2s)$$

$$+ \frac{4}{2^{\ell-1}} \frac{(-1)^{\ell-2}}{(\ell-2)!} \prod_{s=0}^{\ell-3} (x+\ell-2s) + \frac{4}{2^{\ell}} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=0}^{\ell-2} (x+\ell-2s)$$

$$= -\frac{1}{4} f_{\ell,\ell-1}(x) + \frac{4}{2^{\ell-2}} \sum_{i=2}^{\ell-1} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2} (x+\ell-2s)$$

$$- \frac{4}{2^{\ell-2}} \sum_{i=3}^{\ell-1} \frac{(-1)^{i-3}}{(i-3)!} \prod_{s=1}^{i-3} (x+\ell-2s)$$

$$+ \frac{4}{2^{\ell-1}} \frac{(-1)^{\ell-2}}{(\ell-2)!} \prod_{s=0}^{\ell-3} (x+\ell-2s) + \frac{4}{2^{\ell}} \frac{(-1)^{\ell-1}}{(\ell-1)!} \prod_{s=0}^{\ell-2} (x+\ell-2s)$$

As

$$\sum_{i=2}^{\ell-1} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2} (x+\ell-2s) - \sum_{i=3}^{\ell-1} \frac{(-1)^{i-3}}{(i-3)!} \prod_{s=1}^{i-3} (x+\ell-2s)$$

$$= \frac{(-1)^{\ell-3}}{(\ell-3)!} \prod_{s=1}^{\ell-3} (x+\ell-2s) = \frac{(-1)^{\ell-3}}{(\ell-3)!} (x+\ell-2)(x+\ell-4) q_\ell(x),$$

we have

$$f_{\ell+2,\ell+1}(x)$$

$$= -\frac{1}{4} f_{\ell,\ell-1}(x) + \frac{4}{2^{\ell-2}} \frac{(-1)^{\ell-3}}{(\ell-3)!} (x+\ell-2)(x+\ell-4) q_\ell(x)$$

$$+ \frac{4}{2^{\ell-1}} \frac{(-1)^{\ell-2}}{(\ell-2)!} (x+\ell)(x+\ell-2)(x+\ell-4) q_\ell(x)$$

$$+ \frac{4}{2^{\ell}} \frac{(-1)^{\ell-1}}{(\ell-1)!} (x+\ell)(x+\ell-2)(x+\ell-4)(x-\ell+4) q_\ell(x)$$

$$= -\frac{1}{4} f_{\ell,\ell-1}(x) + \frac{4}{2^{\ell}} \frac{(-1)^{\ell-1}}{(\ell-1)!} q_\ell(x)(x^2-(\ell-2)^2)(x^2-(\ell-4)^2).$$

LEMMA 4.6. *Let $\ell \geq 2$. Then,*

$$f_{\ell,\ell}(x) = (-1)^{\ell} f_{\ell,\ell}(-x) \quad \textit{and} \quad f_{\ell+1,\ell}(x) = (-1)^{\ell} f_{\ell+1,\ell}(-x),$$

for all $x \in \mathbb{R}$.

PROOF. It is easy to check that

$$f_{n,2}(x) = 2^{n-2} = (-1)^2 f_{n,2}(-x), \quad \forall n \geq 2, \tag{13}$$

and

$$f_{n,3}(x) = -2^{n-4}x = (-1)^3 f_{n,3}(-x), \quad \forall n \geq 3. \tag{14}$$

We will prove by induction that $f_{\ell,\ell}(x) = (-1)^\ell f_{\ell,\ell}(-x)$, for all $\ell \geq 2$. For $\ell = 2, 3$, it follows from the equalities above.

For $\ell = 4, 5, 6$, an easy computation shows that

$$f_{4,4}(x) = \frac{1}{8}(x^2 - 2) = (-1)^4 f_{4,4}(-x),$$

$$f_{5,5}(x) = \frac{1}{48}(-x^3 + 7x) = (-1)^5 f_{5,5}(-x),$$

and

$$f_{6,6}(x) = \frac{1}{384}(x^4 - 16x^2 + 24) = (-1)^6 f_{6,6}(-x).$$

Let $\ell \geq 6$. We suppose that $f_{i,i}(x) = (-1)^i f_{i,i}(-x)$, for all $i \leq \ell$. By Lemma 4.4, we have

$$f_{\ell+1,\ell+1}(x) = c_1 f_{\ell-1,\ell-1}(x) + c_2(x^3 + c_3 x) p_\ell(x),$$

therefore $f_{\ell+1,\ell+1}(x) = (-1)^{\ell+1} f_{\ell+1,\ell+1}(-x)$, for all $x \in \mathbb{R}$.

Now, we will prove by induction that $f_{\ell+1,\ell}(x) = (-1)^\ell f_{\ell+1,\ell}(-x)$, for all $\ell \geq 2$. For $\ell = 2, 3$, it follows from (13) and (14). For $\ell = 4, 5, 6, 7$, an easy computation shows that

$$f_{5,4}(x) = \frac{1}{4}(x^2 - 3) = (-1)^4 f_{5,4}(-x),$$

$$f_{6,5}(x) = \frac{1}{24}(-x^3 + 10x) = (-1)^5 f_{6,5}(-x),$$

$$f_{7,6}(x) = \frac{1}{192}(x^4 - 22x^2 + 45) = (-1)^6 f_{7,6}(-x),$$

and

$$f_{8,7}(x) = \frac{1}{1920}(-x^5 + 40x^3 - 264x) = (-1)^7 f_{8,7}(-x).$$

For $\ell \geq 7$, we have shown that

$$f_{\ell+2,\ell+1}(x) = a_1 f_{\ell,\ell-1}(x) + a_2 q_\ell(x)(x^2 - a_3)(x^2 - a_4)$$

in Lemma 4.5. So, if we suppose that $f_{i+1,i}(x) = (-1)^i f_{i+1,i}(-x)$, for all $i \leq \ell$, then $f_{\ell+2,\ell+1}(x) = (-1)^\ell f_{\ell+2,\ell+1}(-x)$, for all $x \in \mathbb{R}$.

Proof of Proposition 4.1. We have the equalities $\binom{n-i}{n-\ell} = \binom{n-i}{\ell-i}$ and $\binom{n-i}{\ell-i} = \binom{n-i-1}{\ell-i} + \binom{n-i-1}{\ell-i-1}$, where by definition $\binom{n-\ell-1}{-1} = 0$. So, by the definition of $f_{n,\ell}$ for $4 \leq \ell \leq n$, we get

$$\begin{aligned} f_{n,\ell}(x) &= 2\frac{4^{n-1-\ell}}{2^{n-1-2}} \sum_{i=2}^{\ell} \binom{n-1-i}{\ell-i} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2}(x+1+n-1-2s) \\ &\quad + \frac{1}{2}\frac{4^{n-\ell}}{2^{n-1-2}} \sum_{i=2}^{\ell-1} \binom{n-1-i}{\ell-1-i} \frac{(-1)^{i-2}}{(i-2)!} \prod_{s=1}^{i-2}(x+1+n-1-2s) \\ &= 2f_{n-1,\ell}(x+1) + \frac{1}{2}f_{n-1,\ell-1}(x+1), \quad \forall x \in \mathbb{R}. \end{aligned}$$

Finally, let us verify by induction that $f_{n,\ell}(x) = (-1)^\ell f_{n,\ell}(-x)$.

Given $\ell \geq 4$, we suppose that $f_{j,i}(x) = (-1)^i f_{j,i}(-x)$, for $i \leq \ell$, $j < n$. Then, by using suitably the equality above we have

$$\begin{aligned} f_{n,\ell}(-x) &= 2f_{n-1,\ell}(-x+1) + \frac{1}{2}f_{n-1,\ell-1}(-x+1) \\ &= 2(-1)^\ell f_{n-1,\ell}(x-1) + (-1)^{\ell-1}\frac{1}{2}f_{n-1,\ell-1}(x-1) \\ &= 2(-1)^\ell\Big(2f_{n-2,\ell}(x) + \frac{1}{2}f_{n-2,\ell-1}(x)\Big) \\ &\qquad + (-1)^{\ell-1}\frac{1}{2}\Big(2f_{n-2,\ell-1}(x) + \frac{1}{2}f_{n-2,\ell-2}(x)\Big) \\ &= 4(-1)^\ell f_{n-2,\ell}(x) - \frac{1}{4}(-1)^{\ell-2}f_{n-2,\ell-2}(x) \end{aligned}$$

So, by the inductive hypothesis and the equality above we have

$$f_{n,\ell}(-x) = 4f_{n-2,\ell}(-x) - \frac{1}{4}f_{n-2,\ell-2}(-x) = (-1)^\ell f_{n,\ell}(x).$$

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