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PSEUDO-SKOLEM SEQUENCES AND GRAPH SKOLEM LABELLING

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Abstract

Pseudo-Skolem sequences, which are similar to Skolem-type sequences in their structure and applications, are introduced. Constructions of such sequences, either directly or via the use of known Skolem-type sequences, are presented. The applicability of these sequences to Skolem labelled graphs, in particular classes of rail-siding graphs and caterpillars, are also discussed.

1. Introduction

A Skolem-type sequence is a sequence $(s_1, s_2, \dots, s_m)$ of $i \in D$, where D is a set of positive integers called differences, such that for each $i \in D$ there is exactly one $j \in \{1, 2, \dots, m-i\}$ for which $s_j = s_{j+i} = i$. A Skolem sequence of order n, denoted by $\mathscr{S}_n$, is a partition of the set $\{1, 2, \dots, 2n\}$ into a collection of disjoint ordered pairs $\{(a_i, b_i)\}_{i=1}^{n}$ such that $a_i < b_i$ and $b_i - a_i = i$ [11]. Equivalently, a Skolem sequence is a Skolem-type sequence with $m = 2n$ and $D = \{1, 2, \dots, n\}$. Positions in the sequence not occupied by integers $i \in D$ contain null elements or hooks. A Skolem sequence is called k-extended if it contains exactly one hook located in position k. If this hook is in the penultimate position, then the sequence is called a hooked Skolem sequence $h\mathscr{S}_n$.

In 1991, Mendelsohn and Shalaby introduced the notion of Skolem labelling of graphs [6]. A strongly d-Skolem labelled graph is a triple (G, φ, d), where G is a graph and $\varphi\colon V \to \{d, d+1, \dots, d+n-1\}$ such that:

(a) for every $i \in \{d, d+1, \dots, d+n-1\}$, there are exactly two vertices $u, v \in V(G)$ such that $\varphi(u) = \varphi(v) = i$ and $d_G(u, v) = i$,

(b) if G' is a proper spanning subgraph of G, (G', φ, d) is not a Skolem labelled graph.

If condition (b) (which states that the removal of any edge of G destroys the labelling) is not satisfied, then (G, φ, d) is called a weakly Skolem labelled

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graph. When $d = 1$, the labelling is called Skolem labelling of G. Roughly speaking, a graph on $2n$ vertices can be (weakly) Skolem-labelled if each of the vertices can be assigned a label from the set $D = \{1, \ldots, n\}$ such that exactly two vertices at distance i are labelled i for each $i \in D$. If some of the vertices are not labelled (or are labelled by 0 or $*$), then the labelling is called hooked Skolem. Various classes of graphs such as k-windmills, ladder graphs and Cartesian products of paths have been investigated to determine if they admit a (hooked) Skolem labelling [1], [2], [5], [7].

In this paper we introduce the concept of pseudo-Skolem sequences, in which we permit some of the pairs of the sequence to overlap with other pairs in locations termed as *pockets*. We also introduce a class of graphs called rail-siding graphs and show how to (weakly) Skolem label several of their subclasses by taking advantage of a natural correlation between such labellings and pseudo-Skolem sequences. The techniques which are developed in this paper also provide new ways to Skolem label other classes of graphs, most notably graphs known as caterpillars.

2. Skolem labellings, antenna arrays and rail-siding graphs

Our interest in what we call rail-siding graphs is motivated by Skolem labelling of graphs, which itself can be motivated by applications in astronomy [3]. In the context of linear antenna arrays, each antenna is given a location such that the antennae are collinear and the distance between any pair of them is an integer multiple of some unit distance. A desirable property in the placement of the antennae is for several distinct distances to occur, so that the radio wavelengths corresponding to these distances can be observed by the array. In cases where each pair of antennae can operate together simultaneously and in parallel, then with n antennae the optimal scenario is for them to be spaced so that $\binom{n}{2}$ distinct distances occur, for otherwise some distances will have redundant occurrences. When $n = 3$ this problem can be solved by placing antennae at positions $0, 1, 3$ and for $n = 4$ a perfect solution has positions $0, 1, 4, 6$. For $n \geq 5$, such redundance-avoiding solutions do not exist.

Rather than assuming that all $\binom{n}{2}$ pairs of antennae can operate in parallel, consider instead the scenario in which each antenna must operate with a dedicated partner. In this context it is now permissible for distances to be repeated when placing the antennae. The problem becomes one of partitioning the antennae into pairs such that each distance is represented by only one of the pairs of the partition. To cast this situation as a problem in graph theory, consider a path P_n with n vertices, each deemed to be distance 1 from each of its neighbours. Now, given a set D of desirable distances, we wish to label the vertices of the path so that

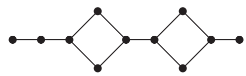


FIGURE 1. A rail-siding graph with the fourth and seventh vertices inflated.

(1) each vertex has at most one label,
(2) for each $i \in D$ exactly two vertices have label i,
(3) for each $i \in D$ the distance between the two vertices with label i is i.

If n is even and $D = \{1, 2, \ldots, n/2\}$ then this labelling corresponds to a Skolem labelling of the path. Other choices of n and D can also be considered, provided that $\max D \leq n - 1$, to yield such labellings as hooked Skolem labellings, labellings with deficiencies, etc.

If $D = \{d_1, d_2, \ldots, d_t\}$ then the length of any linear antenna array that can accommodate D is at least $2t - 1$ since the array must have at least $2t$ antennae partitioned into t pairs. The corresponding path may be excessively long in relation to the distances that are sought to be covered. If there should happen to be physical constraints that inhibit the construction of an array with a length of $2t - 1$, then a compromise may be reached by modifying some number, say p, of the antennae to function as if there were multiple antennae in the same location. If we let μ denote the number of antennae being emulated by each of the enhanced antennae, $n = 2t$ no longer represents the number of functional antennae (this value is now $2t - p + p\mu$) and so the same $2t$ antennae locations can potentially accommodate $(2t - p + p\mu)/2$ pairs of wavelengths (which is somewhat greater than the previous number of t).

This new approach can also be modelled as a Skolem labelled graph, whereby we start with a path P_n of n vertices and then inflate p of the vertices. We require these inflated vertices to be independent, so that if v is a vertex of the original path P_n and is inflated, then its inflation has a closed neighbourhood that is either $K_{2,\mu}$ if v was not an end-vertex of P_n or $K_{1,\mu}$ otherwise. Owing to the similarity between the resulting graph and a railway track with occasional sidings built into it, we call such a graph a rail-siding graph, and the original path P_n is called the main rail of the graph. In this article we focus on the case $\mu = 2$, and hence the closed neighbourhood of each inflated vertex is either a C_4 (i.e., a diamond) if the vertex was internal or a P_3 if the vertex was an end-vertex. We say that a diamond subgraph is at position k if it corresponds to the kth vertex of the original path being inflated. Figure 1 illustrates an example of a rail-siding graph with two diamonds in positions 4 and 7.

3. Existence results

Skolem-type sequences and classes of Skolem labelled graphs are the tools used to construct our sequences. Here we present the results used in the subsequent sections.

THEOREM 3.1 ([8], [11]). *A Skolem sequence of order n exists if and only if* $n \equiv 0, 1 \pmod 4$. *A hooked Skolem sequence of order n exists if and only if* $n \equiv 2, 3 \pmod 4$.

A k-near Skolem sequence of order n, denoted k-near $\mathscr{S}_n$, is a Skolem-type sequence $(s_1, \ldots, s_m)$ with $D = \{1, 2, \ldots, n\} \setminus \{k\}$. For $k \leq n$, a k-near Rosa sequence of order n, denoted k-near $\mathscr{R}_n$, is a k-near Skolem-type sequence of order n with $m = 2n - 1$ and $s_n = 0$.

THEOREM 3.2 ([9]). *A k-near Skolem sequence of order n exists if and only if* $n \equiv 0, 1 \pmod 4$ *when k is odd, and* $n \equiv 2, 3 \pmod 4$ *when k is even. A hooked k-near Skolem sequence of order n exists if and only if* $n \equiv 0, 1 \pmod 4$ *when k is even, and* $n \equiv 2, 3 \pmod 4$ *when k is odd.*

THEOREM 3.3 ([10]). *There exists a k-near Rosa sequence of order n if and only if either* $n \equiv 0, 3 \pmod 4$ *and k is even, or* $n \equiv 1, 2 \pmod 4$ *and k is odd, with the exceptions when* $(n, k) = (3, 2), (4, 2)$.

A k-windmill is a tree consisting of k paths of equal length, called vanes, which meet at a central vertex called the pivot. In a generalised k-windmill, denoted gk-windmill, the vanes may have different lengths.

THEOREM 3.4 ([2]). *A $g3$-windmill T has a Skolem labelling if and only if T satisfies the Skolem parity condition stated as follows: either*

(i) $n \equiv 0, 3 \pmod 4$ *and the parity of T is even, or*

(ii) $n \equiv 1, 2 \pmod 4$ *and the parity of T is odd.*

4. Pseudo-Skolem sequences

In a Skolem sequence the pairs (a_i, b_i) are disjoint for $i \in \{1, 2, \ldots, n\}$. If we allow some of the pairs to share a point, then we have what we call a *pseudo-Skolem sequence*.

DEFINITION 4.1. Suppose that $\{k, n\} \subset \mathbb{N}$ such that $n \geq 2$ and $1 \leq k \leq 2n - 1$. A *k-pseudo-Skolem sequence* of order n, denoted k-pseudo-$\mathscr{S}_n$, is a distribution of the elements of the multiset $\{1, 2, \ldots, 2n-1, k\}$ into a collection of ordered pairs $\{ (a_i, b_i) : i = 1, 2, \ldots, n \}$ such that $a_i < b_i$ and $b_i - a_i = i$ and the pairs that do not contain k are mutually disjoint (there are exactly two pairs containing k). We may display a k-pseudo-$\mathscr{S}_n$ by $(s_1, s_2, \ldots, s_{k-1}, \overset{s'_k}{s_k},$

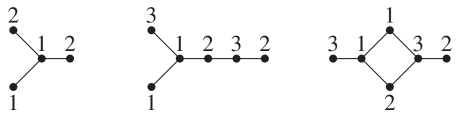


FIGURE 2. Skolem labelled graphs corresponding to $(\overset{2}{1}, 1, 2)$, $(\overset{3}{1}, 1, 2, 3, 2)$ and $(3, 1, \overset{2}{1}, 3, 2)$.

$s_{k+1}, \ldots, s_{2n-1})$ of positive integers $i \in \{1, 2, \ldots, n\}$ such that for each i there is exactly one $j \in \{1, 2, \ldots, 2n-1-i\}$ such that $s_j = s_{j+i} = i$, $s'_j = s_{j+i} = i$ or $s_j = s'_{j+i} = i$. The integer k is called the pocket of the sequence.

EXAMPLE 4.2. For $n = 2$, $\{(1, 2), (1, 3)\}$ or equivalently $(\overset{2}{1}, 1, 2)$ is a 1-pseudo-$\mathscr{S}_2$ with 1 being the pocket of the sequence. For $n = 3$, $\{(1, 2), (3, 5), (1, 4)\}$ and $\{(2, 3), (3, 5), (1, 4)\}$ (or equivalently $(\overset{3}{1}, 1, 2, 3, 2)$ and $(3, 1, \overset{2}{1}, 3, 2)$) are 1-pseudo-$\mathscr{S}_3$ and 3-pseudo-$\mathscr{S}_3$ (resp.), and with 1 and 3 being the pockets of the sequences (resp.). These three pseudo-Skolem sequences are equivalent to Skolem labellings of the rail-siding graphs in Figure 2.

Similarly, we define pseudo-Skolem sequences with p pockets for every $p \geq 2$.

DEFINITION 4.3. Suppose that $\{k_1, k_2, \ldots, k_p, n\} \subset \mathbb{N}$ such that $n \geq 2$ and $1 \leq k_\ell \leq 2n - p$ for each $1 \leq \ell \leq p$. A $\{k_1, k_2, \ldots, k_p\}$-*pseudo-Skolem sequence* of order n, denoted $\{k_1, k_2, \ldots, k_p\}$-pseudo-$\mathscr{S}_n$, is a distribution of the elements of the multiset $\{1, 2, \ldots, 2n - p, k_1, k_2, \ldots, k_p\}$ into a collection of ordered pairs $\{\, (a_i, b_i) : i = 1, 2, \ldots, n \,\}$ such that $a_i < b_i$ and $b_i - a_i = i$ and the pairs that do not contain k_ℓ, $1 \leq \ell \leq p$, are mutually disjoint (there are exactly p pairs with k_ℓ, $1 \leq \ell \leq p$, as an element). We may show a $\{k_1, k_2, \ldots, k_p\}$-pseudo-$\mathscr{S}_n$ by $(s_1, s_2, \ldots, \overset{s'_{k_1}}{s_{k_1}}, s_{k_1+1}, \ldots, \overset{s'_{k_2}}{s_{k_2}}, s_{k_2+1}, \ldots, \overset{s'_{k_p}}{s_{k_p}}, s_{k_p+1}, \ldots, s_{2n-p})$ of positive integers $i \in \{1, 2, \ldots, n\}$ such that for each i there is exactly one $j \in \{1, 2, \ldots, 2n-p-i\}$ such that $s_j = s_{j+i} = i$, $s'_j = s_{j+i} = i$, $s'_j = s'_{j+i} = i$ or $s_j = s'_{j+i} = i$. The integers k_ℓ for $1 \leq \ell \leq p$ are called the pockets of the sequence.

EXAMPLE 4.4. The collection $\{(2, 3), (4, 6), (3, 6), (1, 5)\}$ or equivalently $(4, 1, \overset{3}{1}, 2, 4, \overset{3}{2})$ is a $\{3, 6\}$-pseudo-$\mathscr{S}_4$ with 3 and 6 being the pockets of the sequence.

The focus of this paper is the sequences with two elements in each pocket; however, the definitions above can be easily generalised to allow the pockets to hold more than two elements. We refer to such sequences as stacked pseudo-Skolem sequences. As well, we can define sequences such that some of the positions are filled by null elements.

A significant consequence of the results that we establish in this section is that by using known Skolem-type sequences we can obtain pseudo-Skolem sequences and thereby Skolem label classes of rail-siding graphs. For example, using the 3-near $\mathscr{S}_5$ (4, 5, 1, 1, 4, 2, 5, 2) we can build pseudo-Skolem sequences $(3, 4, 5, \overset{3}{1}, 1, 4, 2, 5, 2)$, $(\overset{3}{4}, 5, 1, \overset{3}{1}, 4, 2, 5, 2)$, $(4, \overset{3}{5}, 1, 1, \overset{3}{4}, 2, 5, 2)$, $(4, 5, \overset{3}{1}, 1, 4, \overset{3}{2}, 5, 2)$ and hence obtain Skolem labellings for the graphs in Figure 3.

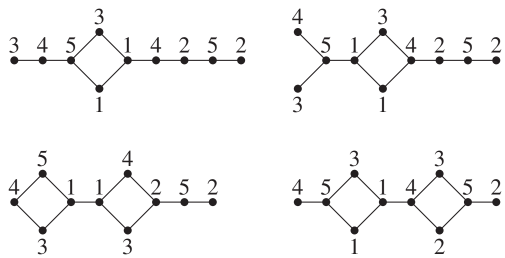


FIGURE 3. Skolem labelling of rail-siding graphs using 3-near $\mathscr{S}_5$ (4, 5, 1, 1, 4, 2, 5, 2).

We can repeat this process and obtain infinite families of (stacked) pseudo-Skolem sequences and therefore obtain infinite families of Skolem labelled graphs. For example, by assigning labels "6" to suitable positions of the pseudo-Skolem sequence $(3, 4, 5, \overset{3}{1}, 1, 4, 2, 5, 2)$ above, we can build $(6, 3, 4, 5, \overset{3}{1}, 1, \overset{6}{4}, 2, 5, 2)$, $(\overset{6}{3}, 4, 5, \overset{3}{1}, 1, 4, \overset{6}{2}, 5, 2)$, $(3, \overset{6}{4}, 5, \overset{3}{1}, 1, 4, 2, \overset{6}{5}, 2)$, $(3, 4, \overset{6}{5}, \overset{3}{1}, 1, 4, 2, 5, \overset{6}{2})$ and $(3, 4, 5, \overset{6}{\overset{3}{1}}, 1, 4, 2, 5, 2, 6)$; the last sequence is a stacked pseudo-Skolem sequence and is equivalent to a Skolem labelling of the graph in Figure 4(a). A graph with a Skolem labelling equivalent to $(3, 4, \overset{6}{5}, \overset{3}{1}, 1, 4, 2, 5, \overset{6}{2})$ is shown in Figure 4(b).

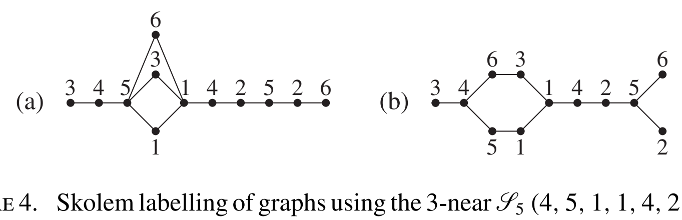


FIGURE 4. Skolem labelling of graphs using the 3-near $\mathscr{S}_5$ (4, 5, 1, 1, 4, 2, 5, 2).

On the other hand however, having a Skolem labelled rail-siding graph is not necessarily equivalent to having a pseudo-Skolem sequence. This particularly happens when the corresponding sequence has the two labels “2” in the same pocket (or equivalently when two 2-vertices of a diamond are labelled “2”). In the following subsections we investigate the necessary and sufficient conditions for having pseudo-Skolem sequences with up to three pockets. Throughout, we assume that b_i and a_i are the largest and smallest positions of element i in a sequence, respectively.

4.1. *Pseudo-Skolem sequences with one pocket*

We begin with finding the necessary and sufficient conditions for the existence of a k-pseudo-$\mathscr{S}_n$.

THEOREM 4.5. *Let* $\{k, n\} \subset \mathbb{N}$ *such that* $1 \leq k \leq 2n-1$ *and* $n \geq 2$. *If a* k*-pseudo-*$\mathscr{S}_n$ *exists then* k *is odd and* $n \equiv 2, 3 \pmod 4$*, or* k *is even and* $n \equiv 0, 1 \pmod 4$.

PROOF. Suppose that $\{k, n\} \subset \mathbb{N}$ and there is a k-pseudo-$\mathscr{S}_n$. We will first find the necessary conditions for having such sequences.

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{2n-1} i + k = n(2n-1) + k \tag{4.1}$$

and

$$\sum_{i=1}^{n}(b_i - a_i) = \sum_{i=1}^{n} i = \frac{1}{2}n(n+1). \tag{4.2}$$

By (4.1) and (4.2), $\sum_{i=1}^{n} b_i = \frac{5n^2-n}{4} + \frac{k}{2}$. Since $\sum_{i=1}^{n} b_i \in \mathbb{N}$, then we conclude that when k is odd, $n \equiv 2, 3 \pmod 4$ and when k is even, $n \equiv 0, 1 \pmod 4$.

Now we will find that the necessary conditions for having a k-pseudo-$\mathscr{S}_n$ are also sufficient by presenting such sequences for any suitable pair of k and n. In order to clarify the idea of the constructions, we give constructions for $k \in \{1, 2\}$ before presenting the theorem and its proof.

Case $k = 1$: Finding a 1-pseudo-$\mathscr{S}_n$ is equivalent to Skolem labelling a $g3$-windmill, say $\mathscr{W}_3$, with two vanes of lengths one and a vane of length $2n-3$.

The parity of $\mathscr{W}_3$ is $\sum_{i=1}^{2n-3} i + 2 = (n-1)(2n-3) + 2$. When $n \equiv 2 \pmod 4$, then the parity of $\mathscr{W}_3$ is odd and when $n \equiv 3 \pmod 4$, then the parity of $\mathscr{W}_3$ is even. So by Theorem 3.4, $\mathscr{W}_3$ can be Skolem labelled and hence there is a 1-pseudo-$\mathscr{S}_n$ if $n \equiv 2, 3 \pmod 4$.

REMARK 4.6. We can find a 1-pseudo-$\mathscr{S}_n$ using a hooked 1-near $\mathscr{S}_n$ as well.

Case $k = 2$: When $k = 2$, if a 2-pseudo-$\mathscr{S}_n$ exists, then $n \equiv 0, 1 \pmod 4$ and $n \neq 1$ by Theorem 4.5. If we have an $(n-2)$-near $\mathscr{R}_n$, then we will have a 2-pseudo-$\mathscr{S}_n$ by filling the hook by "$n-2$" and having the pocket at the second position with one of the labels being "$n-2$". If $n \equiv 1 \pmod 4$ then $n-2$ is odd and by Theorem 3.3 there exists an $(n-2)$-near $\mathscr{R}_n$. If $n \equiv 0 \pmod 4$, then $n-2$ is even and by Theorem 3.3 there exists an $(n-2)$-near $\mathscr{R}_n$. If $(R_1, R_2, \ldots, 0, \ldots, R_{2n-1})$ is such an $(n-2)$-near $\mathscr{R}_n$, then $(R_1, \overset{n-2}{R_2}, \ldots, n-2, \ldots, R_{2n-1})$ with "$n-2$" being in the second and nth positions of the sequence is the desired 2-pseudo-$\mathscr{S}_n$.

We will now generalise this idea for every pocket position k.

THEOREM 4.7. *A k-pseudo-$\mathscr{S}_n$ exists when k is odd and $n \equiv 2, 3 \pmod 4$, and when k is even and $n \equiv 0, 1 \pmod 4$.*

PROOF. The case $k = 1$ and, by symmetry, the case $k = 2n-1$ have already been established. We discuss the rest of the cases in two parts; $2 \leq k \leq n-1$ and $k = n$.

For $2 \leq k \leq n-1$, we only need to show that for any suitable pair (n, k) there exists an $(n-k)$-near $\mathscr{R}_n$. Assume that k is odd and $n \equiv 2, 3 \pmod 4$. If $n \equiv 2 \pmod 4$, then $(n-k)$ is odd and so there is an $(n-k)$-near $\mathscr{R}_n$ by Theorem 3.3. If $n \equiv 3 \pmod 4$, then $(n-k)$ is even and so there is an $(n-k)$-near $\mathscr{R}_n$ for $n \geq 4$ by Theorem 3.3.

Assume that k is even and $n \equiv 0, 1 \pmod 4$. If $n \equiv 0 \pmod 4$, then $(n-k)$ is even and so there is an $(n-k)$-near $\mathscr{R}_n$ by Theorem 3.3. If $n \equiv 1 \pmod 4$, then $(n-k)$ is odd and so there is an $(n-k)$-near $\mathscr{R}_n$ for $n \geq 4$ by Theorem 3.3.

Since there are no k-near $\mathscr{R}_n$ for $(n, k) = (4, 2)$, we need to investigate the existence of a 2-pseudo-$\mathscr{S}_4$ separately; $(3, \overset{1}{4}, 1, 3, 2, 4, 2)$ is such a sequence. Having an $(n-k)$-near $\mathscr{R}_n$ we can get a k-pseudo-$\mathscr{S}_n$ by putting label "$n-k$" in positions n and k. If $n+1 \leq k \leq 2n-2$, then we can get a k-pseudo-$\mathscr{S}_n$ by considering the reverse of a $(2n-k)$-pseudo-$\mathscr{S}_n$ we just obtained.

For $n = k$, we only need to show that there exists a hooked $(n-2)$-near $\mathscr{S}_n$. If $n = k$, then n and k have the same parity and by Theorem 4.5, $n \equiv 0, 3 \pmod 4$. If we have a hooked $(n-2)$-near $\mathscr{S}_n$, then we can obtain an n-pseudo-$\mathscr{S}_n$ by assigning label "$n-2$" to the second and the nth position. Now if $n \equiv 0 \pmod 4$, then $n-2$ is even and when $n \equiv 3 \pmod 4$, then $n-2$ is odd and so by Theorem 3.2 there exists a hooked $(n-2)$-near $\mathscr{S}_n$.

As examples, given $(6-3)$-near $\mathscr{R}_6$ $(1, 1, 4, 5, 6, *, 4, 2, 5, 2, 6)$ we can obtain $(1, 1, \overset{3}{4}, 5, 6, 3, 4, 2, 5, 2, 6)$, which is a 3-pseudo-$\mathscr{S}_6$, and given hooked

2-near $\mathscr{S}_4$ $(3, 0, 4, 3, 1, 1, 4)$, we can obtain $(3, 2, 4, \overset{2}{3}, 1, 1, 4)$, which is a 4-pseudo-$\mathscr{S}_4$. By Theorems 4.5 and 4.7 we conclude the following result.

THEOREM 4.8. *Let* $\{k, n\} \subset \mathbb{N}$ *such that* $1 \leq k \leq 2n - 1$ *and* $n \geq 2$. *A* k*-pseudo-*$\mathscr{S}_n$ *exists if and only if* k *is odd and* $n \equiv 2, 3 \pmod 4$*, or* k *is even and* $n \equiv 0, 1 \pmod 4$.

4.1.1. Rail-siding graphs with one vertex inflated. A k-pseudo-Skolem sequence of order n is equivalent to Skolem labelling of a rail siding graph with $2n$ vertices, one of which is inflated. On the other hand, a Skolem labelling of a rail siding graph with one vertex inflated may be equivalent to a sequence with two labels "2" in the same pocket. In this case, $\sum_{i=1}^{2n}(b_i - a_i) = \sum_{i=1}^{n} i - 2 = \frac{1}{2}n(n+1) - 2$ and $\sum_{i=1}^{n} b_i = \frac{5n^2-n}{4} + \frac{k}{2} - 1$. Therefore, when k is odd, then $n \equiv 2, 3 \pmod 4$ and when k is even, then $n \equiv 0, 1 \pmod 4$. By these and Theorem 4.8, the following result is obvious.

COROLLARY 4.9. *Suppose that* $n \geq 2$ *and* G *is a rail-siding graph on* $2n$ *vertices with main rail of length* $2n-2$. *The graph* G *can be Skolem labelled if and only if* k *is odd and* $n \equiv 2, 3 \pmod 4$*, or* k *is even and* $n \equiv 0, 1 \pmod 4$.

In Figure 5, a Skolem labelled rail-siding graph with one diamond subgraph is depicted. This labelling is equivalent to the 6-pseudo-$\mathscr{S}_5$ $(5, 4, 1, 1, 3, \overset{5}{4}, 2, 3, 2)$.

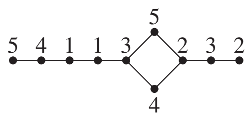


FIGURE 5. Skolem labelled rail-siding graph with one diamond subgraph.

REMARK 4.10. If b_i is the largest position of label i when no two labels "2" are in the same pocket and b'_i is largest position of label i when the two labels "2" are in the same pocket, then $\sum_{i=1}^{n} b'_i = \sum_{i=1}^{n} b_i - 1$. Therefore, when studying the necessary conditions, we only need to find one of $\sum_{i=1}^{n} b_i$ or $\sum_{i=1}^{n} b'_i$. This implies that the necessary conditions for the existence of a Skolem labelling for a rail-siding graph is independent of whether the labels "2" are in the same pocket or not.

4.2. Pseudo-Skolem sequences with two pockets

It is now natural to study the existence of pseudo-Skolem sequences with two pockets.

THEOREM 4.11. *Let* $\{k_1, k_2, n\} \subset \mathbb{N}$ *such that* $1 \le k_1 < k_2 \le 2n-2$ *and* $n \ge 2$. *A* $\{k_1, k_2\}$*-pseudo-*$\mathscr{S}_n$ *exists only if* $n \equiv 0, 1 \pmod 4$ *and* k_1 *and* k_2 *have different parities, or* $n \equiv 2, 3 \pmod 4$ *and* k_1 *and* k_2 *have the same parity. These conditions are sufficient when*

(i) $1 \le k_2 - k_1 \le n-1$ *and* $n \equiv 0, 3 \pmod 4$*, or*

(ii) $k_2 - k_1 = n$ *for* $n \equiv 1, 2 \pmod 4$.

PROOF. If there exists a $\{k_1, k_2\}$-pseudo-$\mathscr{S}_n$, then $\sum_{i=1}^{n} b_i = \frac{1}{4}(5n^2 - 5n + 2) + \frac{k_1}{2} + \frac{k_2}{2}$. If $n \equiv 0 \pmod 4$ or $n = 4k$ for some $k \in \mathbb{Z}$, then $\frac{1}{4}(5n^2 - 5n + 2) = \frac{1}{4}(80k^2 - 20k + 2) = 20k^2 - 5k + \frac{1}{2}$. Therefore, exactly one of k_1 or k_2 is odd. If $n \equiv 1 \pmod 4$, so $n = 4k + 1$ for some $k \in \mathbb{Z}$, then $\frac{1}{4}(5n^2 - 5n + 2) = \frac{1}{4}(80k^2 + 20k + 2) = 20k^2 + 5k + \frac{1}{2}$. Therefore, exactly one of k_1 or k_2 is odd.

If $n \equiv 2 \pmod 4$, so $n = 4k + 2$ for some $k \in \mathbb{Z}$, then $\frac{1}{4}(5n^2 - 5n + 2) = \frac{1}{4}(80k^2 + 60k + 12) = 20k^2 + 15k + 3$. Thus k_1 and k_2 have the same parity. Finally, if $n \equiv 3 \pmod 4$, so $n = 4k + 3$ for some $k \in \mathbb{Z}$, then $\frac{1}{4}(5n^2 - 5n + 2) = \frac{1}{4}(80k^2 + 100k + 32) = 20k^2 + 25k + 8$. Thus k_1 and k_2 have the same parity.

For (i) $1 \le k_2 - k_1 \le n - 1$: if $n \equiv 0, 1 \pmod 4$ then k_1 and k_2 must have different parities and $k_2 - k_1$ is odd. Now if we have a $(k_2 - k_1)$-near $\mathscr{S}_n$, then we can construct a $\{k_1, k_2\}$-pseudo-$\mathscr{S}_n$ by putting label "$k_2 - k_1$" in positions k_2 and k_1 of the sequence. By Theorem 3.2 such near Skolem sequences exist.

Similarly, if $n \equiv 2, 3 \pmod 4$ then k_1 and k_2 have the same parity and $k_2 - k_1$ is even. If we have a $(k_2 - k_1)$-near $\mathscr{S}_n$, then we can construct a $\{k_1, k_2\}$-pseudo-$\mathscr{S}_n$ by putting the label "$k_2 - k_1$" in positions k_2 and k_1 of the sequence. By Theorem 3.2 such near Skolem sequences exist, and therefore (i) is true.

For (ii) $k_2 - k_1 = n$: if we have a $\mathscr{S}_{n-1}$, then we can construct a $\{k_1, k_2\}$-pseudo-$\mathscr{S}_n$ by assigning labels "n" to the positions k_1 and k_2. A $\mathscr{S}_{n-1}$ exists when $n - 1 \equiv 0, 1 \pmod 4$ by Theorem 3.1.

When $n \equiv 1 \pmod 4$, then k_1 and k_2 must have different parities. Since $n - 1 \equiv 0 \pmod 4$, there exists a $\mathscr{S}_{n-1}$. Similarly, when $n \equiv 2 \pmod 4$, then k_1 and k_2 must have the same parity. Since $n - 1 \equiv 1 \pmod 4$, there exists a $\mathscr{S}_{n-1}$. By putting labels "n" in positions k_1 and k_2 we get a $\{k_1, k_2\}$-pseudo-$\mathscr{S}_n$. This proves (ii).

For example, $(4, 1, 1, 2, 4, 2)$ is a $(6-3)$-near $\mathscr{S}_4$, from which a $\{3, 6\}$-pseudo-$\mathscr{S}_4$ $(4, 1, \overset{3}{1}, 2, 4, \overset{3}{2})$ can be obtained. Similarly, $(1, 1, 3, 4, 2, 3, 2, 4)$ is a $\mathscr{S}_4$, from which we can construct the $\{3, 8\}$-pseudo-$\mathscr{S}_5$ $(1, 1, \overset{5}{3}, 4, 2, 3, 2, \overset{5}{4})$.

COROLLARY 4.12. *Let* $\{k_1, k_2, n\} \subset \mathbb{N}$ *such that* $1 \le k_1 < k_2 \le 2n-2$ *and* $n \ge 2$. *There is no* $\{k_1, k_2\}$*-pseudo-*$\mathcal{S}_n$ *for the following cases:*

(i) $k_2 - k_1 = n$ *and* $n \equiv 0, 3 \pmod 4$,

(ii) $k_2 - k_1 = n + i$ *for* $1 \le i \le n-3$, *where* $n \equiv 0, 3 \pmod 4$ *and* i *is even, and where* $n \equiv 1, 2 \pmod 4$ *and* i *is odd.*

PROOF. When $n \equiv 0 \pmod 4$, then since $n = k_2 - k_1$, k_1 and k_2 have the same parity. So, there is no $\{k_1, k_2\}$-pseudo-$\mathcal{S}_n$. Similarly, when $n \equiv 3 \pmod 4$, then since $n = k_2 - k_1$, k_1 and k_2 have different parities. So, there is no $\{k_1, k_2\}$-pseudo-$\mathcal{S}_n$. This proves case (i).

Assume that $n + 1 \le k_2 - k_1 \le 2n - 3$ and let $k_2 - k_1 = n + i$, where $1 \le i \le n-3$. If $n \equiv 0 \pmod 4$, then k_1 and k_2 have different parities and $k_2 - k_1 = n + i$ is odd, which implies that i is odd. If $n \equiv 1 \pmod 4$, then k_1 and k_2 have different parities and $k_2 - k_1 = n + i$ is odd, which implies that i is even. If $n \equiv 2 \pmod 4$, then k_1 and k_2 have the same parity and $k_2 - k_1 = n + i$ is even, which implies that i is even. If $n \equiv 3 \pmod 4$, then k_1 and k_2 have the same parity and $k_2 - k_1 = n + i$ is even, which implies that i is odd. These imply part (ii) of the theorem.

COROLLARY 4.13. *Let* G *be a rail-siding graph on* $2n$ *vertices and with the main rail of length* $2n-3$ *such that the vertices at positions* k_1 *and* k_2, *where* $k_1 < k_2$, *are inflated. The graph* G *can be Skolem labelled if*

(i) $k_2 - k_1 \le n - 1$: $n \equiv 0 \pmod 4$ *and* k_1 *and* k_2 *have different parities, or* $n \equiv 3 \pmod 4$ *and* k_1 *and* k_2 *have the same parity, or*

(ii) $k_2 - k_1 = n$: $n \equiv 1 \pmod 4$ *and* k_1 *and* k_2 *have different parities, or* $n \equiv 2 \pmod 4$ *and* k_1 *and* k_2 *have the same parity.*

4.3. *Pseudo-Skolem sequences with three pockets*

Lastly, we discuss the existence of pseudo-Skolem sequences with three pockets.

THEOREM 4.14. *Let* $\{k_1, k_2, k_3, n\} \subset \mathbb{N}$ *such that* $1 \le k_1 < k_2 < k_3 \le 2n-3$ *and* $n \ge 3$. *A* $\{k_1, k_2, k_3\}$*-pseudo-Skolem sequence of order* n *exists only if* $n \equiv 0, 1 \pmod 4$ *and either only one or each* k_i *is odd for* $i \in \{1, 2, 3\}$, *or* $n \equiv 2, 3 \pmod 4$ *and either only one or each* k_i *is even for* $i \in \{1, 2, 3\}$.

PROOF. Assuming that a $\{k_1, k_2, k_3\}$-pseudo-$\mathcal{S}_n$ exists, we have $\sum_{i=1}^{n} b_i = \frac{1}{4}n(n+1) + \frac{1}{2}(2n-3)(n-1) + \frac{k_1}{2} + \frac{k_2}{2} + \frac{k_3}{2}$. If $n \equiv 0 \pmod 4$, or $n = 4k$ for some $k \in \mathbb{Z}$, then $\frac{1}{4}n(n+1) + \frac{1}{2}(2n-3)(n-1) = 20k^2 - 9k + \frac{3}{2}$. Therefore, either only one or each k_i is odd for $i \in \{1, 2, 3\}$. Similarly, if $n \equiv 1 \pmod 4$, or $n = 4k+1$ for some $k \in \mathbb{Z}$, then $\frac{1}{4}n(n+1) + \frac{1}{2}(2n-3)(n-1) = 20k^2 + k + \frac{1}{2}$. Hence, either only one or each k_i is odd for $i \in \{1, 2, 3\}$.

(a)

$$\left(\frac{\overset{n-4}{n-2}}{1}, \frac{n-6}{2}, \overset{-2}{\cdots}, \frac{4}{\frac{n}{2}-3}, \frac{2}{\frac{n}{2}-2}, \frac{n}{\frac{n}{2}-1}, \frac{2}{\frac{n}{2}}, \frac{4}{\frac{n}{2}+1}, \overset{+2}{\cdots}, \frac{n-6}{n-4}, \frac{n-4}{n-3}, \frac{n-1}{n-2}, \frac{\overset{n-3}{n-2}}{n-1}, \frac{n-5}{n}, \frac{n-3}{n+1}, \overset{-2}{\cdots}, \frac{\frac{n}{2}+3}{\frac{5n}{4}-4}, \frac{\frac{n}{2}+1}{\frac{5n}{4}-3}, \frac{1}{\frac{5n}{4}-2}, \frac{1}{\frac{5n}{4}-1},\right.$$

$$\left.\frac{\frac{n}{2}-3}{\frac{5n}{4}}, \overset{-2}{\cdots}, \frac{3}{\frac{3n}{2}-3}, \frac{\frac{n}{2}-1}{\frac{3n}{2}-2}, \frac{n}{\frac{3n}{2}-1}, \frac{3}{\frac{3n}{2}+1}, \overset{+2}{\cdots}, \frac{\frac{n}{2}-3}{\frac{7n}{4}-3}, \frac{\frac{n}{2}+1}{\frac{7n}{4}-2}, \overset{+2}{\cdots}, \frac{n-3}{2n-4}, \frac{\overset{\frac{n}{2}-1}{n-1}}{2n-3}\right)$$

(b)

$$\left(\frac{\frac{n}{2}-1}{1}, \frac{\overset{n-1}{n-3}}{2}, \frac{n-5}{3}, \overset{-2}{\cdots}, \frac{\frac{n}{2}+1}{\frac{n}{4}}, \frac{\frac{n}{2}-3}{\frac{n}{4}+1}, \frac{\frac{n}{2}-5}{\frac{n}{4}+2}, \overset{-2}{\cdots}, \frac{3}{\frac{n}{2}-2}, \frac{n}{\frac{n}{2}-1}, \frac{\frac{n}{2}-1}{\frac{n}{2}}, \frac{3}{\frac{n}{2}+1}, \overset{+2}{\cdots}, \frac{\frac{n}{2}-3}{\frac{3n}{4}-2}, \frac{1}{\frac{3n}{4}-1}, \frac{1}{\frac{3n}{4}},\right.$$

$$\left.\frac{\frac{n}{2}+1}{\frac{3n}{4}+1}, \overset{+2}{\cdots}, \frac{n-5}{n-2}, \frac{\overset{n-2}{n-3}}{n-1}, \frac{n-4}{n}, \frac{n-1}{n+1}, \frac{n-6}{n+2}, \overset{-2}{\cdots}, \frac{2}{\frac{3n}{2}-2}, \frac{n}{\frac{3n}{2}-1}, \frac{2}{\frac{3n}{2}}, \frac{4}{\frac{3n}{2}+1}, \overset{+2}{\cdots}, \frac{n-8}{2n-5}, \frac{\overset{n-4}{n-6}}{2n-4}, \frac{n-2}{2n-3}\right)$$

FIGURE 6. (a) $\{1, n-1, 2n-3\}$-pseudo-$\mathscr{S}_n$ for $n = 4k$, (b) $\{2, n-1, 2n-4\}$-pseudo-$\mathscr{S}_n$ for $n = 4k$.

If $n \equiv 2 \pmod 4$, or $n = 4k+2$ for some $k \in \mathbb{Z}$, then $\frac{1}{4}n(n+1) + \frac{1}{2}(2n-3)(n-1) = 20k^2 + 11k + 2$. Therefore, either only one or each k_i is even for $i \in \{1, 2, 3\}$. Lastly, if $n \equiv 3 \pmod 4$, or $n = 4k+3$ for some $k \in \mathbb{Z}$, then $\frac{1}{4}n(n+1) + \frac{1}{2}(2n-3)(n-1) = 20k^2 + 21k + 6$. Therefore, either only one or each k_i is even for $i \in \{1, 2, 3\}$.

For example, $(\overset{4}{3}, 1, \overset{2}{1}, 3, \overset{4}{2})$ is the $\{1, 3, 5\}$-pseudo-$\mathscr{S}_4$ (unique) and $(\overset{4}{3}, \overset{2}{1}, 1, \overset{3}{2}, 4)$ is a $\{1, 2, 4\}$-pseudo-$\mathscr{S}_4$. As well, $(\overset{5}{3}, 1, \overset{4}{1}, 3, 2, 5, \overset{4}{2})$ and $(2, 5, \overset{4}{3}, \overset{2}{1}, \overset{4}{1}, 3, 5)$ are $\{1, 3, 7\}$- and $\{1, 3, 5\}$-pseudo-$\mathscr{S}_5$.

Next we present some constructions for prescribed pseudo-Skolem sequences.

(i) Pockets in positions $i, n-1$ and $n+i$ where $1 \le i \le n-3$: we are looking to find $\{i, n-1, n+i\}$-pseudo-$\mathscr{S}_n$. By Theorem 4.14, $\{i, n-1, n+i\}$-pseudo-$\mathscr{S}_n$ exists only if $n \equiv 0, 1 \pmod 4$. These conditions are sufficient. If we have an $\{n-1\}$-pseudo-$\mathscr{S}_{n-1}$, then we can construct an $\{i, n-1, n+i\}$-pseudo-$\mathscr{S}_n$ by assigning labels "n" to the positions i and $n+i$. By Theorem 4.7 an $\{n-1\}$-pseudo-$\mathscr{S}_{n-1}$ exists for $n \equiv 0, 1 \pmod 4$.

Example 4.15. From 4-pseudo-$\mathscr{S}_4$ $(2, 4, 2, \overset{3}{1}, 1, 4, 3)$, we can obtain $(\overset{5}{2}, 4, 2, \overset{3}{1}, 1, \overset{5}{4}, 3)$ and $(2, \overset{5}{4}, 2, \overset{3}{1}, 1, 4, \overset{5}{3})$ which are $\{1, 4, 6\}$- and $\{2, 4, 7\}$-pseudo-$\mathscr{S}_5$.

(ii) Pockets are in the first, last and middle positions: we are looking to construct $\{1, n-1, 2n-3\}$-pseudo-$\mathscr{S}_n$. By the necessary conditions obtained above, such sequences exist only if $n \equiv 0 \pmod 4$ (which implies that $1, n-1$ and $2n-3$ are odd integers). For $n = 4k$, the sequence in Figure 6(a) is a $\{1, n-1, 2n-3\}$-pseudo-$\mathscr{S}_n$ where each line segment indicates a position and the labels under them are the positions.

Example 4.16. The following examples are $\{1, n-1, 2n-3\}$-pseudo-$\mathscr{S}_n$ for $n \in \{4, 8, 12, 16\}$.

$n = 4$: $(\overset{3}{4}, 1, \overset{1}{2}, 3, \overset{2}{4})$.

$n = 8$: $(\overset{4}{6}, 2, 8, 2, 4, 7, \overset{5}{6}, 1, 1, 3, 8, 5, \overset{3}{7})$.

$n = 12$: $(\overset{8}{10}, 6, 4, 2, 12, 2, 4, 6, 8, 11, \overset{9}{10}, 7, 1, 1, 3, 5, 12, 3, 7, 9, \overset{5}{11})$.

$n = 16$: $(\overset{12}{14}, 10, 8, 6, 4, 2, 16, 2, 4, 6, 8, 10, 12, 15, \overset{13}{14},$
$11, 9, 1, 1, 5, 3, 7, 16, 3, 5, 9, 11, 13, \overset{7}{15})$.

(iii) Pockets are in the second, middle and penultimate positions: we are looking to construct $\{2, n-1, 2n-4\}$-pseudo-$\mathscr{S}_n$. By the necessary conditions obtained above, such sequences exist only if $n \equiv 0, 3 \pmod 4$. The sequence in Figure 6(b) is a $\{2, n-1, 2n-4\}$-pseudo-$\mathscr{S}_n$ for $n \equiv 0 \pmod 4$ in which each line segment indicates a position and the labels under them are the positions.

EXAMPLE 4.17. The following examples are $\{2, n-1, 2n-2\}$-pseudo-$\mathscr{S}_n$ for $n \in \{4, 8, 12, 16\}$.

$n = 4$: None.

$n = 8$: $(2, \overset{6}{7}, 2, 3, 8, 4, \overset{3}{5}, 6, 7, 4, 1, \overset{1}{5}, 8)$
or $(3, \overset{7}{5}, 8, 3, 1, 1, \overset{6}{5}, 4, 7, 2, 8, \overset{4}{2}, 6)$.

$n = 12$: $(5, \overset{11}{9}, 7, 3, 12, 5, 3, 1, 1, 7, \overset{10}{9}, 8, 11, 6, 4, 2, 12, 2, 4, \overset{8}{6}, 10)$.

$n = 16$: $(7, \overset{15}{13}, 11, 9, 5, 3, 16, 7, 3, 5, 1, 1, 9, 11, \overset{14}{13},$
$12, 15, 10, 8, 6, 4, 2, 16, 2, 4, 6, 8, \overset{12}{10}, 14)$.

While we have no construction for $\{2, n-1, 2n-4\}$-pseudo-$\mathscr{S}_n$ when $n \equiv 3 \pmod 4$, we conjecture that these sequences exist for any such integer n.

5. Skolem labelling of classes of rail-siding graphs

In this section we consider (hooked) Skolem label classes of rail-siding graphs. We let d denote the number of diamond subgraphs of a rail-siding graph.

FIGURE 7. Type I rail-siding graph $\mathscr{D}_d$.

FIGURE 8. Skolem labelled graphs $\mathscr{D}_1$ and $\mathscr{D}_2$.

5.1. Type I rail-siding graphs

A type I rail-siding graph $\mathscr{D}_d$, is a graph with P_{3d+2} being its main rail, $\Delta = 3$ and the end vertices are not inflated. The general form of such graphs is shown

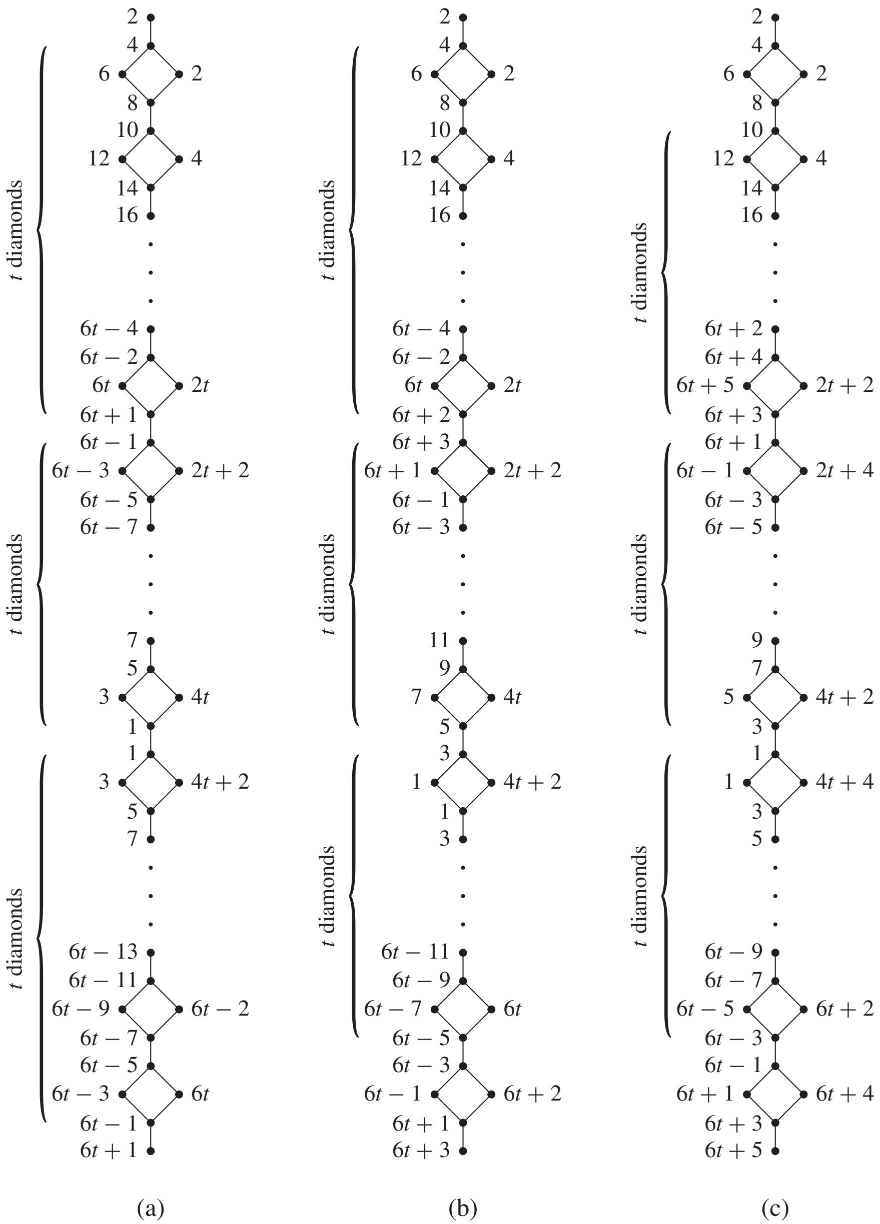


FIGURE 9. Skolem labelled graph (a) $\mathscr{D}_d$ with $d \equiv 0 \pmod 4$, (b) $\mathscr{D}_d$ with $d \equiv 1 \pmod 4$ and (c) $\mathscr{D}_d$ with $d \equiv 2 \pmod 4$.

in Figure 7 and examples of two Skolem labelled graphs of this type are presented in Figure 8.

We present a Skolem labelling of $\mathscr{D}_d$ according to the value of d. Note that $|V(\mathscr{D}_d)| = 4d + 2$ and so the set of labels is $\{1, 2, 3, \ldots, 2d + 1\}$.

(i) $d \equiv 0 \pmod 4$: if $d = 3t$ for some $t \in \mathbb{N}$, then the greatest label is $6t+1 = 3(2t)+1$. A Skolem labelling of $\mathscr{D}_d$ is presented in Figure 9(a).

(ii) $d \equiv 1 \pmod 3$: if $d = 3t + 1$ for some $t \in \mathbb{N}$, then the greatest label is $6t + 3 = 3(2t + 1)$. A Skolem labelling of these graphs is presented in Figure 9(b).

(iii) $d \equiv 2 \pmod 4$: if $d = 3t + 2$ for some $t \in \mathbb{N}$, then the greatest label is $6t + 5 = 3(2t + 1) + 2$. A Skolem labelling of $\mathscr{D}_d$ is presented in Figure 9(c).

5.2. Type II rail-siding graphs

A type II rail-siding graph $\mathscr{D}_d^2$, is a graph with P_{3d+1} being its main rail, $\Delta = 3$ and the end vertices are not inflated. The general form of type II rail-siding graphs $\mathscr{D}_d^2$, where d is the number of diamond subgraphs, is shown in Figure 10. It is obvious that $|V(\mathscr{D}_d^2)| = 4d + 1$ which is odd. Figure 11 illustrates two hooked Skolem labelled type II graphs.

FIGURE 10. Type II rail-siding graph $\mathscr{D}_d^2$.

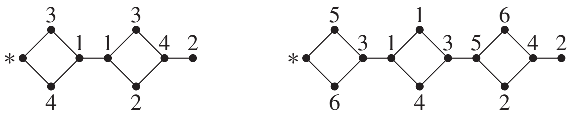


FIGURE 11. Hooked Skolem labelled graphs $\mathscr{D}_2^2$ and $\mathscr{D}_3^2$.

Similar to the type I graphs, we discuss according the value of d and present a hooked Skolem labelling of type II graphs.

(i) $d \equiv 0 \pmod 3$: if $d = 3t$ for some $t \in \mathbb{N}$, then the greatest label is $6t$. Figure 12(a) presents a hooked Skolem labelling of $\mathscr{D}_d^2$ for such d in general.

(ii) $d \equiv 1 \pmod 3$: if $d = 3t + 1$ for some $t \in \mathbb{N}$, then the greatest label is $6t + 2$. Figure 12(b) presents a hooked Skolem labelling of $\mathscr{D}_d^2$ for such d in general.

(iii) $d \equiv 2 \pmod 3$: if $d = 3t + 2$ for some $t \in \mathbb{N}$, then the greatest label is $6t + 4$. Figure 12(c) presents a hooked Skolem labelling of $\mathscr{D}_d^2$ for such d in general.

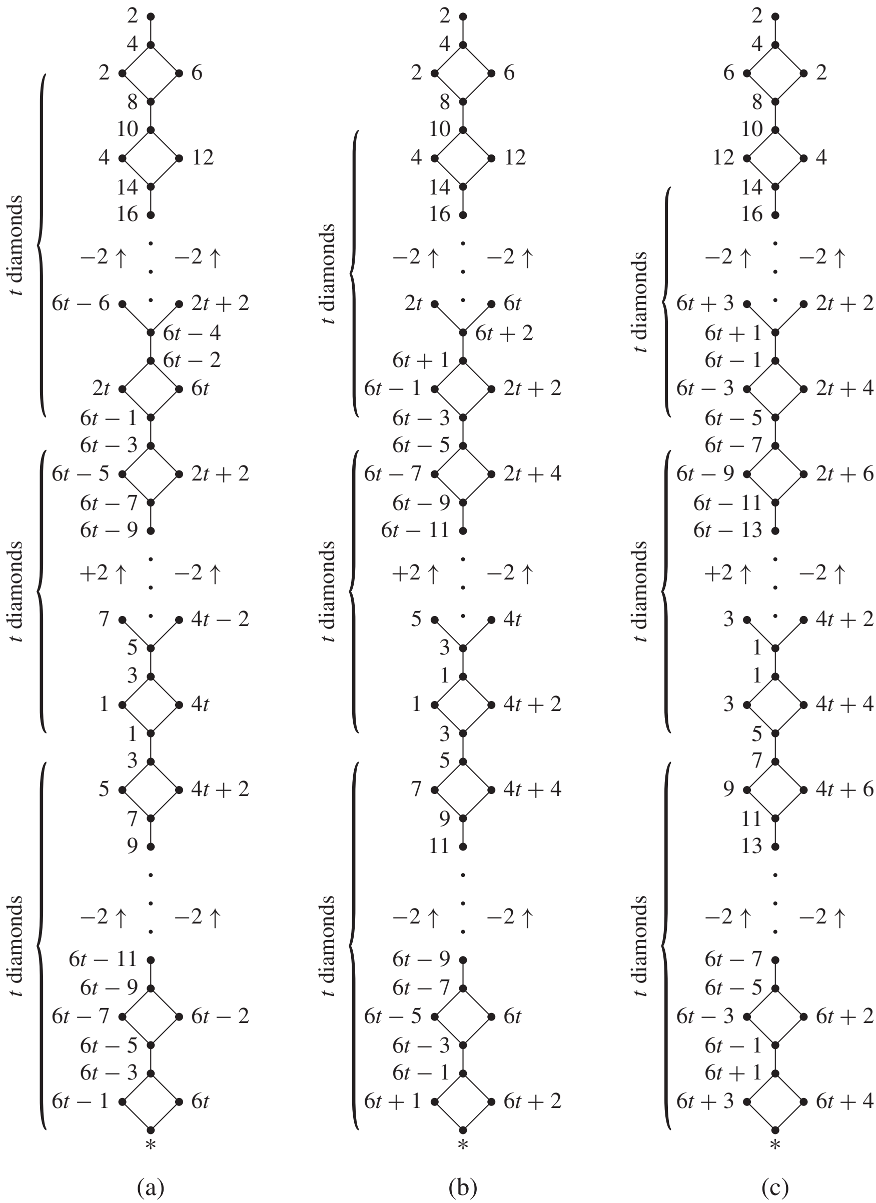


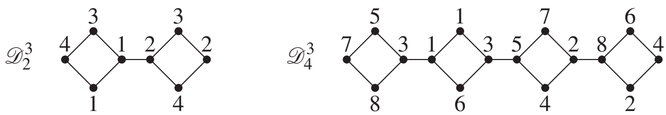
FIGURE 12. Hooked Skolem labelled graph $\mathscr{D}_d^2$ (a) for $d \equiv 0 \pmod 3$, (b) $d \equiv 1 \pmod 3$ and (c) $d \equiv 2 \pmod 3$.

5.3. More rail-siding graphs

While there are infinitely many classes of rail-siding graphs, we introduce two more classes and give necessary conditions for the existence of their Skolem labellings.

A type III rail-siding graph $\mathscr{D}_d^3$, is a graph with P_{3d} being its main rail, $\Delta = 3$ and the end vertices are not inflated; see in Figure 13. Examples of two Skolem labelled type III rail-siding graphs are presented in Figure 14.

FIGURE 13. $\mathscr{D}_d^3$.



P_2^3 $\underline{4}\ \underline{\overset{3}{1}}\ \underline{1}\ \underline{2}\ \underline{\overset{3}{4}}\ \underline{2}$ $\qquad$ P_4^3 $\underline{7}\ \underline{\overset{8}{5}}\ \underline{3}\ \underline{1}\ \underline{\overset{6}{1}}\ \underline{3}\ \underline{5}\ \underline{\overset{7}{4}}\ \underline{2}\ \underline{8}\ \underline{\overset{6}{2}}\ \underline{4}$

FIGURE 14. Skolem labelled graphs $\mathscr{D}_2^3$ and $\mathscr{D}_4^3$ and their corresponding sequences.

Having a Skolem labelled graph $\mathscr{D}_d^3$ is equivalent to a sequence with d pockets such that:

(1) the set of labels is $\{1, 2, 3, \ldots, 2d\}$ (since $|V(\mathscr{D}_d^3)| = 4d$),
(2) the pockets' positions are $3k + 2$ for $0 \le k \le d - 1$.

If a_i and b_i are the smallest and largest positions of the label i in the sequence, then

$$\begin{aligned}\sum_{i=1}^{2d}(a_i + b_i) &= \sum_{i=1}^{3d} i + \sum_{i=0}^{d-1}(3i + 2) = \frac{1}{2}3d(3d + 1) + 3\sum_{i=1}^{d-1} i + 2d \\ &= 6d^2 + 2d\end{aligned} \tag{5.1}$$

and

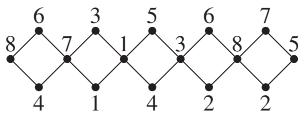
$$\sum_{i=1}^{2d}(b_i - a_i) = \sum_{i=1}^{2d} i = 2d^2 + d. \tag{5.2}$$

By (5.1) and (5.2), $\sum_{i=1}^{2d} b_i = \frac{1}{2}d(8d+3)$. Since $\sum_{i=1}^{2d} b_i \in \mathbb{N}$, we conclude that $d \equiv 0 \pmod 2$. Therefore, we get the following necessary conditions for the graph $\mathscr{D}_d^3$ having a Skolem labelling.

THEOREM 5.1. *The graph $\mathscr{D}_d^3$ has a Skolem labelling only if $d \equiv 0 \pmod 2$.*

The fourth class of graphs that we consider is the class of quadrangular cacti which are denoted by $\mathscr{D}_d^4$ where d is the number of diamond subgraphs. A quadrangular cactus $\mathscr{D}_d^4$, see Figure 15, is a rail-siding graph with P_{2d+1} being its main rail, $\Delta = 4$ and the end vertices are not inflated.

FIGURE 15. Quadrangular cactus $\mathscr{D}_d^4$.



$$\underline{8}\ \underline{\overset{6}{4}}\ \underline{7}\ \underline{\overset{3}{1}}\ \underline{1}\ \underline{\overset{5}{4}}\ \underline{3}\ \underline{\overset{6}{2}}\ \underline{8}\ \underline{\overset{7}{2}}\ \underline{5}$$

FIGURE 16. Skolem labelled graph $\mathscr{D}_5^4$ and its corresponding sequence (another possible sequence is $(7, \overset{2}{2}, 8, \overset{6}{4}, 5, \overset{3}{1}, 1, \overset{7}{4}, 3, \overset{6}{5}, 8)$).

We will find the necessary conditions for the graph $\mathscr{D}_d^4$ to admit a Skolem labelling. Having a Skolem labelled graph $\mathscr{D}_d^4$ is equivalent to having a sequence with d pockets such that:

(1) the set of labels is $\{1, 2, 3, \dots, \frac{1}{2}(3d+1)\}$ (because $|V(\mathscr{D}_d^4)| = 3d+1$),
(2) the pockets' positions are $2k$ for $1 \le k \le d$.

If a_i and b_i are the smallest and largest positions of the label i in the sequence, then

$$\sum_{i=1}^{(3d+1)/2} (a_i + b_i) = \sum_{i=1}^{2d+1} i + \sum_{i=1}^{d} 2i = 3d^2 + 4d + 1 \tag{5.3}$$

and

$$\sum_{i=1}^{(3d+1)/2} (b_i - a_i) = \sum_{i=1}^{\frac{3d+1}{2}} i = \frac{1}{8}(3d+1)(3d+3). \tag{5.4}$$

By (5.3) and (5.4), we have $\sum_{i=1}^{(3d+1)/2} b_i = \frac{11}{16}(3d+1)(d+1)$. Since $\sum_{i=1}^{(3d+1)/2} b_i \in \mathbb{N}$, we conclude that $d \equiv 5, 7 \pmod 8$.

THEOREM 5.2. *The graph $\mathscr{D}_d^4$ admits a Skolem labelling only if $d \equiv 5, 7$ $\pmod 8$.*

The study of the sufficient conditions for these two classes is left as an open question for future studies.

6. Graph theory applications

In this section we see how adding or deleting edges to or from Skolem labelled rail-siding graphs will produce more Skolem labelled graph classes.

6.1. Skolem labelling caterpillars

Caterpillars are trees for which removing all the leaves and incident edges produces a path. Dyer and McKay have established the Skolem labelling of 3-regular caterpillars [4]. A k-regular caterpillar is a caterpillar such that all non-leaf vertices are of degree k.

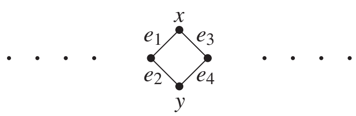


FIGURE 17. A diamond of a Skolem labelled rail-siding graph.

A Skolem labelling of a rail-siding graph gives a Skolem labelling of some caterpillars. To see this, suppose that we have a Skolem labelled rail-siding graph $\mathscr{R}$. Consider a diamond subgraph of $\mathscr{R}$ with two vertices u and v labelled x and y as illustrated in Figure 17. There are two more vertices of $\mathscr{R}$, say u' and v', with labels x and y respectively. If u' and v' are to the left of u and v, then we can omit either e_3 or e_4. A similar argument holds if u' and v' are to the right of u and v. If u' is to the left of u and v' is to the right of v, then we can omit either e_2 or e_3. Doing so for each diamond of the rail-siding graph, we get a Skolem labelled caterpillar.

For example consider the graph $\mathscr{D}_4^3$ in Figure 14. We have re-drawn it in Figure 18 with some of the edges drawn as dashed lines. In each diamond subgraph, one of the dashed lines can be deleted without violating the Skolem labelling. So, having Skolem labelling of $\mathscr{D}_4^3$, we can get four Skolem labelled caterpillars (and more Skolem labelled graphs).

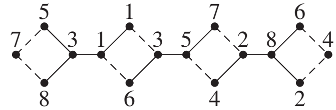
$\mathscr{D}_4^3$


FIGURE 18. One dashed edge per diamond can be removed without violating the Skolem labelling.

6.2. Skolem labelling of other graph classes

We make one more observation that enables us to Skolem label a wide range of graphs. Consider Skolem labelling of a rail-siding graph equivalent to a pseudo-Skolem sequence (no two labels “2” are in the same pocket). By adding an edge between any pair of vertices of a diamond that represent a pocket of the sequence, we get a graph that is still properly Skolem labelled. As it is illustrated in Figure 19, by adding any collection of the dashed edges, we get new graphs that are Skolem labelled.

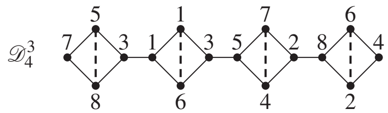


FIGURE 19. Adding any collection of the dashed edges produces new graph which is Skolem labelled by the present labelling.

7. Discussion and open problems

A major advantage of the Skolem-type sequences, besides their wide range of applications, is that they can be manipulated in order to find the type that is useful for a given purpose. The recently defined pseudo-Skolem sequences have similar properties and advantages of Skolem-type sequences. They have demonstrated their potential in Skolem labelling of graphs and we have Skolem labelled classes of rail-siding graphs and some of their spanning subgraphs, caterpillars for example, using them. Constructing more classes of pseudo-Skolem sequences will result in having more graph classes Skolem labelled. Constructing hooked pseudo-Skolem sequences is another potential research focus. It is also rewarding to answer the question of whether pseudo-Skolem sequences can be used to construct other combinatorial structures like combinatorial designs.

REFERENCES

1. Baker, C., Kergin, P., and Bonato, A., *Skolem arrays and Skolem labellings of ladder graphs*, Ars Combin. 63 (2002), 97–107.
2. Baker, C. A., and Manzer, J. D. A., *Skolem-labeling of generalized three-vane windmills*, Australas. J. Combin. 41 (2008), 175–204.
3. Biraud, F., Blum, E. J., and Ribes, J. C., *On optimum synthetic linear arrays with application to radioastronomy*, IEEE Trans. Antennas and Propagation 22 (1974), no. 1, 108–109.
4. Dyer, D., and McKay, N., *Skolem-labeling of k-regular caterpillars*, preprint, 2006.
5. Graham, A. J., Pike, D. A., and Shalaby, N., *Skolem labelled trees and $P_s \square P_t$ Cartesian products*, Australas. J. Combin. 38 (2007), 101–115.

6. Mendelsohn, E., and Shalaby, N., *Skolem labelled graphs*, Discrete Math. 97 (1991), no. 1-3, 301–317.
7. Mendelsohn, E., and Shalaby, N., *On Skolem labelling of windmills*, Ars Combin. 53 (1999), 161–172.
8. O'Keefe, E. S., *Verification of a conjecture of Th. Skolem*, Math. Scand. 9 (1961), 80–82.
9. Shalaby, N., *The existence of near-Skolem and hooked near-Skolem sequences*, Discrete Math. 135 (1994), no. 1-3, 303–319.
10. Shalaby, N., *The existence of near-Rosa and hooked near-Rosa sequences*, Discrete Math. 261 (2003), no. 1-3, 435–450.
11. Skolem, Th., *On certain distributions of integers in pairs with given differences*, Math. Scand. 5 (1957), 57–68.

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