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# FREE RESOLUTION OF POWERS OF MONOMIAL IDEALS AND GOLOD RINGS

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**Abstract**

Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. In this paper we present a criterion for componentwise linearity of powers of monomial ideals. In particular, we prove that if a square-free monomial ideal $I$ contains no variable and some power of $I$ is componentwise linear, then $I$ satisfies the gcd condition. For a square-free monomial ideal $I$ which contains no variable, we show that $S/I$ is a Golod ring provided that for some integer $s \geq 1$, the ideal $I^s$ has linear quotients with respect to a monomial order.

## 1. Introduction and preliminaries

Over the last 20 years the study of algebraic, homological and combinatorial properties of powers of ideals has been one of the major topics in Commutative Algebra. In this paper we study the minimal free resolution of the powers of monomial ideals. First we give some definitions and basic facts.

Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. For any finitely generated $\mathbb{Z}^n$-graded $S$-module $M$ and every $\mathbf{a} \in \mathbb{Z}^n$, let $M_{\mathbf{a}}$ denote its graded piece of degree $\mathbf{a}$ and let $M(\mathbf{a})$ denote the *twisting* of $M$ by $\mathbf{a}$, i.e. the module where $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$. As usual for every degree $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we denote by $|\mathbf{a}|$ the total degree of $\mathbf{a}$ which is equal to $a_1 + \cdots + a_n$.

A *minimal graded free resolution* of $M$ is an exact complex

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where each $F_i$ is a $\mathbb{Z}^n$-graded free $S$-module of the form $\oplus_{\mathbf{a}\in\mathbb{Z}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}(M)}$ such that the number of basis elements is minimal and each map is graded. The value $\beta_{i,\mathbf{a}}(M)$ is called the $i$th $\mathbb{Z}^n$-graded Betti number of degree $\mathbf{a}$. The number $\beta_{i,j}(M) = \sum_{|\mathbf{a}|=j} \beta_{i,\mathbf{a}}(M)$ is called the $i$th $\mathbb{Z}$-graded Betti number of degree $j$. These numbers are independent of the choice of resolution of $M$, thus yielding important numerical invariants of $M$. To simplify the notation, for every monomial $u = \mathbf{x}^{\mathbf{a}} = x_1^{a_1} \dots x_n^{a_n}$, we write $\beta_{i,\mathbf{u}}(M)$ instead of $\beta_{i,\mathbf{a}}(M)$.

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Let $I$ be a monomial ideal of $S$ and let $G(I)$ denote the set of minimal monomial generators of $I$. Then $I$ is said to have a *linear resolution*, if there is an integer $d$ such that $\beta_{i,i+t}(I) = 0$ for all $i$ and every $t \neq d$. It is clear from the definition that if a monomial ideal has a linear resolution, then all the minimal monomial generators of $I$ have the same degree.

There have been many attempts to characterize the monomial ideals with a linear resolution. One of the most important results in this direction is due to Fröberg [4, Theorem 1], who characterized all square-free monomial ideals generated by quadratic monomials that have linear resolutions. It is also interesting to ask whether some powers of a given monomial ideal $I$ has a linear resolution. It is known [13] that polymatroidal ideals have linear resolutions and that powers of polymatroidal ideals are again polymatroidal (see [9]). In particular they have again linear resolutions. In general however, powers of ideals with linear resolution need not to have linear resolutions. The first example of such an ideal was given by Terai. He showed that over a base field of characteristic $\neq 2$ the Stanley-Reisner ideal

$$I = (x_1x_2x_3, x_1x_2x_5, x_1x_3x_6, x_1x_4x_5, x_1x_4x_6, \\ x_2x_3x_4, x_2x_4x_6, x_2x_5x_6, x_3x_4x_5, x_3x_5x_6)$$

of the minimal triangulation of the projective plane has a linear resolution, while $I^2$ has no linear resolution. This example depends on the characteristic of the base field. If the base field has characteristic 2, then $I$ itself has no linear resolution. Another example, namely

$$I = (x_4x_5x_6, x_3x_5x_6, x_3x_4x_6, x_3x_4x_5, x_2x_5x_6, x_2x_3x_4, x_1x_3x_6, x_1x_4x_5)$$

is given by Sturmfels [18]. Again $I$ has a linear resolution, while $I^2$ has no linear resolution. However, Herzog, Hibi and Zheng [10] prove that a monomial ideal $I$ generated in degree 2 has linear resolution if and only if every power of $I$ has linear resolution.

Componentwise linear ideals are introduced by Herzog and Hibi [8] and they are defined as follows. For a monomial ideal $I$ we write $I_{\langle j \rangle}$ for the ideal generated by all monomials of degree $j$ belonging to $I$. A monomial ideal $I$ is called *componentwise linear* if $I_{\langle j \rangle}$ has a linear resolution for all $j$. In Section 2 we study the componentwise linearity of powers of monomial ideals. For a monomial $u$, the *support* of $u$, denoted by $\mathrm{supp}(u)$, is the set of variables which divide $u$. We prove that if $I$ is a monomial ideal, which contains no variable and some power of $I$ is componentwise linear, then for every pair of monomials $u, v \in G(I)$ with $\gcd(u, v) = 1$, there exists a monomial $w \in G(I)$ such that $w \neq u, v$ and $\mathrm{supp}(w) \subseteq \mathrm{supp}(u) \cup \mathrm{supp}(v)$ (see Theorem 2.3). Let $I$ be a square-free monomial ideal which contains no variable. We prove that if some power of $I$ has linear quotients with respect to a monomial order (see Definition 2.7), then $S/I$ is a Golod ring (see Theorem 2.11).

One of the main tools which is used in this paper is the lcm-lattice, defined by Gasharov, Peeva and Welker [5].

DEFINITION 1.1. Let $I$ be a monomial ideal minimally generated by monomials $m_1, \dots, m_d$. We denote by $L_I$ the lattice with elements labeled by the least common multiples of $m_1, \dots, m_d$ ordered by divisibility. In particular, the atoms in $L_I$ are $m_1, \dots, m_d$, the maximal element is $\operatorname{lcm}(m_1, \dots, m_d)$, and the minimal element is 1 regarded as the lcm of the empty set. The least common multiple of elements in $L_I$ is their join, i.e. their least common upper bound in the poset $L_I$. We call $L_I$ the *lcm-lattice* of $I$.

Gasharov, Peeva and Welker proved that the lcm-lattice of a monomial ideal determines its free resolution. For an open interval $(1, m)$ in $L_I$, we denote by $\Delta(1, m)$ the order complex of the interval, i.e. the simplicial complex whose facets are the maximal chains in $(1, m)$. Let $\widetilde{H}_i(\Delta(1, m); \mathbb{K})$ denote the $i$th reduced homology of $\Delta(1, m)$ with coefficients in $\mathbb{K}$. By [5, Theorem 3.3], the $\mathbb{Z}^n$-graded Betti numbers of $I$ can be computed by the homology of the open intervals in $L_I$ as follows: if $m \notin L_I$ then $\beta_{i,m}(I) = 0$ for every $i$, and if $m \in L_I$ and $i \geq 1$, we have

$$\beta_{i,m}(I) = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(\Delta(1, m); \mathbb{K}).$$

We use this formula to prove Theorem 2.3.

## 2. Componentwise linearity and Golod rings

As the first main result of this section, we provide a criterion for the componentwise linearity of powers of monomial ideals. We begin by the definition of the gcd condition and the strong gcd condition which are defined by Jöllenbeck as follows.

DEFINITION 2.1 ([15, Definition 3.8]).

(i) A monomial ideal $I$ is said to satisfy the *gcd condition*, if for any two monomials $u, v \in G(I)$ with $\gcd(u, v) = 1$ there exists a monomial $w \neq u, v$ in $G(I)$ with $w \mid \operatorname{lcm}(u, v) = uv$.

(ii) A monomial ideal $I$ is said to satisfy the *strong gcd condition*, if there exists a linear order $\prec$ on $G(I)$ such that for any two monomials $u \prec v \in G(I)$ with $\gcd(u, v) = 1$ there exists a monomial $w \neq u, v$ in $G(I)$ with $u \prec w$ and $w \mid \operatorname{lcm}(u, v) = uv$.

As generalizations of the gcd condition and the strong gcd condition, we define the concepts of the semi-gcd condition and the strong semi-gcd condition.

DEFINITION 2.2.

(i) A monomial ideal $I$ is said to satisfy the *semi-gcd condition*, if for any two monomials $u, v \in G(I)$ with $\gcd(u, v) = 1$ there exists a monomial $w \neq u, v$ in $G(I)$ with $\mathrm{supp}(w) \subseteq \mathrm{supp}(u) \cup \mathrm{supp}(v)$.

(ii) A monomial ideal $I$ is said to satisfy the *strong semi-gcd condition*, if there exists a linear order $\prec$ on $G(I)$ such that for any two monomials $u \prec v \in G(I)$ with $\gcd(u, v) = 1$ there exists a monomial $w \neq u, v$ in $G(I)$ with $u \prec w$ and $\mathrm{supp}(w) \subseteq \mathrm{supp}(u) \cup \mathrm{supp}(v)$.

It is clear that every monomial ideal with the gcd condition (resp. the strong gcd condition) satisfies the semi-gcd condition (resp. the strong semi-gcd condition).

The following theorem is the first main result of this paper. It provides a criterion for componentwise linearity of powers of a monomial ideal.

THEOREM 2.3. *Let $I$ be a monomial ideal, which contains no variable. Assume that there exists an integer $s \geq 1$ such that $I^s$ is componentwise linear. Then $I$ satisfies the semi-gcd condition.*

PROOF. For a contradiction, suppose that there exist $u, v \in G(I)$ such that $\gcd(u, v) = 1$ and there is no monomial $w \in G(I)$ with $\mathrm{supp}(w) \subseteq \mathrm{supp}(u) \cup \mathrm{supp}(v)$. Without loss of generality assume that $\deg(u) = d' \leq \deg(v) = d$. Note that $v^s \in I^s_{\langle ds \rangle}$. On the other hand $uv^{s-1} \in I^s$. So if we multiply $uv^{s-1}$ by a divisor of $v$ of degree $d - d'$, we obtain a monomial $u' \in I^s_{\langle ds \rangle}$ with $\mathrm{lcm}(u', v^s) = uv^s$.

Consider the open interval $(1, uv^s)$ in the lcm-lattice of $I^s_{\langle ds \rangle}$. We claim that the atom $v^s$ is an isolated vertex of $\Delta(1, uv^s)$. Assume that this is not true. Then there exists an atom $w' \in (1, uv^s)$ such that $w' \neq v^s$ and $\mathrm{lcm}(v^s, w')$ strictly divides $uv^s$. This implies that $\gcd(w', u) \neq 1$. Since $w' \in I^s$ and since there is no monomial $w \in G(I)$ such that

$$\mathrm{supp}(w) \subseteq \mathrm{supp}(u) \cup \mathrm{supp}(v),$$

it follows that $u \mid w'$. Thus $\mathrm{lcm}(v^s, w') = uv^s$, which is a contradiction. This proves our claim. Now $u'$ is another vertex of $\Delta(1, uv^s)$ and thus $\Delta(1, uv^s)$ is disconnected. Hence by [5, Theorem 3.3]

$$\beta_{1,uv^s}(I^s_{\langle ds \rangle}) = \dim_{\mathbb{K}} \tilde{H}_0(\Delta(1, uv^s); \mathbb{K}) \geq 1$$

This in particular shows that $\beta_{1,ds+d'}(I^s_{\langle ds \rangle}) \neq 0$. Since $I$ contains no variable, it follows that $d' \geq 2$ and therefore the minimal free resolution of $I^s_{\langle ds \rangle}$ is not linear, which is a contradiction.

One should note that a square-free monomial ideal $I$ satisfies the gcd condition if and only if it satisfies the semi-gcd condition. Thus as a consequence of Theorem 2.3 we conclude the following corollary.

COROLLARY 2.4. *Let $I$ be a square-free monomial ideal which contains no variable. Assume that $I^s$ is componentwise linear for some $s \geq 1$. Then $I$ satisfies the gcd condition.*

To any finite simple graph $G$ with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$, one associates an ideal $I(G)$ generated by all quadratic monomials $x_i x_j$ such that $\{v_i, v_j\} \in E(G)$. The ideal $I(G)$ is called the *edge ideal of $G$*. We recall that for a graph $G = (V(G), E(G))$, its *complementary graph $G^c$* is a graph with $V(G^c) = V(G)$ and $E(G^c)$ consists of those 2-element subsets $\{v_i, v_j\}$ of $V(G)$ for which $\{v_i, v_j\} \notin E(G)$. If we restrict ourselves to edge ideal of graphs, we obtain the following unpublished result which is due to Francisco, Hà and Van Tuyl.

COROLLARY 2.5. *Let $I = I(G)$ be the edge ideal of a graph $G$. If $I^s$ has linear resolution for some $s \geq 1$, then $G^c$ has no induced* 4*-cycle.*

PROOF. Assume that the assertion is not true, so $G^c$ has an induced 4-cycle, say $v_1, v_2, v_3, v_4$. Set $u = x_1 x_3$ and $v = x_2 x_4$. Then $u, v \in G(I)$ and $\gcd(u, v) = 1$. But there is no monomial $w \in G(I)$ such that $w \mid uv$. Hence it follows from Corollary 2.4 that no power of $I$ can have linear resolution.

REMARK 2.6. A short proof is presented by Nevo and Peeva in [16, Proposition 1.8]; note that there is a repeated typo in their proof and $(x_p x_q)^s$ has to be replaced by $x_p x_q$ throughout.

Ideals with linear quotients were first considered in [13] and they are a large subclass of componentwise linear ideals.

DEFINITION 2.7. Let $I$ be a monomial ideal and let $G(I)$ be the set of minimal monomial generators of $I$. Assume that $u_1 \prec u_2 \prec \cdots \prec u_t$ is a linear order on $G(I)$. We say that $I$ has *linear quotients with respect to $\prec$*, if for every $2 \leq i \leq t$, the ideal $(u_1, \dots, u_{i-1}) : u_i$ is generated by a subset of variables. We say that $I$ has *linear quotients*, if it has linear quotients with respect to a linear order on $G(I)$.

In the following theorem we examine linear quotients for powers of monomial ideals.

THEOREM 2.8. *Let $I$ be a monomial ideal which contains no variable and let $s \geq 1$ be an integer. Assume that $I^s$ has linear quotients with respect to a monomial order $<$ on $G(I^s)$. Then $I$ satisfies the strong semi-gcd condition.*

PROOF. Let $G(I) = \{u_1, \dots, u_t\}$ and assume that $u_1 < u_2 < \cdots < u_t$. We consider the linear order $u_1 \succ u_2 \succ \cdots \succ u_t$ on $G(I)$. We prove that using this order, $I$ satisfies the desired property. So suppose that there exist $1 \le i < j \le t$ such that $\gcd(u_i, u_j) = 1$. We have to show that there exists $k$ with $k \ne i$ and $k < j$ such that $\mathrm{supp}(u_k) \subseteq \mathrm{supp}(u_i) \cup \mathrm{supp}(u_j)$. Since $<$ is a monomial order, it follows that $u_i^s < u_j^s$. Since $I^s$ has linear quotients with respect to $<$, we conclude that there exists a monomial $w \in G(I^s)$ such that $w < u_j^s$ and

$$\frac{w}{\gcd(u_j^s, w)} = x_\ell, \qquad (\dagger)$$

for some $1 \le \ell \le n$, and moreover

$$x_\ell \,\Big|\, \frac{u_i^s}{\gcd(u_j^s, u_i^s)}. \qquad (\ddagger)$$

Since $w \in I^s$, we can write $w = u_{k_1} \dots u_{k_s}$ for some integers $1 \le k_1 \le k_2 \le \cdots \le k_s \le t$. Since $w < u_j^s$, it follows that $u_{k_1} < u_j$ and thus $k_1 < j$. To simplify the notation, we denote $k_1$ by $k$. It follows from (†) that

$$\mathrm{supp}(u_k) \subseteq \mathrm{supp}(u_j) \cup \{x_\ell\}.$$

We consider two cases.

*Case 1.* $x_\ell \nmid u_k$ or $x_\ell \mid u_j$. In this case

$$\mathrm{supp}(u_k) \subseteq \mathrm{supp}(u_j) \subseteq \mathrm{supp}(u_j) \cup \mathrm{supp}(u_i),$$

and it follows from the first inclusion that $\gcd(u_k, u_j) \ne 1$ and therefore $u_k \ne u_i$. Hence the assertion is true in this case.

*Case 2.* $x_\ell \mid u_k$ and $x_\ell \nmid u_j$. It follows from (‡) that $x_\ell \in \mathrm{supp}(u_i)$. Therefore

$$\mathrm{supp}(u_k) \subseteq \mathrm{supp}(u_j) \cup \mathrm{supp}(u_i).$$

Since $I$ contains no variable, $u_k \ne x_\ell$. On the other hand since $x_\ell \nmid u_j$, using (†), we conclude that $x_\ell^2 \nmid w$ and therefore $x_\ell^2 \nmid u_k$. This shows that $\mathrm{supp}(u_k) \ne \{x_\ell\}$ and hence

$$\mathrm{supp}(u_k) \cap \mathrm{supp}(u_j) \ne \varnothing.$$

Thus $\gcd(u_j, u_k) \ne 1$. Therefore $u_k \ne u_i$ and this completes the proof.

If we restrict ourselves to the class of square-free monomial ideals, we obtain the following result.

COROLLARY 2.9. *Let $I$ be a square-free monomial ideal which contains no variable. Assume that for some integer $s \ge 1$, the ideal $I^s$ has linear quotients with respect to a monomial order. Then $I$ satisfies the strong gcd condition.*

PROOF. One should only note that for every pair of square-free monomials $u, v$ with $\gcd(u, v) = 1$, a square-free monomial $w$ divides $uv$ if and only if

$$\operatorname{supp}(w) \subseteq \operatorname{supp}(u) \cup \operatorname{supp}(v).$$

This shows that a square-free monomial ideal $I$ satisfies the strong gcd condition if and only if it satisfies the strong semi-gcd condition. Now the assertion follows from Theorem 2.8.

Let $I$ be a square-free monomial ideal which contains no variable. In Corollary 2.9 we proved that if for some integer $s \geq 1$, the ideal $I^s$ has linear quotients with respect to a suitable order, then $I$ itself satisfies the strong gcd condition. Based on this result, it is natural to ask whether the same assertion holds if $I^s$ has linear quotients with respect to an arbitrary order. Indeed, we can ask the following question.

QUESTION 2.10. Let $I$ be a square-free monomial ideal which contains no variable. Assume that for some integer $s \geq 1$, the ideal $I^s$ has linear quotients. Does $I$ satisfy the strong gcd condition?

We note that by [2, Proposition 6], the answer of Question 2.10 is positive when $I$ itself has linear quotients. However, in Example 2.13 we give a negative answer to this question.

For a monomial ideal $I$ in $S$ the ring $S/I$, is called *Golod* if all Massey operations on the Koszul complex of $S/I$ with respect of $\mathrm{x} = x_1, \ldots, x_n$ vanish. Golod [6] showed that the vanishing of the Massey operations is equivalent to the equality case in the following coefficientwise inequality of power-series which was first derived by Serre:

$$\sum_{i\geq 0} \dim_{\mathbb{K}} \operatorname{Tor}_i^{S/I}(\mathbb{K}, \mathbb{K}) t^i \leq \frac{(1+t)^n}{1 - t\sum_{i\geq 1} \dim_{\mathbb{K}} \operatorname{Tor}_i^S(S/I, \mathbb{K}) t^i}$$

We refer the reader to [1] and [7] for further information on the Golod property.

By [3, Theorem 5.5], we know that if a monomial ideal $I$ satisfies the strong gcd condition, then $S/I$ is a Golod ring. Thus as a consequence of Corollary 2.9, we obtain the following result.

THEOREM 2.11. *Let $I$ be a square-free monomial ideal which contains no variable. Assume that for some integer $s \geq 1$, the ideal $I^s$ has linear quotients with respect to a monomial order. Then $S/I$ is a Golod ring.*

Let $I$ be a monomial ideal. Jöllenbeck [15, Lemma 7.4] proves that if $S/I$ is a Golod ring, then $I$ satisfies the gcd condition. In Corollary 2.4 we proved that if $I$ is a square-free monomial ideal which contains no variable, such that $I^s$ is componentwise linear for some $s \geq 1$, then $I$ satisfies the gcd condition. So it is natural to ask whether an stronger result is true (see Question 2.12).

QUESTION 2.12. Let $I$ be a square-free monomial ideal which contains no variable. Assume that $I^s$ is componentwise linear for some $s \geq 1$. Is $S/I$ a Golod ring?

We note that by [12, Theorem 4] the answer to Question 2.12 is positive in the case of $s = 1$. We also note that by [17, Theorem 1.1], for every integer $s \geq 2$ and every monomial ideal $I$, the ring $S/I^s$ is Golod (see also [11]). However, in Example 2.13 we give a negative answer to this question.

EXAMPLE 2.13. Let $G$ be the complement of the cycle with six vertices (i.e. $G = C_6^c$) and $I$ be the edge ideal of $G$. It is shown in [14, Example 4.3] that $I^2$ has linear quotients (and thus has linear resolution). But it follows from [2, Proposition 15] that $S/I$ is not a Golod ring and hence $I$ does not satisfy the strong gcd condition. This shows that the answers to Questions 2.10 and 2.12 are negative. This also shows that if some powers of a monomial ideal has linear quotients (resp. linear resolution), then the ideal itself does not necessarily have linear quotients (resp. linear resolution).

We close the paper with the following remark.

REMARK 2.14. In all results of the paper, we assume that the considered ideal contains no variable. The results are not true without this assumption. For example the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ has linear resolution (even linear quotients), but it does not satisfy the gcd condition.

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