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DECOMPOSABILITY OF BIMODULE MAPS

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Abstract

Consider a unital C^*-algebra A, a von Neumann algebra M, a unital sub-C^*-algebra $C \subset A$ and a unital $*$-homomorphism $\pi\colon C \to M$. Let $u\colon A \to M$ be a decomposable map (i.e. a linear combination of completely positive maps) which is a C-bimodule map with respect to π. We show that u is a linear combination of C-bimodule completely positive maps if and only if there exists a projection $e \in \pi(C)'$ such that u is valued in eMe and $e\pi(\cdot)e$ has a completely positive extension $A \to eMe$. We also show that this condition is always fulfilled when C has the weak expectation property.

1. Introduction and preliminaries

Let A be a unital C^*-algebra, let $C \subset A$ be a sub-C^*-algebra containing the unit, let M be another unital C^*-algebra and let $\pi\colon C \to M$ be a unital $*$-homomorphism. We say that a bounded linear map $u\colon A \to M$ is a C-bimodule map with respect to π if

$$u(c_1ac_2) = \pi(c_1)u(a)\pi(c_2) \tag{1.1}$$

for any $a \in A$ and any $c_1, c_2 \in C$. In the sequel we will simply say a 'C-bimodule map' if the homomorphism π to which it refers is clear.

It was shown in [11] that if M is injective, then any C-bimodule completely bounded map $u\colon A \to M$ can be decomposed as a linear combination of four completely positive C-bimodule maps from A into M. The aim of this note is to extend that result to the more general context of decomposable operators. Note that in the sequel, M is no longer assumed to be injective.

We will assume that the reader is familiar with the notions of completely positive maps and completely bounded maps, see [8] for an introduction. A linear map $u\colon A \to M$ is called decomposable if there exist four completely positive maps $u_1, u_2, u_3, u_4\colon A \to M$ such that $u = (u_1 - u_2) + i(u_3 - u_4)$. Let $u_*\colon A \to M$ be defined by $u_*(a) = u(a^*)^*$. According to [6], u is decomposable if and only if there exist two completely positive maps $S_1, S_2\colon A \to M$

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such that the mapping

$$\begin{pmatrix} a_1 & t \\ s & a_2 \end{pmatrix} \longmapsto \begin{pmatrix} S_1(a_1) & u(t) \\ u_*(s) & S_2(a_2) \end{pmatrix}$$

from $M_2(A)$ into $M_2(M)$ is completely positive. Furthermore letting $\|u\|_{\rm dec} = \inf\big\{\|S_1\|, \|S_2\|\big\}$ with infimum taken over all possible pairs (S_1, S_2) satisfying the above property defines a norm on the space of decomposable operators. Moreover $\|u\|_{\rm cb} \leq \|u\|_{\rm dec}$ for any decomposable operator $u\colon A \to M$ and $\|u\|_{\rm cb} = \|u\|_{\rm dec}$ when M is injective.

A natural question is whether a C-bimodule decomposable map from A into M can be decomposed as a linear combination of completely positive C-bimodule maps $A \to M$. We do not know if this always holds true. In the case when M is a von Neumann algebra, we will show that this holds when π extends to a completely positive map from A into M. This extension property is automatically satisfied when M is injective (so that our result is formally an extension of Wittstock's Theorem) and also when C has the weak expectation property (see Corollary 2.3). Further we will show in Proposition 2.4 that this extension property is somehow unavoidable.

We end this section with a C-bimodule version of the 2×2 matrix characterization of decomposability reviewed above. Its proof is similar to the one in the classical case so we omit it. We regard the direct sum $C \oplus C$ as a unital sub-C^*-algebra of $M_2(A)$ by identifying that algebra with the set of diagonal matrices with entries in C. Then we let $\pi \oplus \pi\colon C \oplus C \to M_2(M)$ be the $*$-homomorphism sending $\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ to $\begin{pmatrix} \pi(c_1) & 0 \\ 0 & \pi(c_2) \end{pmatrix}$ for any c_1, c_2 in C.

LEMMA 1.1. *Let $u\colon A \to M$ be a C-bimodule map, the following assertions are equivalent.*

(i) *There exist C-bimodule completely positive maps $u_1, u_2, u_3, u_4\colon A \to M$ such that $u = (u_1 - u_2) + i(u_3 - u_4)$.*

(ii) *There exist two C-bimodule completely positive maps $S_1, S_2\colon A \to M$ such that*

$$\begin{pmatrix} a_1 & t \\ s & a_2 \end{pmatrix} \longmapsto \begin{pmatrix} S_1(a_1) & u(t) \\ u_*(s) & S_2(a_2) \end{pmatrix}$$

is a completely positive map from $M_2(A)$ into $M_2(M)$.

(iii) *There exist two bounded maps $S_1, S_2\colon A \to M$ such that the map*

$$\begin{pmatrix} a_1 & t \\ s & a_2 \end{pmatrix} \longmapsto \begin{pmatrix} S_1(a_1) & u(t) \\ u_*(s) & S_2(a_2) \end{pmatrix}$$

from $M_2(A)$ into $M_2(M)$ is completely positive and is a $(C \oplus C)$-bimodule map with respect to $\pi \oplus \pi$.

2. Decomposition into C-bimodule completely positive maps

We will use the following classical lemma (see e.g. [1, 1.3.12] or [9, Lemma 14.3]).

LEMMA 2.1. *Let $\mathscr{A}, \mathscr{B}$ be C^*-algebras, let $\mathscr{D} \subset \mathscr{B}$ be a sub-C^*-algebra and let $V\colon \mathscr{B} \to \mathscr{A}$ be a contractive completely positive map such that the restriction $V_{|\mathscr{D}}$ is a $*$-homomorphism. Then*

$$V(bd) = V(b)V(d) \quad \textit{and} \quad V(db) = V(d)V(b), \qquad b \in \mathscr{B},\ d \in \mathscr{D}.$$

Our main result is the following theorem. Its proof combines and extends techniques from [10] and [9, Chapter 14]. We will use the maximal tensor product of C^*-algebras, see e.g. [8, Chapter 12] or [9, Chapter 11] for some background.

THEOREM 2.2. *Let M be a von Neumann algebra, let A be a unital C^*-algebra, let $C \subset A$ be a sub-C^*-algebra containing the unit and let $\pi\colon C \to M$ be a unital $*$-homomorphism. Let $u\colon A \to M$ be a C-bimodule map (with respect to π) and assume that u is decomposable. If π admits a completely positive extension $A \to M$ then there exist C-bimodule completely positive maps $u_1, u_2, u_3, u_4\colon A \to M$ such that $u = (u_1 - u_2) + i(u_3 - u_4)$.*

PROOF. We may assume that $\|u\|_{\text{dec}} = 1$. Consider $M \subset B(H)$ for some Hilbert space H and let $M' \subset B(H)$ be the corresponding commutant. Let $w\colon A \otimes M' \to B(H)$ be the linear map taking $a \otimes y$ to $u(a)y$ for any $a \in A$ and any $y \in M'$. Since u is decomposable, w extends to a completely bounded map

$$w\colon A \otimes_{\max} M' \longrightarrow B(H),$$

with $\|w\|_{\text{cb}} \leq 1$; see [9, Eq. (11.6) and Thm. 14.1].

Let $\widehat{\pi}\colon A \to M$ be a completely positive extension of π (as given by the assumption). Similarly there exists a completely bounded map $\sigma\colon A \otimes_{\max} M' \to B(H)$ such that $\sigma(a \otimes y) = \widehat{\pi}(a)y$ for any $a \in A$ and any $y \in M'$.

We let α be the C^*-norm on $C \otimes M'$ induced by $A \otimes_{\max} M'$, so that

$$C \otimes_\alpha M' \subset A \otimes_{\max} M'.$$

For simplicity we let $B = A \otimes_{\max} M'$ and $D = C \otimes_\alpha M'$. Next we let

$$\rho\colon D \longrightarrow B(H)$$

be the restriction of σ to D. It is clearly a unital $*$-representation.

For any $a \in A$, $c \in C$ and $y, z \in M'$, we have

$$\begin{aligned} w\big((c \otimes z)(a \otimes y)\big) &= w(ca \otimes zy) = u(ca)zy \\ &= \pi(c)u(a)zy = \pi(c)zu(a)y = \rho(c \otimes z)w(a \otimes y), \end{aligned}$$

because u is a C-bimodule map and z commutes with $u(a) \in M$. Likewise we have

$$w\big((a \otimes y)(c \otimes z)\big) = w(a \otimes y)\rho(c \otimes z).$$

By linearity and continuity, this implies that w is a D-bimodule map with respect to ρ.

We consider the 'bimodule Paulsen system' associated to $D \subset B$, i.e.

$$\mathcal{S} = \left\{ \begin{pmatrix} d_1 & x \\ y & d_2 \end{pmatrix} : x, y \in B,\ d_1, d_2 \in D \right\} \subset M_2(B).$$

It is well-known that the bimodule property of w and the norm condition $\|w\|_{\rm cb} \leq 1$ imply that the map $W \colon \mathcal{S} \to M_2(B(H))$ defined by

$$W \colon \begin{pmatrix} d_1 & x \\ y & d_2 \end{pmatrix} \longmapsto \begin{pmatrix} \rho(d_1) & w(x) \\ w_*(y) & \rho(d_2) \end{pmatrix}$$

is a unital completely positive map (see e.g. [1, 3.6.1]). By Arveson's Extension Theorem (see e.g. [8, Thm. 7.5]), W admits a completely positive extension

$$\widehat{W} \colon M_2(B) \longrightarrow M_2(B(H)).$$

Regard $D \oplus D \subset \mathcal{S}$ as the subspace of 2×2 diagonal matrices with entries in D. By construction, the restriction of $\widehat{W}$ to that unital C^*-algebra is the unital $*$-representation

$$\rho \oplus \rho \colon D \oplus D \longrightarrow M_2(B(H)).$$

By Lemma 2.1, this implies that $\widehat{W}$ is a $D \oplus D$-bimodule map with respect to $\rho \oplus \rho$.

In particular $\widehat{W}$ is a $\mathbb{C} \oplus \mathbb{C}$-bimodule map, which implies that it is 'corner preserving' in the sense of [1, 2.6.15]. More precisely, there exist two completely positive maps $\Gamma_1, \Gamma_2 \colon B \to B(H)$ such that

$$\widehat{W} \begin{pmatrix} b_1 & x \\ y & b_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1(b_1) & w(x) \\ w_*(y) & \Gamma_2(b_2) \end{pmatrix}$$

for any x, y, b_1, b_2 in B (see [1, 2.6.17]). Now the fact that $\widehat{W}$ is a $D \oplus D$-bimodule map ensures that Γ_1 and Γ_2 are both D-bimodule maps (with respect to ρ).

For $j = 1, 2$, we define $S_j\colon A \to B(H)$ by letting $S_j(a) = \Gamma_j(a \otimes 1)$ for any $a \in A$. Since $w(a \otimes 1) = u(a)$, the linear map $\Phi\colon M_2(A) \to M_2(B(H))$ given by

$$\Phi\colon \begin{pmatrix} a_1 & t \\ s & a_2 \end{pmatrix} \longmapsto \begin{pmatrix} S_1(a_1) & u(t) \\ u_*(s) & S_2(a_2) \end{pmatrix}$$

is the restriction of $\widehat{W}$ to $A \simeq A \otimes 1$. Thus Φ is completely positive.

Let $a \in A$. For any $z \in M'$ we have

$$(a \otimes 1)(1 \otimes z) = a \otimes z = (1 \otimes z)(a \otimes 1).$$

Since Γ_1 is a D-bimodule map, this implies that

$$\Gamma_1(a \otimes 1)\rho(1 \otimes z) = \rho(1 \otimes z)\Gamma_1(a \otimes 1).$$

Equivalently, $S_1(a)z = zS_1(a)$. This shows that $S_1(a) \in M$. The same holds for S_2 and hence Φ is valued in $M_2(M)$.

According to Lemma 1.1, it therefore suffices to show that S_1, S_2 are C-bimodule maps. For that purpose consider $a \in A$, $c_1, c_2 \in C$. Since Γ_1, Γ_2 are D-bimodule maps, we have

$$\begin{aligned} S_j(c_1ac_2) = \Gamma_j(c_1ac_2 \otimes 1) &= \Gamma_j\big((c_1 \otimes 1)(a \otimes 1)(c_2 \otimes 1)\big) \\ &= \rho(c_1 \otimes 1)\Gamma_j(a \otimes 1)\rho(c_2 \otimes 1) \\ &= \pi(c_1)S_j(a)\pi(c_2), \end{aligned}$$

which proves the result.

If M is injective, then π has a completely positive extension $A \to M$. Hence Theorem 2.2 reduces to Wittstock's Theorem in this case.

Let $\kappa_C\colon C \hookrightarrow C^{**}$ be the canonical embedding. A C^*-algebra C has the weak expectation property (WEP in short) provided that for one (equivalently for any) C^*-algebra embedding $C \subset B(K)$, there exists a contractive and completely positive map $P\colon B(K) \to C^{**}$ whose restriction to C equals κ_C. This definition is equivalent to the original one going back to Lance [7] and Effros-Lance [4]. See [5] and [1, 7.1.3] for more on this definition. The class of C^*-algebras with the WEP includes nuclear ones and injective ones.

Corollary 2.3. *Let A, C, M, π as in Theorem 2.2. Assume that C has the WEP. Let $u\colon A \to M$ be a C-bimodule decomposable map. Then there exist C-bimodule completely positive maps $u_1, u_2, u_3, u_4\colon A \to M$ such that $u = (u_1 - u_2) + i(u_3 - u_4)$.*

Proof. By Theorem 2.2 it suffices to show that π has a completely positive extension $\widehat{\pi}\colon A \to M$. Let $J\colon A \hookrightarrow B(K)$ be a C^*-algebra embedding for a suitable K. By assumption there exists a completely positive $P\colon B(K) \to C^{**}$ such that $PJ_{|C} = \kappa_C$.

Since M is a dual algebra, π admits a (necessarily unique) w^*-continuous extension $\widetilde{\pi}\colon C^{**} \to M$, which is a $*$-homomorphism (see e.g. [1, 2.5.5]). Then

$$\widehat{\pi} = \widetilde{\pi} \circ P_{|A}$$

is a completely positive extension of π.

We note that the above proof cannot be applied beyond the WEP case.

We now turn to an observation which will imply that the extension property assumption in Theorem 2.2 is actually necessary for the conclusion to hold, up to changing M into a smaller von Neumann algebra containing the range of u.

In the rest of this section we consider A, C, M, π as in Theorem 2.2 and we let $\pi(C)' \subset M$ be the commutant of the range of π. If $e \in \pi(C)'$ is a projection, we let $\pi_e\colon C \to eMe$ be defined by $\pi_e(c) = e\pi(c)e$; this is a unital $*$-homomorphism.

PROPOSITION 2.4. *Let $\phi\colon A \to M$ be a C-bimodule completely positive map. Then there exists a projection $e \in \pi(C)'$ such that $\phi(A) \subset eMe$ and $\pi_e\colon C \to eMe$ admits a completely positive extension $A \to eMe$.*

PROOF. Let $b = \phi(1)$, then b belongs to M_+. Let e be the support projection of b. Since ϕ is a C-bimodule map, we have

$$\pi(c)\phi(1) = \phi(c) = \phi(1)\pi(c)$$

for any $c \in C$. Hence $b \in \pi(C)'$, which implies that $b^{\frac{1}{2}} \in \pi(C)'$ and $e \in \pi(C)'$. Thus we have

$$\phi(c) = b^{1/2}\pi(c)b^{1/2}, \qquad c \in C. \tag{2.1}$$

For any integer $n \geq 1$ let $\psi_n\colon A \to M$ be defined by

$$\psi_n(x) = \big(b + \tfrac{1}{n}\big)^{-1/2}\phi(x)\big(b + \tfrac{1}{n}\big)^{-1/2}, \qquad x \in A.$$

According to the proof of [2, Lem. 2.2], $\big(b + \frac{1}{n}\big)^{-1/2}b^{1/2} \to e$ strongly and $\big(\psi_n(x)\big)_{n\geq 1}$ has a strong limit for any $x \in A$. Let $\psi\colon A \to M$ be the resulting point-strong limit of ψ_n. Then ψ is completely positive and valued in eMe. By (2.1), we have

$$\psi_n(c) = \big(b + \tfrac{1}{n}\big)^{-1/2}b^{1/2}\pi(c)b^{1/2}\big(b + \tfrac{1}{n}\big)^{-1/2}$$

for any $c \in C$. Hence

$$\psi(c) = e\pi(c)e, \qquad c \in C,$$

which means that ψ extends π_e.

Let $x \in A$ with $0 \leq x \leq 1$. By positivity, $0 \leq \phi(x) \leq b$, which implies that $\phi(x) \in eMe$. We deduce that $\phi(A) \subset eMe$.

COROLLARY 2.5. *Let $u: A \to M$ be a linear combination of C-bimodules completely positive maps. Then there exists a projection $e \in \pi(C)'$ such that $u(A) \subset eMe$ and $\pi_e: C \to eMe$ admits a completely positive extension $A \to eMe$.*

PROOF. By assumption, we may write $u = (u_1 - u_2) + i(u_3 - u_4)$, where for any $j = 1, \dots, 4$, u_j is a C-bimodule completely positive map. Then the sum

$$\phi = u_1 + u_2 + u_3 + u_4$$

is a C-bimodule completely positive map. Applying Proposition 2.4, we obtain a projection $e \in \pi(C)'$ such that $\pi_e: C \to eMe$ admits a unital completely positive extension to A and $\phi(A) \subset eMe$.

Let $x \in A$ with $0 \leq x \leq 1$. For any j, we have $0 \leq u_j(x) \leq \phi(x)$, hence $u_j(x) \in eMe$. Thus $u_j(A) \subset eMe$, and hence $u(A) \subset eMe$.

Combining Theorem 2.2 and Corollary 2.5, we obtain the following characterization.

COROLLARY 2.6. *Let $u: A \to M$ be a C-bimodule map and assume that u is decomposable. Then the following assertions are equivalent.*

(i) *There exist C-bimodule completely positive maps $u_1, u_2, u_3, u_4: A \to M$ such that $u = (u_1 - u_2) + i(u_3 - u_4)$.*

(ii) *There exists a projection $e \in \pi(C)'$ such that $\pi_e: A \to eMe$ admits a completely positive extension $\widehat{\pi}_e: A \to eMe$ and $u(A) \subset eMe$.*

REMARK 2.7. It is easy to modify the proof of Theorem 2.2 to obtain the following extension property: consider A, C, M, π as in Theorem 2.2, let $F \subset A$ be a linear subspace which is an operator C-bimodule in the sense that $c_1 f c_2 \in F$ for any $f \in F$ and any $c_1, c_2 \in C$. Let $u: F \to M$ be a C-bimodule bounded map. Assume that u admits a decomposable extension $\widehat{u}: A \to M$ and π admits a completely positive extension $\widehat{\pi}: A \to M$. Then u admits an extension $v: A \to M$ which is a linear combination of C-bimodule completely positive maps $A \to M$.

Theorem 2.2 corresponds to the case $F = A$. In the case when M is injective, the above result corresponds to Wittstock's extension Theorem [11] (see also [1, Thm. 3.6.2]).

3. Additional remarks

In this final section we provide supplementary observations on the decomposition problem considered in this paper. We first note a uniqueness property about the $*$-homomorphism π with respect to which a completely positive map $\phi\colon A \to M$ can be considered as a C-bimodule map. In the sequel we write $e^{\perp} = 1 - e$ for a projection $e \in M$.

COROLLARY 3.1. *Consider a von Neumann algebra M, a C^*-algebra A and a unital sub-C^*-algebra $C \subset A$. Let $\pi_1\colon A \to M$ and $\pi_2\colon A \to M$ be unital $*$-homomorphisms and let $\phi\colon A \to M$ be a completely positive map. Assume that ϕ is a C-bimodule map with respect to π_1. Then the following assertions are equivalent.*

(i) *ϕ is a C-bimodule map with respect to π_2;*

(ii) *There exists a projection $e \in \pi_1(C)' \cap \pi_2(C)'$ such that $\phi(A) \subset eMe$ and $(\pi_1 - \pi_2)(C) \subset e^{\perp}Me^{\perp}$.*

PROOF. Suppose firstly that $\phi\colon A \to M$ is a C-bimodule map with respect to π_2. Then as in the proof of Proposition 2.4, the support projection e of $\phi(1)$ lies in $\pi_1(C)' \cap \pi_2(C)'$ and the map $\psi\colon A \to eMe$ is a completely positive extension of the maps $(\pi_1)_e$ and $(\pi_2)_e$. Consequently, for all $c \in C$,

$$\begin{aligned}\pi_1(c) - \pi_2(c) &= e\pi_1(c)e + e^{\perp}\pi_1(c)e^{\perp} - e\pi_2(c)e - e^{\perp}\pi_2(c)e^{\perp}\\ &= e^{\perp}(\pi_1(c) - \pi_2(c))e^{\perp},\end{aligned}$$

and, therefore, $(\pi_1 - \pi_2)(C) \subset e^{\perp}Me^{\perp}$.

Conversely, let $e \in \pi_1(C)' \cap \pi_2(C)'$ be a projection such that $\phi(A) \subset eMe$ and $(\pi_1 - \pi_2)(C) \subset e^{\perp}Me^{\perp}$. Then given any $c \in C$ there exists $z \in e^{\perp}Me^{\perp}$ such that $\pi_1(c) = \pi_2(c) + z$. Hence for all $a \in A$,

$$\begin{aligned}\phi(ca) &= \pi_1(c)\phi(a) = (\pi_2(c) + z)\phi(a)\\ &= \pi_2(c)\phi(a) + z\phi(a) = \pi_2(c)\phi(a) + e^{\perp}ze^{\perp}\phi(a).\end{aligned}$$

Since ϕ is valued in eMe, this implies

$$\phi(ca) = \pi_2(c)\phi(a).$$

Similarly we have

$$\phi(ac) = \phi(a)\pi_2(c).$$

Our second observation is about the non unital case. Let A be a non unital C^*-algebra. Let $M(A)$ denote its multiplier algebra, that we may regard as a unital sub-C^*-algebra of A^{**} (see e.g. [1, 2.6.7]). Let $C \subset M(A)$ be a unital

C^*-algebra, let M be a von Neumann algebra and let $\pi : C \to M$ be a unital $*$-homomorphism. The definition of a C-bimodule map $u : A \to M$ with respect to π as given by (1.1) extends to that setting. Then Theorem 2.2 extends as follows.

COROLLARY 3.2. *Let $u : A \to M$ be a C-bimodule map (with respect to π) and assume that u is decomposable. If π admits a completely positive extension $A^{**} \to M$ then there exist C-bimodule completely positive maps $u_1, u_2, u_3, u_4 : A \to M$ such that $u = (u_1 - u_2) + i(u_3 - u_4)$.*

PROOF. Let $\widetilde{u} : A^{**} \to M$ be the unique w^*-continuous extension of u (see e.g. [1, Lem. A.2.2]. It is easy to check that $\widetilde{u}$ is decomposable, with $\|\widetilde{u}\|_{\text{dec}} = \|u\|_{\text{dec}}$. Since the product is separately w^*-continuous on von Neumann algebras, $\widetilde{u}$ is a C-bimodule map with respect to π. Indeed let $c_1, c_2 \in C$, let $\eta \in A^{**}$ and let $(a_i)_i$ be a net in A such that $a_i \to \eta$ in the w^*-topology of A^{**}. Then $c_1 a_i c_2 \to c_1 \eta c_2$ in the w^*-topology of A^{**}, hence

$$\widetilde{u}(c_1 \eta c_2) = w^* - \lim_i u(c_1 a_i c_2).$$

Further $u(c_1 a_i c_2) = \pi(c_1) u(a_i) \pi(c_2)$ and $u(a_i) \to \widetilde{u}(\eta)$ in the w^*-topology of M. Hence

$$w^* - \lim_i \pi(c_1) u(a_i) \pi(c_2) = \pi(c_1) \widetilde{u}(\eta) \pi(c_2).$$

This yields $\widetilde{u}(c_1 \eta c_2) = \pi(c_1) \widetilde{u}(\eta) \pi(c_2)$.

Applying Theorem 2.2 to $\widetilde{u}$ we obtain a decomposition of that map into completely positive C-bimodule maps $A^{**} \to M$. Restricting to A, we find the desired decomposition of u.

Our final observation is that the decomposability problem considered in this paper always has a positive solution if π is a $*$-isomorphism. This is a direct consequence of a remarkable projection result of Christensen-Sinclair [3]. We thank Éric Ricard for pointing out this result to us.

PROPOSITION 3.3. *Let A, C, M as in Theorem 2.2 and let $\pi : C \to M$ is a $*$-isomorphism. Then for any decomposable and C-bimodule map $u : A \to M$, there exist C-bimodule completely positive maps $u_1, u_2, u_3, u_4 : A \to M$ such that $u = (u_1 - u_2) + i(u_3 - u_4)$.*

PROOF. Let $\mathrm{CB}(A, M)$ be the space of completely bounded maps from A into M, equipped with $\|\cdot\|_{\text{cb}}$. Let $\mathrm{BIMOD}(A, M) \subset \mathrm{CB}(A, M)$ be the subspace of C-bimodule completely bounded maps. According to [3, Thm. 4.1] (to be applied with $\theta = \pi^{-1}$), there exists a contractive idempotent map

$Q\colon \mathrm{CB}(A, M) \to \mathrm{CB}(A, M)$ with range equal to $\mathrm{BIMOD}(A, M)$, such that $Q(v)$ is completely positive for any completely positive $v\colon A \to M$.

Let $u\colon A \to M$ be decomposable and write it as $u = (v_1 - v_2) + i(v_3 - v_4)$ for some completely positive maps $v_j\colon A \to M$. Then $Q(u) = (Q(v_1) - Q(v_2)) + i(Q(v_3) - Q(v_4))$ and each $Q(v_j)$ is a completely positive and C-bimodule map. If u is assumed to be C-bimodule, then $Q(u) = u$ and the above decomposition proves the result.

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