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HERMITIAN SYMMETRIC SPACES OF TUBE TYPE AND MULTIVARIATE MEIXNER-POLLACZEK POLYNOMIALS

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Abstract

Harmonic analysis on Hermitian symmetric spaces of tube type is a natural framework for introducing multivariate Meixner-Pollaczek polynomials. Their main properties are established in this setting: orthogonality, generating and determinantal formulae, difference equations. For proving these properties we use the composition of the following transformations: Cayley transform, Laplace transform, and spherical Fourier transform associated to Hermitian symmetric spaces of tube type. In particular the difference equation for the multivariate Meixner-Pollaczek polynomials is obtained from an Euler type equation on a bounded symmetric domain.

1. Introduction

The one variable Meixner-Pollaczek polynomials $P_m^{\alpha}(\lambda;\phi)$ can be defined by the Gaussian hypergeometric representation as

$$P_m^{(\nu/2)}(\lambda;\phi)=\frac{(\nu)_m}{m!}e^{im\phi}\,{}_2F_1\Big(-m,\frac{\nu}{2}+i\lambda;\nu;1-e^{-2i\phi}\Big).$$

For $\phi=\pi/2$ the Meixner-Pollaczek polynomials $P_m^{(\nu/2)}(\lambda;\pi/2)$ are also obtained as Mellin transforms of Laguerre functions. Their main properties follow from this fact: hypergeometric representation above, orthogonality, generating formula, difference equation, and three terms relation (see [1, pp. 348–349]).

These polynomials $P_m^{(\nu/2)}(\lambda;\pi/2)$ have been generalized to the multivariate case. In fact, the multivariable Meixner-Pollaczek (symmetric) polynomials have been essentially considered in the setting of the Fourier analysis on Riemannian symmetric spaces in several papers: See Peetre-Zhang [12, Appendix 2: A class of hypergeometric orthogonal polynomials] , Ørsted-Zhang [11, section 3.4], Zhang [15] and Davidson-Ólafsson-Zhang [5]. Also, see the papers by Davidson-Ólafsson [4] and Aristidou-Davidson-Ólafsson [2]. Further, for an arbitrary real value of the multiplicity d, the multivariate

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Meixner-Pollaczek polynomials are defined by Sahi-Zhang [13] in the setting of Heckman-Opdam and Cherednik-Opdam transforms, related to symmetric and non-symmetric Jack polynomials, and generating formulae for them are established. However the case where the parameter ϕ is involved has not been studied so far. Moreover, once we define the multivariate Meixner-Pollaczek polynomials with parameter ϕ, it is also important to clarify a geometric meaning of the parameter. Establishing a natural setting for the study of multivariate Meixner-Pollaczek polynomials with such parameter, one can expect to obtain wider applications such as a study of multi-dimensional Lévi-process, in particular, introducing multi-dimensional Meixner process (see [14] for the one-dimensional case).

The purpose of this article is to provide a geometric framework for introducing the multivariate Meixner-Pollaczek polynomials (with parameter ϕ) and study their fundamental properties. Our analysis may explain much simpler geometric understanding of several basic properties of the multivariate Meixner-Pollaczek polynomials than ever, even in the case $\phi = \pi/2$. For instance, the $\mathfrak{S}_n$-invariant difference operator of which the multivariate Meixner-Pollaczek polynomials are eigenfunctions can be understood by an image of the Euler operator under the composition of three intertwiners: the Cayley transform, the Laplace transform and the spherical Fourier transform. In particular, the multivariate Meixner-Pollaczek polynomials are spherical Fourier transforms of multivariate Laguerre functions.

In Section 2 we recall the basic facts about the spherical Fourier analysis on a symmetric cone. In Section 3 we define the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu)}(\mathbf{s})$ (the case $\phi = \pi/2$), where $\mathbf{m}$ is a partition, prove that they are orthogonal with respect to a measure M_ν on $\mathbb{R}^n$, and establish a generating formula.

In Section 4, adding a real parameter θ (instead of $\phi = \theta + \frac{\pi}{2}$), we introduce the symmetric polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s})$ in the variables $\mathbf{s} = (s_1, \ldots, s_n)$, $Q_{\mathbf{m}}^{(\nu)} = Q_{\mathbf{m}}^{(\nu,0)}$. In the one variable case

$$q_m^{(\nu,\theta)}(s) = (-i)^m \, P_m^{(\nu/2)}\left(-is; \theta + \frac{\pi}{2}\right).$$

The orthogonality property for the polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s})$ is obtained by using a Gutzmer formula for the spherical Fourier transform. A generating formula is obtained for these polynomials. In case of the multiplicity $d = 2$, we establish in Section 5 determinantal formulae for multivariate Laguerre and Meixner-Pollaczek polynomials. Sections 6, 7, and 8 are devoted to a difference equation satisfied by the polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s})$. Starting from an Euler-type equation involving the parameter θ, this difference equation is obtained in three steps, corresponding to a Cayley transform, an inverse Laplace transform, and a spherical Fourier transform for symmetric cones. The symmetry $\theta \mapsto -\theta$ in the parameter is related to geometric symmetries and to a generalized Tricomi theorem for the Hankel transform on a symmetric cone. In the last section we show that multivariate Meixner-Pollaczek polynomials satisfy a Pieri's formula. In the one variable case it reduces to the three terms relation satisfied by the classical Meixner-Pollacek polynomials.

2. Spherical Fourier analysis on a symmetric cone

A reference for this preliminary section is [8]. We consider an irreducible symmetric cone Ω in a Euclidean Jordan algebra V. We denote by G the identity component in the group $G(\Omega)$ of linear automorphisms of Ω, and $K \subset G$ is the isotropy subgroup of the unit element $e \in V$.

The Gindikin gamma function Γ_Ω of the cone Ω will be the cornerstone of the analysis we will develop. It is defined, for $\mathbf{s} \in \mathbb{C}^n$, with $\operatorname{Re} s_j > \frac{d}{2}(j-1)$, by

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr}(u)} \Delta_{\mathbf{s}}(u) \Delta(u)^{-N/n}\, m(du).$$

The notation $\operatorname{tr}(u)$ and $\Delta(u)$ denote the trace and the determinant with respect to the Jordan algebra structure, $\Delta_{\mathbf{s}}$ is the power function, N and n are the dimension and the rank of V, and m is the Euclidean measure associated to the Euclidean structure on V given by $(u \mid v) = \operatorname{tr}(uv)$. Its evaluation gives

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{(N-n)/2} \prod_{j=1}^{n} \Gamma\left(s_j - \frac{d}{2}(j-1)\right),$$

where d is the multiplicity, related to N and n by the relation $N = n + \frac{d}{2}n(n-1)$. The spherical function $\varphi_{\mathbf{s}}$, for $\mathbf{s} \in \mathbb{C}^n$, is defined on Ω by

$$\varphi_{\mathbf{s}}(u) = \int_K \Delta_{\mathbf{s}+\rho}(k \cdot u)\, dk,$$

where $\rho = (\rho_1, \ldots, \rho_n)$, $\rho_j = \frac{d}{4}(2j - n - 1)$, and dk is the normalized Haar measure on the compact group K. The algebra $\mathbb{D}(\Omega)$ of G-invariant differential operators on Ω is commutative, and the spherical function $\varphi_{\mathbf{s}}$ is an eigenfunction of every $D \in \mathbb{D}(\Omega)$:

$$D\varphi_{\mathbf{s}} = \gamma_D(\mathbf{s})\varphi_{\mathbf{s}}.$$

The function γ_D is a symmetric polynomial function, and the map $D \mapsto \gamma_D$ is an algebra isomorphism from $\mathbb{D}(\Omega)$ onto the algebra $\mathscr{P}(\mathbb{C}^n)^{\mathfrak{S}_n}$ of symmetric polynomial functions, a special case of the Harish-Chandra isomorphism. The spherical Fourier transform $\mathscr{F}\psi$ of a K-invariant function ψ on Ω is given by

$$\mathscr{F}\psi(\mathbf{s}) = \int_{\Omega} \psi(u)\varphi_{\mathbf{s}}(u)\Delta^{-N/n}(u)\, m(du).$$

Hence, for $\psi(u) = e^{-\operatorname{tr} u}\Delta^{\nu/2}(u)$, $\nu > \frac{d}{2}(n-1)$, we have

$$\mathscr{F}\psi(\mathbf{s}) = \Gamma_{\Omega}\left(\mathbf{s} + \frac{\nu}{2} + \rho\right) = (2\pi)^{(N-n)/2} \prod_{j=1}^{n} \Gamma\left(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\right).$$

For $D \in \mathbb{D}(\Omega)$ an invariant differential operator, $\mathscr{F}(D\psi)(\mathbf{s}) = \gamma_D(-\mathbf{s})\mathscr{F}\psi(\mathbf{s})$ holds. The space $\mathscr{P}(V)$ of polynomials on V decomposes under G as the multiplicity-free representation

$$\mathscr{P}(V) = \bigoplus_{\mathbf{m}} \mathscr{P}_{\mathbf{m}},$$

where $\mathscr{P}_{\mathbf{m}}$ is a finite dimensional subspace, irreducible under G. The parameter $\mathbf{m}$ is a partition: $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, $m_1 \geq \cdots \geq m_n$. The polynomials in $\mathscr{P}_{\mathbf{m}}$ are homogeneous of degree $|\mathbf{m}| := m_1 + \cdots + m_n$. The subspace $\mathscr{P}_{\mathbf{m}}^K$ of K-invariant polynomials in $\mathscr{P}_{\mathbf{m}}$ is one-dimensional, generated by the spherical polynomial $\Phi_{\mathbf{m}}$, normalized by the condition $\Phi_{\mathbf{m}}(e) = 1$, and so $\Phi_{\mathbf{m}} = \varphi_{\mathbf{m}-\rho}$. There is a unique invariant differential operator $D^{\mathbf{m}}$ such that

$$D^{\mathbf{m}}\psi(e) = \left(\Phi_{\mathbf{m}}\left(\frac{\partial}{\partial u}\right)\psi\right)(e).$$

We will write $\gamma_{\mathbf{m}} = \gamma_{D^{\mathbf{m}}}$. For $n = 1$, observe that $\Phi_m(u) = u^m$,

$$D^m = u^m \left(\frac{d}{du}\right)^m \quad \text{and} \quad \gamma_m(s) = [s]_m := s(s-1)\ldots(s-m+1).$$

The classical Pochhammer symbol $(\alpha)_m := \alpha(\alpha+1)\ldots(\alpha+m-1)$ generalizes as follows: for $\alpha \in \mathbb{C}$ and a partition $\mathbf{m}$,

$$(\alpha)_{\mathbf{m}} = \frac{\Gamma_{\Omega}(\mathbf{m}+\alpha)}{\Gamma_{\Omega}(\alpha)} = \prod_{i=1}^{n} \left(\alpha - (i-1)\frac{d}{2}\right)_{m_i}.$$

If a K-invariant function ψ is analytic in a neighborhood of e, it admits a spherical Taylor expansion near e:

$$\psi(e+v) = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{m}}} D^{\mathbf{m}}\psi(e)\Phi_{\mathbf{m}}(v),$$

where $d_{\mathbf{m}}$ is the dimension of $\mathscr{P}_{\mathbf{m}}$. In particular, for $\psi = \varphi_{\mathbf{s}}$, a spherical function,

$$\varphi_{\mathbf{s}}(e+v) = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \gamma_{\mathbf{m}}(\mathbf{s}) \Phi_{\mathbf{m}}(v).$$

For $\psi = \Phi_{\mathbf{m}} = \varphi_{\mathbf{m}-\rho}$, we get the spherical binomial formula

$$\Phi_{\mathbf{m}}(e+v) = \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(v).$$

In fact the generalized binomial coefficient

$$\binom{\mathbf{m}}{\mathbf{k}} = d_{\mathbf{k}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)$$

vanishes if $\mathbf{k} \not\subset \mathbf{m}$.

3. Multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu)}$

For $n = 1$, we define the Meixner-Pollaczek polynomial $q_m^{(\nu)}$ as follows:

$$q_m^{(\nu)}(s) = \frac{(\nu)_m}{m!} \, {}_2F_1\left(-m, s+\frac{\nu}{2}; \nu; 2\right).$$

This definition differs slightly from the classical one $P_m^{\alpha}(\lambda; \phi)$, as

$$q_m^{(\nu)}(i\lambda) = (-i)^m P_m^{\nu/2}(\lambda; \pi/2)$$

(see for instance [1, p. 348].) Its expansion can be written

$$q_m^{(\nu)}(s) = \frac{(\nu)_m}{m!} \sum_{k=0}^{m} \frac{[m]_k\left[-s-\frac{\nu}{2}\right]_k}{(\nu)_k} \frac{1}{k!} 2^k.$$

The polynomials $q_m^{(\nu)}(i\lambda)$ are orthogonal with respect to the weight on $\mathbb{R}$

$$\left|\Gamma\left(i\lambda+\frac{\nu}{2}\right)\right|^2 \qquad (\nu > 0).$$

We define the multivariate Meixner-Pollaczek polynomial $Q_{\mathbf{m}}^{(\nu)}$ as the following symmetric polynomial in n variables:

$$Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}) = \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}\left(-\mathbf{s}-\frac{\nu}{2}\right)}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} 2^{|\mathbf{k}|}.$$

For $\nu > \frac{d}{2}(n-1)$ let us denote by $M_\nu(d\lambda)$ the probability measure on $\mathbb{R}^n$ given by

$$M_\nu(d\lambda) = \frac{1}{Z_\nu} \prod_{j=1}^{n} \left| \Gamma\left(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1) \right) \right|^2 \frac{1}{|c(i\lambda)|^2} \, m(d\lambda),$$

where

$$Z_\nu = \int_{\mathbb{R}^n} \prod_{j=1}^{n} \left| \Gamma\left(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1) \right) \right|^2 \frac{1}{|c(i\lambda)|^2} \, m(d\lambda),$$

and c is the Harish-Chandra function for the symmetric cone Ω:

$$c(\mathbf{s}) = c_0 \prod_{j<k} B\left(s_j - s_k, \frac{d}{2} \right).$$

(Here B is the Euler beta function, the constant c_0 is such that $c(-\rho) = 1$, see Section XIV.5 in [8].) The constant Z_ν can be evaluated by using the spherical Plancherel formula, applied to the function $\psi(u) = e^{-\operatorname{tr} u} \Delta(u)^{\nu/2}$:

$$\begin{aligned} &\int_\Omega e^{-2\operatorname{tr} u} \Delta(u)^{\nu - \frac{N}{n}} \, m(du) \\ &\quad = (2\pi)^{N-2n} \int_{\mathbb{R}^n} \prod_{j=1}^{n} \left| \Gamma(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1) \right|^2 \frac{1}{|c(i\lambda)|^2} \, m(d\lambda). \end{aligned}$$

Therefore

$$Z_\nu = (2\pi)^{2n-N} 2^{-n\nu} \Gamma_\Omega(\nu).$$

The next statement involves the geometry of the Hermitian symmetric space of tube type associated to the symmetric cone Ω. The map $z \mapsto (z-e)(z+e)^{-1}$ maps the tube domain $T_\Omega = \Omega + iV \subset V_{\mathbb{C}}$ onto the bounded Hermitian symmetric domain $\mathscr{D}$. Its inverse is the Cayley transform

$$c(w) = (e + w)(e - w)^{-1}.$$

THEOREM 3.1. *Assume* $\nu > \frac{d}{2}(n-1)$.

(i) *The multivariate Meixner-Pollaczek polynomials* $Q_{\mathbf{m}}^{(\nu)}(i\lambda)$ *form an orthogonal basis of* $L^2(\mathbb{R}^n, M_\nu)^{\mathfrak{S}_n}$. *The norm of* $Q_{\mathbf{m}}^{(\nu)}$ *is given by*

$$\int_{\mathbb{R}^n} |Q_{\mathbf{m}}^{(\nu)}(i\lambda)|^2 \, M_\nu(d\lambda) = \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}.$$

(ii) *The polynomials* $Q_{\mathbf{m}}^{(\nu)}$ *admit the following generating formula: for* $\mathbf{s} \in \mathbb{C}^n$, $w \in \mathscr{D}$,

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta(e - w^2)^{-\nu/2} \varphi_{\mathbf{s}}\big(c(w)^{-1}\big).$$

We divide the proof into several steps.

a) For $\nu > 2\frac{N}{n} - 1 = 1 + d(n-1)$, $\mathscr{H}_\nu^2(\mathscr{D})$ denotes the weighted Bergman space of holomorphic functions f on $\mathscr{D}$ such that

$$\|f\|_\nu^2 := a_\nu^{(1)} \int_{\mathscr{D}} |f(w)|^2 h(w)^{\nu - 2\frac{N}{n}}\, m(dw) < \infty.$$

The constant

$$a_\nu^{(1)} = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega\big(\nu - \frac{N}{n}\big)}$$

is such that the function $\Phi_0 \equiv 1$ has norm 1. Recall that $h(w) = h(w, w)$, where $h(w, w')$ is a polynomial holomorphic in w, anti-holomorphic in w', such that, for w invertible, $h(w, w') = \Delta(w)\Delta(w^{-1} - \overline{w}')$, where $\overline{w}'$ is the complex conjugate of w' with respect to the real form V of $V_{\mathbb{C}}$. The spherical polynomials $\Phi_{\mathbf{m}}$ form an orthogonal basis of the space $\mathscr{H}_\nu^2(\mathscr{D})^K$ of K-invariant functions in $\mathscr{H}_\nu^2(\mathscr{D})$, and

$$\|\Phi_{\mathbf{m}}\|_\nu^2 = \frac{1}{d_{\mathbf{m}}} \frac{\big(\frac{N}{n}\big)_{\mathbf{m}}}{(\nu)_{\mathbf{m}}}. \tag{3.1}$$

The reproducing kernel of $\mathscr{H}_\nu^2(\mathscr{D})$ is given by $\mathscr{K}_\nu(w, w') = h(w, w')^{-\nu}$. By an integration over K one obtains

$$\mathscr{G}_\nu^{(1)}(\zeta, w) := \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{(\nu)_{\mathbf{m}}}{\big(\frac{N}{n}\big)_{\mathbf{m}}} \Phi_{\mathbf{m}}(\zeta)\Phi_{\mathbf{m}}(w) = \int_K h(w, k\overline{\zeta})^{-\nu}\, dk. \tag{3.2}$$

b) For a function f holomorphic in $\mathscr{D}$, one defines the function $F = C_\nu f$ on T_Ω by

$$F(z) = (C_\nu f)(z) = \Delta\left(\frac{z+e}{2}\right)^{-\nu} f\big((z-e)(z+e)^{-1}\big).$$

The map C_ν is a unitary isomorphism from $\mathscr{H}_\nu^2(\mathscr{D})$ onto the space $\mathscr{H}_\nu^2(T_\Omega)$ of holomorphic functions on T_Ω such that

$$\|F\|_\nu^2 := a_\nu^{(2)} \int_{T_\Omega} |F(z)|^2 \Delta(x)^{\nu - 2\frac{N}{n}}\, m(dz) < \infty.$$

The constant
$$a_\nu^{(2)} = \frac{1}{(4\pi)^n} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega\big(\nu - \frac{N}{n}\big)},$$
is such that the function
$$F_0^{(\nu)} = C_\nu \Phi_0, \quad \text{i.e. } F_0^{(\nu)}(z) = \Delta\Big(\frac{z+e}{2}\Big)^{-\nu},$$
has norm 1. The functions $F_{\mathbf{m}}^{(\nu)} = C_\nu \Phi_{\mathbf{m}}$ form an orthogonal basis of the space $\mathscr{H}_\nu^2(T_\Omega)^K$ of K-invariant functions in $\mathscr{H}_\nu^2(T_\Omega)$, and it follows from (3.1) that
$$\|F_{\mathbf{m}}^{(\nu)}\|_\nu^2 = \frac{1}{d_{\mathbf{m}}} \frac{\big(\frac{N}{n}\big)_{\mathbf{m}}}{(\nu)_{\mathbf{m}}}. \tag{3.3}$$
Performing the transform C_ν with respect to ζ in (3.2) we get a generating formula for the functions $F_{\mathbf{m}}^{(\nu)}$: for $w \in \mathscr{D}$, $z \in T_\Omega$,
$$\begin{aligned} \mathscr{G}_\nu^{(2)}(z, w) &:= \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{(\nu)_{\mathbf{m}}}{\big(\frac{N}{n}\big)_{\mathbf{m}}} \Phi_{\mathbf{m}}(w) F_{\mathbf{m}}^{(\nu)}(z) \\ &= \Delta\Big(\frac{e-w}{2}\Big)^{-\nu} \int_K \Delta\big(k \cdot z + c(w)\big)^{-\nu}\, dk. \end{aligned} \tag{3.4}$$

c) The functions in $\mathscr{H}_\nu^2(T_\Omega)$ admit a Laplace integral representation. The modified Laplace transform $\mathscr{L}_\nu$, given, for a function ψ on Ω, by
$$(\mathscr{L}_\nu)\psi(z) = a_\nu^{(3)} \int_\Omega e^{(z|u)} \psi(u) \Delta(u)^{\nu - \frac{N}{n}}\, m(du),$$
is an isometric isomorphism from the space $L_\nu^2(\Omega)$ of measurable functions ψ on Ω such that
$$\|\psi\|_\nu^2 := a_\nu^{(3)} \int_\Omega |\psi(u)|^2 \Delta(u)^{\nu - \frac{N}{n}}\, m(du) < \infty$$
onto $\mathscr{H}_\nu^2(T_\Omega)$. The constant $a_\nu^{(3)} = 2^{n\nu}/\Gamma_\Omega(\nu)$ is such that the function $\Psi_0(u) = e^{-\operatorname{tr} u}$ has norm 1, and then $\mathscr{L}_\nu \Psi_0 = F_0$. By the binomial formula
$$\begin{aligned} F_{\mathbf{m}}^{(\nu)}(z) &= \Delta\Big(\frac{z+e}{2}\Big)^{-\nu} \Phi_{\mathbf{m}}\big((z-e)(z+e)^{-1}\big) \\ &= \Delta\Big(\frac{z+e}{2}\Big)^{-\nu} \Phi_{\mathbf{m}}\big(e - 2(z+e)^{-1}\big) \\ &= \sum_{\mathbf{k} \subset \mathbf{m}} (-1)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}\big(2(z+e)^{-1}\big) \Delta\big(2(e+z)^{-1}\big)^{\nu}. \end{aligned}$$

By Lemma XI.2.3 in [8] we have the following

LEMMA 3.2. $\mathscr{L}_\nu(e^{-\mathrm{tr} u}\Phi_{\mathbf{m}})(z) = (\nu)_{\mathbf{m}}\Phi_{\mathbf{m}}\big((z+e)^{-1}\big)\Delta\big(2(e+z)^{-1}\big)^{\nu}$.

By Lemma 3.2 the function

$$\Psi_{\mathbf{m}}^{(\nu)} = \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}\mathscr{L}_\nu^{-1}(F_{\mathbf{m}}^{(\nu)})$$

is the Laguerre function given by

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = e^{-\mathrm{tr}\, u}L_{\mathbf{m}}^{(\nu-1)}(2u),$$

where $L_{\mathbf{m}}^{(\nu-1)}$ is the multivariate Laguerre polynomial

$$\begin{aligned} L_{\mathbf{m}}^{(\nu-1)}(x) &= \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}}\binom{\mathbf{m}}{\mathbf{k}}\frac{1}{(\nu)_{\mathbf{k}}}\Phi_{\mathbf{k}}(-x) \\ &= \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}}\frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)}{(\nu)_{\mathbf{k}}}\frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}}\Phi_{\mathbf{k}}(-x). \end{aligned}$$

PROPOSITION 3.3.

(i) *The multivariate Laguerre functions* $\Psi_{\mathbf{m}}^{(\nu)}$ *form an orthogonal basis of* $L^2_\nu(\Omega)^K$, *and*

$$\|\Psi_{\mathbf{m}}^{(\nu)}\|_\nu^2 = \frac{1}{d_{\mathbf{m}}}\frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}. \tag{3.5}$$

(ii) *The functions* $\Psi_{\mathbf{m}}^{(\nu)}$ *admit the following generating formula: for* $u\in\Omega$, $w\in\mathscr{D}$,

$$\mathscr{G}_\nu^{(3)}(u,w) := \sum_{\mathbf{m}} d_{\mathbf{m}}\Psi_{\mathbf{m}}^{(\nu)}(u)\Phi_{\mathbf{m}}(w) = \Delta(e-w)^{-\nu}\int_K e^{-(k\cdot u|c(w))}\,dk. \tag{3.6}$$

The generating formula can also be written

$$\Delta(e-w)^{-\nu}\int_K e^{(k\cdot x|w(e-w)^{-1})}\,dk = \sum_{\mathbf{m}} d_{\mathbf{m}}L_{\mathbf{m}}^{(\nu-1)}(x)\Phi_{\mathbf{m}}(w). \tag{3.6'}$$

Formula (3.6') is proposed as an exercise in [8] (Exercise 3, p. 347). It is a special case of formula (4.4) in [3].

PROOF. Part (i) follows from the fact that $\mathscr{L}_\nu$ is a unitary isomorphism from $L^2_\nu(\Omega)$ onto $\mathscr{H}^2_\nu(T_\Omega)$, and from (3.3).

The modified Laplace transform of $\mathscr{G}_\nu^{(3)}(u, w)$ with respect to u is equal to $\mathscr{G}_\nu^{(2)}(z, w)$, and one gets (ii) from (3.4).

d) We will evaluate the spherical Fourier transform of the Laguerre functions $\Psi_{\mathbf{m}}^{(\nu)}$. We introduce now the modified spherical Fourier transform $\mathscr{F}_\nu$ as follows: for a function ψ on Ω,

$$(\mathscr{F}_\nu \psi)(\mathbf{s}) = \frac{1}{\Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big)} \int_\Omega \psi(u) \varphi_{\mathbf{s}}(u) \Delta(u)^{\frac{\nu}{2} - \frac{N}{n}}\, m(du).$$

Observe that $\mathscr{F}_\nu \Psi_0 \equiv 1$.

LEMMA 3.4. *For* $\operatorname{Re} s_j > \frac{d}{4}(n-1) - \frac{\nu}{2}$,

$$\mathscr{F}_\nu(e^{-\operatorname{tr} u} \Phi_{\mathbf{m}})(\mathbf{s}) = (-1)^{|\mathbf{m}|} \gamma_{\mathbf{m}}\left(-\mathbf{s} - \frac{\nu}{2}\right).$$

PROOF. Let $\sigma_D(u, \xi)$ be the symbol of $D \in \mathbb{D}(\Omega)$ and $p(\xi) = \sigma_D(e, \xi)$ (see [8], p. 290). By the invariance property of σ_D, we have $\sigma_D(u, -e) = p(-u)$, and therefore $De^{-\operatorname{tr} u} = p(-\xi) e^{-\operatorname{tr} u}$. Hence, for $p(\xi) = \Phi_{\mathbf{m}}(\xi)$,

$$\begin{aligned}\mathscr{F}_\nu(e^{-\operatorname{tr} u} \Phi_{\mathbf{m}})(s) &= (-1)^{|\mathbf{m}|} \mathscr{F}_\nu(D^{\mathbf{m}} e^{-\operatorname{tr} u})(s) \\ &= (-1)^{|\mathbf{m}|} \gamma_{\mathbf{m}}\left(-\mathbf{s} - \frac{\nu}{2}\right) \mathscr{F}_\nu(e^{-\operatorname{tr} u}) \\ &= (-1)^{|\mathbf{m}|} \gamma_{\mathbf{m}}\left(-\mathbf{s} - \frac{\nu}{2}\right).\end{aligned}$$

From Lemma 3.4 we obtain the evaluation of the spherical Fourier transform of the Laguerre functions: for $\operatorname{Re} s_j > \frac{d}{4}(n-1) - \frac{\nu}{2}$,

$$\mathscr{F}_\nu(\Psi_{\mathbf{m}}^{\nu})(\mathbf{s}) = Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}).$$

By the spherical Plancherel formula and part (i) of Proposition 3.3, this proves part (i) of Theorem 3.1, for $\nu > 1 + d(n-1)$:

$$\int_{\mathbb{R}^n} |Q_{\mathbf{m}}^{(\nu)}(i\lambda)|^2\, M_\nu(d\lambda) = \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}. \tag{3.7}$$

By analytic continuation it holds for $\nu > \frac{d}{2}(n-1)$. For proving part (ii) of Theorem 2.1 one performs the spherical Fourier transform to both sides of part (ii) in Proposition 3.3:

$$\mathscr{G}_\nu^{(4)}(\mathbf{s}, w) := \sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta(e - w^2)^{-\nu/2} \varphi_{\mathbf{s}}\big(c(w)^{-1}\big).$$

This finishes the proof of Theorem 3.1.

We remark that, in [5], a different notation is used for the Meixner-Pollaczek polynomials: their polynomials $p_{\nu,\mathbf{m}}$ (p. 179), are defined through the generating formula above and $p_{\nu,\mathbf{m}}(i\mathbf{s}) = d_{\mathbf{m}} Q^{(\nu)}_{\mathbf{m}}(\mathbf{s})$.

4. Multivariate Meixner-Pollaczek polynomials $Q^{(\nu,\theta)}_{\mathbf{m}}$

The Meixner-Pollaczek polynomials $q^{(\nu)}_m$ we have considered at the beginning of Section 3 correspond to the special value $\phi = \frac{\pi}{2}$ with the classical notation. Using instead $\theta = \phi - \frac{\pi}{2}$, the more general one variable Meixner-Pollaczek polynomials can be written

$$\begin{aligned} q^{(\nu,\theta)}_m(s) &= e^{im\theta}\frac{(\nu)_m}{m!}\,{}_2F_1\left(-m, s+\frac{\nu}{2}; \nu; 2e^{-i\theta}\cos\theta\right) \\ &= e^{im\theta}\frac{(\nu)_m}{m!}\sum_{k=0}^{m}\frac{[m]_k\left[-s-\frac{\nu}{2}\right]_k}{(\nu)_k}\frac{1}{k!}(2e^{-i\theta}\cos\theta)^k. \end{aligned}$$

In terms of the classical notation $P^{\alpha}_m(\lambda;\phi)$

$$q^{(\nu,\theta)}_m(i\lambda) = (-i)^m P^{\nu/2}_m\left(\lambda; \theta+\frac{\pi}{2}\right).$$

For $\nu > 0$, $|\theta| < \frac{\pi}{2}$, the polynomials $q^{(\nu,\theta)}_m(i\lambda)$ are orthogonal with respect to the weight

$$e^{2\theta\lambda}\left|\Gamma\left(i\lambda+\frac{\nu}{2}\right)\right|^2.$$

In this section we consider the multivariate Meixner-Pollaczek polynomials $Q^{(\nu,\theta)}_{\mathbf{m}}$ defined by

$$Q^{(\nu,\theta)}_{\mathbf{m}}(\mathbf{s}) = e^{i|\mathbf{m}|\theta}\frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}}\frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}\left(-\mathbf{s}-\frac{\nu}{2}\right)}{(\nu)_{\mathbf{k}}}\frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}}(2e^{-i\theta}\cos\theta)^{|\mathbf{k}|}.$$

THEOREM 4.1. *Assume $\nu > \frac{d}{2}(n-1)$, $|\theta| < \frac{\pi}{2}$.*

(i) *The multivariate Meixner-Pollaczek polynomials $Q^{(\nu,\theta)}_{\mathbf{m}}(i\lambda)$ form an orthogonal basis of $L^2(\mathbb{R}^n, e^{2\theta(\lambda_1+\cdots+\lambda_n)}M_\nu)^{\mathfrak{S}_n}$. The norm of $Q^{(\nu,\theta)}_{\mathbf{m}}$ is given by:*

$$\int_{\mathbb{R}^n}|Q^{(\nu,\theta)}_{\mathbf{m}}(i\lambda)|^2 e^{2\theta(\lambda_1+\cdots+\lambda_n)}\,M_\nu(d\lambda) = (\cos\theta)^{-n\nu}\frac{1}{d_{\mathbf{m}}}\frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}.$$

(ii) *The polynomials* $Q_{\mathbf{m}}^{(\nu,\theta)}$ *admit the following generating formula: for* $\mathbf{s} \in \mathbb{C}^n$, $w \in \mathscr{D}$,

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta\big((e - e^{i\theta} w)(e + e^{-i\theta} w)\big)^{-\nu/2} \varphi_{\mathbf{s}}\big(c_\theta(w)^{-1}\big),$$

where c_θ *is the modified Cayley transform:*

$$c_\theta(w) = (e + e^{-i\theta} w)(e - e^{i\theta} w)^{-1}.$$

We will prove Theorem 4.1 in several steps.
a) Let us define the Laguerre functions $\Psi_{\mathbf{m}}^{(\nu,\theta)}$:

$$\Psi_{\mathbf{m}}^{(\nu,\theta)}(u) = e^{i|\mathbf{m}|\theta} e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\nu-1)}(2e^{-i\theta} \cos\theta\, u).$$

For functions ψ on V of the form $\psi(u) = e^{-\operatorname{tr} u} p(u)$, where p is a polynomial, define the inner product

$$(\psi_1 \mid \psi_2)_{(\nu,\theta)} = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega \psi_1(e^{i\theta} u)\, \overline{\psi_2(e^{i\theta} u)}\, \Delta(u)^{\nu - \frac{N}{n}}\, m(du).$$

PROPOSITION 4.2.

(i) *The Laguerre functions* $\Psi_{\mathbf{m}}^{(\nu,\theta)}$ *are orthogonal with respect to the inner product* $(\cdot \mid \cdot)_{(\nu,\theta)}$. *Furthermore*

$$\big\|\Psi_{\mathbf{m}}^{(\nu,\theta)}\big\|_{(\nu,\theta)}^2 = (\cos\theta)^{-n\nu} \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}.$$

(ii) *The Laguerre functions* $\Psi_{\mathbf{m}}^{(\nu,\theta)}$ *satisfy the following generating formula: for* $u \in \Omega$, $w \in \mathscr{D}$,

$$\begin{aligned}\mathscr{G}_{\nu,\theta}^{(3)}(u, w) &:= \sum_{\mathbf{m}} d_{\mathbf{m}} \Psi_{\mathbf{m}}^{(\nu,\theta)}(u) \Phi_{\mathbf{m}}(w)\\ &= \Delta(e - e^{i\theta} w)^{-\nu} \int_K e^{(k\cdot u \mid c_\theta(w))}\, dk.\end{aligned}$$

PROOF. (i) Put $\alpha = e^{i\theta}$, $\beta = 2e^{-i\theta}\cos\theta$. For two polynomials p_1 and p_2 consider the functions

$$\psi_1^{(\theta)}(u) = e^{-\operatorname{tr} u} p_1(\beta u), \quad \psi_2^{(\theta)}(u) = e^{-\operatorname{tr} u} p_2(\beta u),$$

and their inner product

$$(\psi_1^{(\theta)} \mid \psi_2^{(\theta)})_{\nu,\theta} = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-\alpha \operatorname{tr} u} p_1(\beta\alpha u) \overline{e^{-\alpha \operatorname{tr} u} p_2(\beta\alpha u)} \Delta(u)^{\nu - \frac{N}{n}}\, m(du).$$

Observe that $\beta\alpha = 2\cos\theta$, $\alpha + \overline{\alpha} = 2\cos\theta$. Hence

$$\begin{aligned}
&(\psi_1^{(\theta)} \mid \psi_2^{(\theta)})_{\nu,\theta} \\
&\quad = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-2\cos\theta \operatorname{tr} u} p_1(2\cos\theta u)\, \overline{p_2(2\cos\theta u)}\, \Delta(u)^{\nu-\frac{n}{N}}\, m(du) \\
&\quad = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} (\cos\theta)^{-n\nu} \int_\Omega e^{-2\operatorname{tr} v} p_1(2v)\, \overline{p_2(2v)}\, \Delta(v)^{\nu-\frac{N}{n}}\, m(dv) \\
&\quad = (\cos\theta)^{-n\nu} (\psi_1^{(0)} \mid \psi_2^{(0)}).
\end{aligned}$$

Take

$$p_1(u) = L_{\mathbf{p}}^{(\nu-1)}(u), \quad p_2(u) = L_{\mathbf{q}}^{(\nu-1)}(u).$$

Then, by part (i) of Proposition 3.3, the statement (i) is proved.

(ii) The sum in the generating formula can be written

$$\sum_{\mathbf{m}} d_{\mathbf{m}} e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\nu-1)}(2e^{-i\theta}\cos\theta u) \Phi_{\mathbf{m}}(e^{i\theta} w).$$

Hence the generating formula follows from part (ii) in Proposition 3.3.

b) By Lemma 3.4 we obtain the following evaluation of the spherical Fourier transform of the Laguerre functions $\Psi_{\mathbf{m}}^{(\nu,\theta)}$:

$$\mathscr{F}_\nu(\Psi_{\mathbf{m}}^{(\nu,\theta)})(\mathbf{s}) = Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s}).$$

We will need a Gutzmer formula for the spherical Fourier transform on a symmetric cone. Let us first state the following Gutzmer formula for the Mellin transform.

PROPOSITION 4.3. *Let ψ be holomorphic in the following open set in $\mathbb{C}$:*

$$\{\zeta = re^{i\theta} \mid r > 0,\ |\theta| < \theta_0\} \quad (0 < \theta_0 < \pi/2).$$

The Mellin transform of ψ is defined by

$$\mathscr{M}\psi(s) = \int_0^\infty \psi(r) r^{s-1}\, dr.$$

Assume that there is a constant $M > 0$ such that, for $|\theta| < \theta_0$,

$$\int_0^\infty |\psi(re^{i\theta})|^2 r^{-1}\, dr \le M.$$

Then

$$\int_0^\infty |\psi(re^{i\theta})|^2 r^{-1}\, dr = \frac{1}{2\pi} \int_{\mathbb{R}} |\mathscr{M}\psi(i\lambda)|^2 e^{2\theta\lambda}\, d\lambda.$$

Using the decomposition of the symmetric cone Ω as $\Omega =]0,\infty[\times\Omega_1$, where $\Omega_1 = \{u\in\Omega \mid \Delta(u)=1\}$, one gets the following Gutzmer formula for Ω:

PROPOSITION 4.4. *Let ψ be a holomorphic function in the tube $T_\Omega = \Omega + iV$. Assume that there are constants $M>0$ and $0<\theta_0<\pi/2$ such that, for $|\theta|<\theta_0$,*

$$\int_\Omega |\psi(e^{i\theta}u)|^2 \Delta(u)^{-N/n}\, m(du) \le M.$$

Then, for $|\theta|<\theta_0$,

$$\int_\Omega |\psi(e^{i\theta}u)|^2 \Delta(u)^{-N/n}\, du = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\mathcal{F}\psi(i\lambda)|^2 e^{2\theta(\lambda_1+\cdots+\lambda_n)} \frac{1}{|c(i\lambda)|^2}\, m(d\lambda).$$

From Proposition 4.2 and Proposition 4.4 we obtain parts (i) and (ii) of Theorem 4.1. A more general Gutzmer formula has been established for the spherical Fourier transform on Riemannian symmetric spaces of non-compact type [7].

5. Determinantal formulae

In the case $d=2$, i.e. $V=\mathrm{Herm}(n,\mathbb{C})$, $K=U(n)$, there are determinantal formulae for the multivariate Laguerre functions $\Psi_{\mathbf{m}}^{(\nu)}$ and for the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$. Consider a Jordan frame $\{c_1,\ldots,c_n\}$ in V, and let $\delta=(n-1,n-2,\ldots,1,0)$.

THEOREM 5.1. *Assume $d=2$. The multivariate Laguerre function $\Psi_{\mathbf{m}}^{(\nu)}$ admits the following determinantal formula involving the one variable Laguerre functions $\psi_m^{(\nu)}$: for $u=\sum_{j=1}^n u_i c_i$,*

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = \delta!\, 2^{-\frac{1}{2}n(n-1)} \frac{\det\bigl(\psi_{m_j+\delta_j}^{(\nu-n+1)}(u_i)\bigr)_{1\le i,j\le n}}{V(u_1,\ldots,u_n)},$$

where V denotes the Vandermonde polynomial:

$$V(u_1,\ldots,u_n)=\prod_{i<j}(u_j-u_i) \quad \textit{and} \quad \delta! = \prod_{i=1}^n (n-i)!.$$

As a result one obtains the following determinantal formula for the multivariate Laguerre polynomials:

$$\mathbf{L}_{\mathbf{m}}^{\nu}(u) = \delta! \frac{\det\bigl(L_{m_j+\delta_j}^{(\nu-n+1)}(u_i)\bigr)}{V(u_1,\ldots,u_n)}.$$

PROOF. We start from the generating formula for the multivariate Laguerre functions (Proposition 3.3):

$$\begin{aligned}\mathscr{G}_\nu^{(3)}(u, w) &= \sum_{\mathbf{m}} d_{\mathbf{m}} \Phi_{\mathbf{m}}(w) \Psi_{\mathbf{m}}^{(\nu)}(u) \\ &= \Delta(e-w)^{-\nu} \int_K e^{-(ku|(e+w)(e-w)^{-1})}\, dk.\end{aligned}$$

In the case $d = 2$, the evaluation of this integral is classical: for $x = \sum_{i=1}^n x_i c_i$, $y = \sum_{j=1}^n y_j c_j$, then

$$\mathscr{I}(x, y) = \int_K e^{(kx|y)}\, dk = \delta! \frac{\det(e^{x_i y_j})}{V(x_1, \dots, x_n) V(y_1, \dots, y_n)}.$$

Therefore, for $u = \sum_{i=1}^n u_i c_i$, $w = \sum_{j=1}^n w_j c_j$,

$$\mathscr{G}_\nu^{(3)}(u, w) = \delta! \prod_{j=1}^n (1-w_j)^{-\nu} \frac{\det\left(e^{-u_i \frac{1+w_j}{1-w_j}}\right)}{V(u_1, \dots, u_n) V\left(\frac{1+w_1}{1-w_1}, \dots, \frac{1+w_n}{1-w_n}\right)}.$$

Noticing that

$$\frac{1+w_j}{1-w_j} - \frac{1+w_k}{1-w_k} = 2\frac{w_j - w_k}{(1+w_j)(1+w_k)},$$

we obtain

$$\mathscr{G}_\nu^{(3)}(u, w) = \delta! 2^{-\frac{1}{2}n(n-1)} \frac{\det\left((1-w_j)^{-(\nu-n+1)} e^{-u_i \frac{1+w_j}{1-w_j}}\right)}{V(u_1, \dots, u_n) V(w_1, \dots, w_n)}.$$

We will expand the above expression in Schur function series by using a formula due to Hua (see [9], Theorem 1.2.1, p. 22).

LEMMA 5.2. *Consider n power series*

$$f_i(w) = \sum_{m=0}^\infty c_m^{(i)} w^m \qquad (i = 1, \dots, n).$$

Then

$$\frac{\det(f_i(w_j))}{V(w_1, \dots, w_n)} = \sum_{\mathbf{m}} a_{\mathbf{m}} s_{\mathbf{m}}(w_1, \dots, w_n),$$

where $s_{\mathbf{m}}$ is the Schur function associated to the partition $\mathbf{m}$, *and*

$$a_{\mathbf{m}} = \det\left(c_{m_j+\delta_j}^{(i)}\right).$$

Let $\nu' = \nu - n + 1$, and consider the n power series

$$f_i(w) := (1-w)^{-\nu'} e^{-u_i \frac{1+w}{1-w}} = \sum_{m=0}^{\infty} \psi_m^{(\nu')}(u_i) w^m.$$

Since

$$d_{\mathbf{m}} \Phi_{\mathbf{m}}\Bigl(\sum_{j=1}^n w_j c_j\Bigr) = s_{\mathbf{m}}(w_1, \ldots, w_n),$$

we obtain

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = \delta! 2^{-\frac{1}{2}n(n-1)} \frac{\det\bigl(\psi_{m_j+\delta_j}^{(\nu-n+1)}(u_i)\bigr)}{V(u_1, \ldots, u_n)}.$$

By using the same method we will obtain a determinantal formula for the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$.

THEOREM 5.3. *Assume $d = 2$. Then*

$$Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s}) = (-2\cos\theta)^{-\frac{1}{2}n(n-1)} \delta! \frac{\det\bigl(q_{m_j+\delta_j}^{(\nu-n+1,\theta)}(s_i)\bigr)_{1\le i,j\le n}}{V(s_1, \ldots, s_n)},$$

where $q_m^{(\nu,\theta)}$ denotes the one variable Meixner-Pollaczek polynomial.

PROOF. We start from the generating formula for the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$ (Theorem 4.1(ii)):

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta\bigl((e - e^{i\theta}w)(e + e^{-i\theta}w)\bigr)^{-\nu/2} \varphi_{\mathbf{s}}\bigl(c_\theta(w)^{-1}\bigr).$$

For $x = \sum_{i=1}^n x_i c_i$, the spherical function $\varphi_{\mathbf{s}}(x)$ is essentially a Schur function in the variables $x_1, \ldots, x_n$:

$$\varphi_{\mathbf{s}}(x) = \delta!(x_1 x_2 \ldots x_r)^{\frac{1}{2}(n-1)} \frac{\det(x_j^{s_i})}{V(s_1, \ldots, s_n) V(x_1, \ldots, x_n)}.$$

Let us compute now, for $w = \sum_{j=1}^n w_j c_j$,

$$\begin{aligned}
&\Delta\bigl((e - e^{i\theta}w)(e + e^{-i\theta}w)\bigr)^{-\nu/2} \varphi_{\mathbf{s}}\bigl(c_\theta(w)^{-1}\bigr) \\
&\quad = \delta! \prod_{j=1}^n (1 - 2i\sin\theta w_j - w_j^2)^{-\nu/2} \\
&\qquad \times \prod_{j=1}^n \bigl(c_\theta(w_j)\bigr)^{\frac{1}{2}(n-1)} \frac{\det\bigl((c_\theta(w_j))^{-s_i}\bigr)}{V(s_1, \ldots, s_n) V\bigl(c_\theta(w_1), \ldots, c_\theta(w_n)\bigr)}.
\end{aligned}$$

In the same way, as for the proof of Theorem 5.1, we obtain

$$\begin{aligned}\Delta\big((e-e^{i\theta}w)(e+e^{-i\theta})\big)^{-\nu/2}&\varphi_{\mathbf{s}}\big(c_\theta(w)^{-1}\big)\\ &=(-2\cos\theta)^{-\frac{1}{2}n(n-1)}\delta!\\ &\quad\times\frac{\det\big((1-e^{i\theta}w_j)^{s_i-\frac{\nu}{2}+\frac{1}{2}(n-1)}(1+e^{-i\theta}w_j)^{-s_i-\frac{\nu}{2}+\frac{1}{2}(n-1)}\big)}{V(s_1,\ldots,s_n)V(w_1,\ldots,w_n)}.\end{aligned}$$

We apply once more Lemma 5.2 to the n power series

$$f_i(w):=(1-e^{i\theta}w)^{s_i-\frac{\nu'}{2}}(1+e^{-i\theta}w)^{-s_i-\frac{\nu'}{2}}=\sum_m^\infty q_m^{(\nu',\theta)}(s_i)w^m$$

with $\nu'=\nu-n+1$, and obtain finally:

$$Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s})=(-2\cos\theta)^{-\frac{1}{2}n(n-1)}\delta!\frac{\det\big(q_{m_j+\delta_j}^{(\nu-n+1,\theta)}(s_i)\big)}{V(s_1,\ldots,s_n)}.$$

6. Difference equation for the Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$

The one variable Meixner-Pollaczek polynomials $q_m=q_m^{(\nu,\theta)}$ satisfy the following difference equation

$$\begin{aligned}e^{-i\theta}\Big(s+\frac{\nu}{2}\Big)&\big(q_m(s+1)-q_m(s)\big)\\ &+e^{i\theta}\Big(-s+\frac{\nu}{2}\Big)\big(q_m(s-1)-q_m(s)\big)=2m\cos\theta q_m.\end{aligned}$$

(See [1], p. 348, 37.(d)). We will establish an analogue of this formula for the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$.

Recall Pieri's formula for spherical functions:

$$\operatorname{tr} u\varphi_{\mathbf{s}}(u)=\sum_{j=1}^n\alpha_j(\mathbf{s})\varphi_{\mathbf{s}+\varepsilon_j}(u),\quad\text{with } \alpha_j(\mathbf{s})=\prod_{k\neq j}\frac{s_j-s_k+\frac{d}{2}}{s_j-s_k},$$

where $\{\varepsilon_i\}$ denotes the canonical basis of $\mathbb{C}^n$. See [6, Proposition 6.1] or [16, Theorem 1] and also [10, p. 320]. We introduce the difference operator $D_{\nu,\theta}$:

$$\begin{aligned}D_{\nu,\theta}f(\mathbf{s})=e^{-i\theta}&\sum_{j=1}^n\Big(s_j+\frac{\nu}{2}-\frac{d}{4}(n-1)\Big)\alpha_j(\mathbf{s})\big(f(\mathbf{s}+\varepsilon_j)-f(\mathbf{s})\big)\\ &+e^{i\theta}\sum_{j=1}^n\Big(-s_j+\frac{\nu}{2}-\frac{d}{4}(n-1)\Big)\alpha_j(-\mathbf{s})\big(f(\mathbf{s}-\varepsilon_j)-f(\mathbf{s})\big).\end{aligned}$$

THEOREM 6.1. *The Meixner-Pollaczek polynomial $Q_{\mathbf{m}}^{(\nu,\theta)}$ is an eigenfunction of the difference operator $D_{\nu,\theta}$:*

$$D_{\nu,\theta} Q_{\mathbf{m}}^{(\nu,\theta)} = 2|\mathbf{m}| \cos\theta \, Q_{\mathbf{m}}^{(\nu,\theta)}.$$

For the proof we will use the scheme we have used in the proof of Theorem 3.1. For $i = 1, 2, 3, 4$, we define the operators $D_{\nu,\theta}^{(i)}$. The operator $D_{\nu,\theta}^{(1)} = D_{\theta}^{(1)}$ is a first order differential operator on the domain $\mathscr{D}$:

$$D_{\theta}^{(1)} f = e^{i\theta} \langle w + e, \nabla f \rangle + e^{-i\theta} \langle w - e, \nabla f \rangle.$$

(For $w_1, w_2 \in V_{\mathbb{C}}$, we put $\langle w_1, w_2 \rangle = \operatorname{tr}(w_1 w_2)$.) The operators $D_{\nu,\theta}^{(i)}$, for $i = 2, 3, 4$, are defined by the relations:

$$D_{\nu,\theta}^{(2)} C_\nu = C_\nu D_{\nu,\theta}^{(1)}, \quad \mathscr{L}_\nu D_{\nu,\theta}^{(3)} = D_{\nu,\theta}^{(2)} \mathscr{L}_\nu, \quad \mathscr{F}_\nu D_{\nu,\theta}^{(3)} = D_{\nu,\theta}^{(4)} \mathscr{F}_\nu.$$

The operator $D_{\nu,\theta}^{(2)}$ is a first order differential operator on the tube T_Ω. In Section 8 we will see that $D_{\nu,\theta}^{(3)}$ is a second order differential operator on the cone Ω, and prove that $D_{\nu,\theta}^{(4)}$ is the difference operator $D_{\nu,\theta}$ we have introduced above.

The function $\Phi_{\mathbf{m}}^{(\theta)}(w) = \Phi_{\mathbf{m}}(w \cos\theta + ie \sin\theta)$ is an eigenfunction of the operator $D_{\theta}^{(1)}$: $D_{\theta}^{(1)} \Phi_{\mathbf{m}}^{(\theta)} = 2|\mathbf{m}| \cos\theta \, \Phi_{\mathbf{m}}^{(\theta)}$. Hence $F_{\mathbf{m}}^{(\nu,\theta)} = C_\nu \Phi_{\mathbf{m}}^{(\theta)}$ is an eigenfunction of $D_{\nu,\theta}^{(2)}$: $D_{\nu,\theta}^{(2)} F_{\mathbf{m}}^{(\nu,\theta)} = 2|\mathbf{m}| \cos\theta \, F_{\mathbf{m}}^{(\nu,\theta)}$. Further, since

$$\mathscr{L}_\nu \Psi_{\mathbf{m}}^{(\nu,\theta)} = \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} F_{\mathbf{m}}^{(\nu,\theta)},$$

we get $D_{\nu,\theta}^{(3)} \Psi_{\mathbf{m}}^{(\nu,\theta)} = 2|\mathbf{m}| \cos\theta \, \Psi_{\mathbf{m}}^{(\nu,\theta)}$. Finally, since $Q_{\mathbf{m}}^{(\nu,\theta)} = \mathscr{F}_\nu \Psi_{\mathbf{m}}^{(\nu,\theta)}$, then $D_{\nu,\theta}^{(4)} Q_{\mathbf{m}}^{(\nu,\theta)} = 2|\mathbf{m}| \cos\theta \, Q_{\mathbf{m}}^{(\nu,\theta)}$. Hence the proof of Theorem 6.1 amounts to showing that $D_{\nu,\theta}^{(4)} = D_{\nu,\theta}$.

7. The symmetries $S_\nu^{(i)}$ ($i = 1, 2, 3, 4$) and the Hankel transform

The symmetries $S_\nu^{(i)}$ we introduce now will be useful for the computation of the operators $D_{\nu,\theta}^{(i)}$. We start from the symmetry $w \mapsto -w$ of the domain $\mathscr{D}$. Its action on functions is given by $S^{(1)} f(w) = f(-w)$. We carry this symmetry over the tube T_Ω through the Cayley transform and obtain the inversion $z \mapsto z^{-1}$. We define $S_\nu^{(2)}$ such that $S_\nu^{(2)} C_\nu = C_\nu S^{(1)}$. Hence, for a function F on T_Ω, we have $S_\nu^{(2)} F(z) = \Delta(z)^{-\nu} F(z^{-1})$. Further $S_\nu^{(3)}$ is defined by the relation

$\mathscr{L}_\nu S_\nu^{(3)} = S_\nu^{(2)}\mathscr{L}_\nu$. By a generalized Tricomi theorem (Theorem XV.4.1 in [8]), the unitary isomorphism $S_\nu^{(3)}$ of $L^2_\nu(\Omega)$ is the Hankel transform: $S_\nu^{(3)} = U_\nu$,

$$U_\nu \psi(u) = \int_\Omega H_\nu(u, v)\psi(v)\Delta(v)^{\nu-\frac{N}{n}}\, m(dv).$$

The kernel $H_\nu(u, v)$ has the following invariance property: for $g \in G$,

$$H_\nu(g\cdot u, v) = H_\nu(u, g^*\cdot v), \quad \text{and} \quad H_\nu(u, e) = \frac{1}{\Gamma_\Omega(\nu)}\mathscr{J}_\nu(u),$$

where $\mathscr{J}_\nu$ is a multivariate Bessel function.

Finally we define $S_\nu^{(4)}$ acting on symmetric polynomials in n variables such that

$$S_\nu^{(4)}\mathscr{F}_\nu = \mathscr{F}_\nu S_\nu^{(3)}.$$

PROPOSITION 7.1. *For a function ψ on Ω of the form $\psi(u) = e^{-\operatorname{tr} u}q(u)$, where q is a K-invariant polynomial, $\mathscr{F}_\nu(U_\nu\psi)(\mathbf{s}) = \mathscr{F}_\nu\psi(-\mathbf{s})$. It follows that, for a symmetric polynomial p on $\mathbb{C}^n$,*

$$S_\nu^{(4)}p(\mathbf{s}) = p(-\mathbf{s}).$$

PROOF. We will evaluate the spherical Fourier transform $\mathscr{F}_\nu(U_\nu\psi)$. By the invariance property, the kernel $H_\nu(u, v)$ can be written

$$H_\nu(u, v) = h_\nu(P(v^{1/2})u)\Delta(u)^{-\nu/2}\Delta(v)^{-\nu/2},$$

with $h_\nu(u) = H_\nu(u, e)\Delta(u)^{\nu/2}$, and P the so-called quadratic representation of the Jordan algebra V. Let us compute first

$$\int_\Omega H_\nu(u, v)\varphi_{\mathbf{s}}(u)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}}\, m(du) = \Delta(v)^{-\nu/2}\int_\Omega h_\nu(P(v^{1/2})u)\varphi_{\mathbf{s}}(u)\Delta(u)^{-N/n}\, m(du).$$

By letting $P(v^{1/2})u = u'$, we get

$$\int_\Omega H_\nu(u, v)\varphi_{\mathbf{s}}(u)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}}\, m(du) = \Delta(v)^{-\nu/2}\int_\Omega h_\nu(u')\varphi_{\mathbf{s}}(P(v^{-1/2})u')\Delta(u')^{-N/n}\, m(du').$$

By using K-invariance and the functional equation of the spherical function $\varphi_{\mathbf{s}}$,

$$\int_K \varphi_{\mathbf{s}}(P(v^{-1/2})ku')\, dk = \varphi_{\mathbf{s}}(v^{-1})\varphi_{\mathbf{s}}(u'),$$

we get

$$\int_\Omega H_\nu(u,v)\varphi_{\mathbf{s}}(u)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}}\,m(du)=\varphi_{\mathbf{s}}(v^{-1})\Delta(v)^{-\nu/2}\mathscr{F}(h_\nu)(\mathbf{s}).$$

Recall that $\varphi_{\mathbf{s}}(v^{-1})=\varphi_{-\mathbf{s}}(v)$. We multiply both sides by $\psi(v)$ and get by integrating with respect to v that

$$\Gamma_\Omega\left(\mathbf{s}+\frac{\nu}{2}+\rho\right)\mathscr{F}_\nu(U_\nu\psi)(\mathbf{s})=\mathscr{F}h_\nu(\mathbf{s})\Gamma_\Omega\left(-\mathbf{s}+\frac{\nu}{2}+\rho\right)\mathscr{F}_\nu\psi(-\mathbf{s}).$$

Consider the special case $\psi(u)=\Psi_0(u)=e^{-\operatorname{tr}u}$. Since $U_\nu\Psi_0=\Psi_0$ and $\mathscr{F}_\nu\Psi_0\equiv 1$, we get

$$\mathscr{F}(h_\nu)(\mathbf{s})=\frac{\Gamma_\Omega\big(\mathbf{s}+\frac{\nu}{2}+\rho\big)}{\Gamma_\Omega\big(-\mathbf{s}+\frac{\nu}{2}+\rho\big)}.$$

Finally $\mathscr{F}_\nu(U_\nu\psi)(\mathbf{s})=\mathscr{F}_\nu\psi(-\mathbf{s})$, and $S_\nu^{(4)}p(\mathbf{s})=p(-\mathbf{s})$.

Corollary 7.2. $Q_{\mathbf{m}}^{(\nu,\theta)}(-\mathbf{s})=(-1)^{|\mathbf{m}|}Q_{\mathbf{m}}^{(\nu,-\theta)}(\mathbf{s})$.

Proof. This relation follows from

$$\big(S^{(1)}\Phi_{\mathbf{m}}^{(\theta)}\big)(w)=\Phi_{\mathbf{m}}^{(\theta)}(-w)=(-1)^{|\mathbf{m}|}\Phi_{\mathbf{m}}^{(-\theta)}(w),$$

which is easy to check, and Proposition 7.1.

The operators $D_{\nu,\theta}^{(i)}$ $(i=1,2,3,4)$ can be written

$$D_{\nu,\theta}^{(i)}=e^{i\theta}D_\nu^{(i,+)}+e^{-i\theta}D_\nu^{(i,-)}.$$

For $i=1$, $D_\nu^{(1,\pm)}$ does not depend on ν, $D_\nu^{(1,\pm)}=D^{(1,\pm)}$,

$$D^{(1,+)}f(w)=\langle w+e,\nabla f(w)\rangle,\qquad D^{(1,-)}f(w)=\langle w-e,\nabla f(w)\rangle.$$

Observe that $D^{(1,-)}=S^{(1)}D^{(1,+)}S^{(1)}$. Hence, for $i=2,3,4$, we have $D_\nu^{(i,-)}=S_\nu^{(i)}D_\nu^{(i,+)}S_\nu^{(i)}$.

In the next Section we will first compute $D_\nu^{(i,-)}$. The operator $D_\nu^{(i,+)}$ is then obtained by using the above relation. For $i=3$, we will use the following property of the Hankel transform:

Proposition 7.3. $U_\nu(\operatorname{tr} v\,\psi)=-\left(\left\langle u,\left(\frac{\partial}{\partial u}\right)^2\right\rangle+\nu\operatorname{tr}\left(\frac{\partial}{\partial u}\right)\right)U_\nu\psi.$

This is a consequence of Proposition XV.2.3 in [8].

8. Proof of Theorem 6.1

a) Recall that $D^{(1,-)}$ is the first order differential operator on the domain $\mathscr{D}$ given by

$$D^{(1,-)} f(w) = \langle w - e, \nabla f(w)\rangle,$$

and $D_\nu^{(2,-)}$ is the first order differential operator on the tube T_Ω such that

$$D_\nu^{(2,-)} C_\nu = C_\nu D^{(1,-)}.$$

LEMMA 8.1. $D_\nu^{(2,-)} F(z) = -\langle z + e, \nabla F(z)\rangle - n\nu F(z).$

PROOF. Recall that, for a function F on the tube T_Ω,

$$f(w) = (C_\nu^{-1} F)(w) = \Delta(e - w)^{-\nu} F(c(w)),$$

where c is the Cayley transform

$$c(w) = (e + w)(e - w)^{-1} = 2(e - w)^{-1} - e.$$

Its differential is given by

$$(Dc)_w = 2P\big((e - w)^{-1}\big).$$

We get

$$\nabla f(w) = \nabla\big(\Delta(e-w)^{-\nu}\big)F(c(w)) + \Delta(e-w)^{-\nu} 2P\big(e-w)^{-1}\big)\big(\nabla F(c(w))\big).$$

By using $\nabla\big(\Delta(x)^\alpha\big) = \alpha\Delta(x)^\alpha x^{-1}$,

$$\big\langle e - w, (e - w)^{-1}\big\rangle = n \quad \text{and} \quad P\big((e - w)^{-1}\big)(e - w) = (e - w)^{-1},$$

we obtain

$$\begin{aligned} D^{(1,-)} f(w) &= \langle w - e, \nabla f(w)\rangle \\ &= \Delta(e - w)^{-\nu}\big(-n\nu F(c(w)) + 2\big\langle (w - e)^{-1}, \nabla F(c(w))\big\rangle\big) \\ &= (C_\nu^{-1} G)(z), \end{aligned}$$

with

$$G(z) = -\langle z + e, \nabla F(z)\rangle - n\nu F(z).$$

b) Consider now the differential operator $D_\nu^{(3,-)}$ on the cone Ω such that

$$\mathscr{L}_\nu D_\nu^{(3,-)} = D_\nu^{(2,-)} \mathscr{L}_\nu.$$

Recall that the modified Laplace transform $\mathscr{L}_\nu\psi$ of a function ψ, defined on Ω, is given by

$$F(z) = \mathscr{L}_\nu\psi(z) = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-(z|u)}\psi(u)\Delta(u)^{\nu-\frac{N}{n}}\, m(du).$$

LEMMA 8.2. $D_\nu^{(3,-)}\psi(u) = \langle u, \nabla\psi(u)\rangle + \operatorname{tr} u\, \psi(u)$.

PROOF. For $a \in V_{\mathbb{C}}$,

$$\langle a, \nabla F(z)\rangle = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-(z|u)}(-\langle a, u\rangle)\psi(u)\Delta(u)^{\nu-\frac{N}{n}}\, m(du).$$

Observe that $(z \mid u)e^{-(z|u)} = \langle u, \nabla_u\rangle e^{-(z|u)}$. Therefore

$$\langle z, \nabla F(z)\rangle = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega (-\langle u, \nabla_u\rangle e^{-(z|u)})\psi(u)\Delta(u)^{\nu-\frac{N}{n}}\, m(du).$$

An integration by parts gives this is equal to

$$\frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-(z|u)}(\langle u, \nabla\rangle + n\nu)\psi(u)\Delta^{\nu-\frac{N}{n}}\, m(du).$$

Finally

$$(D_\nu^{(2,-)}F)(z) = \mathscr{L}_\nu(\langle u, \nabla\psi\rangle + \operatorname{tr} u\, \psi).$$

c) The operator $D_\nu^{(4,-)}$ acting on symmetric functions on $\mathbb{C}^n$ is such that

$$D_\nu^{(4,-)}\mathscr{F}_\nu = \mathscr{F}_\nu D_\nu^{(3,-)}.$$

Recall that the spherical Fourier transform $f = \mathscr{F}_\nu\psi$ of a function ψ defined on Ω, is given by

$$f(\mathbf{s}) = (\mathscr{F}_\nu\psi)(\mathbf{s}) = \frac{1}{\Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big)} \int_\Omega \varphi_{\mathbf{s}}(u)\psi(u)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}}\, m(du).$$

PROPOSITION 8.3. *The operator $D_\nu^{(4,-)}$ is the following difference operator: for a function f on $\mathbb{C}^n$,*

$$D_\nu^{(4,-)}f(\mathbf{s}) = \sum_{j=1}^n \left(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\alpha_j(\mathbf{s})\right)\big(f(\mathbf{s}+\varepsilon_j) - f(\mathbf{s})\big).$$

PROOF. We will compute $\mathscr{F}_\nu(D_\nu^{(3,-)}\psi) = \mathscr{F}_\nu(\langle u, \nabla\psi\rangle + \operatorname{tr} u\, \psi)$. Consider first

$$\mathscr{F}_\nu(\langle u, \nabla\psi\rangle)(\mathbf{s}) = \frac{1}{\Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big)} \int_\Omega \langle u, \nabla\psi(u)\rangle \varphi_{\mathbf{s}+\frac{\nu}{2}}(u)\Delta(u)^{-N/n}\, m(du).$$

An integration by parts gives, using that the function $\varphi_{\mathbf{s}}$ is homogeneous of degree $\sum_{j=1}^n s_j$ and that $\sum_{j=1}^n \rho_j = 0$, that

$$\begin{aligned}
&\mathscr{F}_\nu(\langle u, \nabla\psi\rangle)(\mathbf{s})\\
&\quad= \frac{1}{\Gamma_\Omega\big(\mathbf{s}+\frac{\nu}{2}+\rho\big)}\int_\Omega \psi(u)\big(-\langle u, \nabla_u\rangle\varphi_{\mathbf{s}+\frac{\nu}{2}}(u)\big)\Delta(u)^{-N/n}\, m(du)\\
&\quad= \frac{1}{\Gamma_\Omega\big(\mathbf{s}+\frac{\nu}{2}+\rho\big)}\int_\Omega \psi(u)\bigg(-\sum_{j=1}^n\Big(s_j+\frac{\nu}{2}\Big)\bigg)\varphi_{\mathbf{s}}(u)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}}\, m(du)\\
&\quad= -\sum_{j=1}^n\Big(s_j+\frac{\nu}{2}\Big)\mathscr{F}_\nu\psi(\mathbf{s}).
\end{aligned}$$

Recall Pieri's formula for spherical functions:

$$\operatorname{tr} u\, \varphi_{\mathbf{s}}(u) = \sum_{j=1}^n \alpha_j(\mathbf{s})\varphi_{\mathbf{s}+\varepsilon_j}(u), \quad \text{with } \alpha_j(\mathbf{s}) = \prod_{k\neq j}\frac{s_j - s_k + \frac{d}{2}}{s_j - s_k}.$$

Hence

$$\begin{aligned}
&\mathscr{F}_\nu(\operatorname{tr} u\, \psi)(\mathbf{s})\\
&\quad= \frac{1}{\Gamma_\Omega\big(\mathbf{s}+\frac{\nu}{2}+\rho\big)}\int_\Omega \psi(u)\bigg(\sum_{j=1}^n \alpha(\mathbf{s})\varphi_{\mathbf{s}+\varepsilon_j}(u)\bigg)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}}\, m(du)\\
&\quad= \sum_{j=1}^n \frac{\Gamma_\Omega\big(\mathbf{s}+\varepsilon_j+\frac{\nu}{2}+\rho\big)}{\Gamma_\Omega\big(\mathbf{s}+\frac{\nu}{2}+\rho\big)}\alpha_j(\mathbf{s})\\
&\qquad\times\frac{1}{\Gamma_\Omega\big(\mathbf{s}+\varepsilon_j+\frac{\nu}{2}+\rho\big)}\int_\Omega \psi(u)\varphi_{\mathbf{s}+\varepsilon_j}(u)\Delta^{\frac{\nu}{2}-\frac{N}{n}}\, m(du)\\
&\quad= \sum_{j=1}^n\Big(s_j+\frac{\nu}{2}-\frac{d}{4}(n-1)\Big)\alpha_j(\mathbf{s})\mathscr{F}_\nu\psi(\mathbf{s}+\varepsilon_j).
\end{aligned}$$

Finally

$$\mathscr{F}_\nu(D_\nu^{(3,-)}\psi)(\mathbf{s}) = \sum_{j=1}^n\Big(s_j+\frac{\nu}{2}-\frac{d}{4}(n-1)\Big)\alpha_j(\mathbf{s})f(\mathbf{s}+\varepsilon_j) - \sum_{j=1}^n\Big(s_j+\frac{\nu}{2}\Big)f(\mathbf{s})$$

with $f = \mathscr{F}_\nu(\psi)$. From $D_\nu^{(3,-)}\Psi_0 = 0$ and $\mathscr{F}_\nu(\Psi_0) = 1$, we get

$$\sum_{j=1}^n\Big(s_j+\frac{\nu}{2}-\frac{d}{4}(n-1)\Big)\alpha_j(\mathbf{s}) = \sum_{j=1}^n\Big(s_j+\frac{\nu}{2}\Big).$$

Therefore

$$\mathscr{F}_\nu(D_\nu^{(3,-)}\psi)(\mathbf{s}) = \sum_{j=1}^n \Big(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\Big)\alpha_j(\mathbf{s})\big(f(\mathbf{s}+\varepsilon_j) - f(\mathbf{s})\big).$$

We now finish the proof of Theorem 6.1. Recall that

$$D_\nu^{(4,+)} = S_\nu^{(4)} D_\nu^{(4,-)} S_\nu^{(4)} \quad \text{and} \quad S_\nu^{(4)} f(\mathbf{s}) = f(-\mathbf{s}).$$

Therefore, by Proposition 8.3,

$$D_\nu^{(4,+)} f(\mathbf{s}) = \sum_{j=1}^n \Big(-s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\Big)\alpha_j(-\mathbf{s})\big(f(\mathbf{s}-\varepsilon_j) - f(\mathbf{s})\big).$$

We have established the formula of Theorem 6.1 since

$$D_{\nu,\theta} = D_{\nu,\theta}^{(4)} = e^{i\theta} D_\nu^{(4,+)} + e^{-i\theta} D_\nu^{(4,-)}.$$

9. Pieri's formula for the Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$

THEOREM 9.1. *The Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$ satisfy the following Pieri formula:*

$$\begin{aligned}(2|\mathbf{s}|\cos\theta - 2i|2\mathbf{m}+\nu|\sin\theta)&Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s}) \\ &= \sum_{j=1}^n \Big(m_j + \nu - 1 - \frac{d}{4}(j-1)\Big)\alpha_j(\mathbf{m}-\varepsilon_j-\rho)d_{\mathbf{m}-\varepsilon_j} Q_{\mathbf{m}-\varepsilon_j}^{(\nu,\theta)}(\mathbf{s}) \\ &\quad - \sum_{j=1}^n \Big(m_j + 1 + \frac{d}{4}(n-j)\Big)\alpha_j(-\mathbf{m}-\varepsilon_j-\rho)d_{\mathbf{m}+\varepsilon_j} Q_{\mathbf{m}+\varepsilon_j}^{(\nu,\theta)}(\mathbf{s}).\end{aligned}$$

PROOF. The generating formula (Theorem 3.1(ii)), with $\mathbf{s} = \mathbf{m} + \frac{\nu}{2} - \rho$ can be written as

$$\sum_{\mathbf{k}} d_{\mathbf{k}} Q_{\mathbf{k}}^{(\nu,\theta)}\Big(\mathbf{m} + \frac{\nu}{2} - \rho\Big)\Phi_{\mathbf{k}}(w) = \Delta(e + e^{-i\theta}w)^{-\nu}\Phi_{\mathbf{m}}\big((e - e^{i\theta}w)(e + e^{-i\theta}w)^{-1}\big).$$

Since

$$F_{\mathbf{m}}^{(\nu,\theta)}(e^{-i\theta}w) = 2^{n\nu}\Delta(e + e^{-i\theta}w)^{-\nu}(-1)^{|\mathbf{m}|}e^{-i|\mathbf{m}|\theta}\Phi_{\mathbf{m}}\big((e - e^{i\theta}w)(e + e^{-i\theta}w)^{-1}\big),$$

we obtain

$$\sum_{\mathbf{k}} Q_{\mathbf{k}}^{(\nu,\theta)}\left(\mathbf{m}+\frac{\nu}{2}-\rho\right)e^{i|\mathbf{k}|\theta}\Phi_{\mathbf{k}}(w)=2^{-n\nu}(-1)^{|\mathbf{m}|}e^{i|\mathbf{m}|\theta}F_{\mathbf{m}}^{(\nu,\theta)}(w).$$

Recall that the function $F_{\mathbf{m}}^{(\nu,\theta)}$ is an eigenfunction of the differential operator $D_{\nu,\theta}^{(2)}$:

$$D_{\nu,\theta}^{(2)}F_{\mathbf{m}}^{(\nu,\theta)}(w)=2|\mathbf{m}|\cos\theta\, F_{\mathbf{m}}^{(\nu,\theta)}(w).$$

It follows that

$$\begin{aligned}(9.1)\quad \sum_{\mathbf{k}} d_{\mathbf{k}} Q_{\mathbf{k}}^{(\nu,\theta)}\left(\mathbf{m}+\frac{\nu}{2}-\rho\right)&e^{i|\mathbf{k}|\theta}D_{\nu,\theta}^{(2)}\Phi_{\mathbf{k}}(w)\\ &=2|\mathbf{m}|\cos\theta\sum_{\mathbf{k}} d_{\mathbf{k}} Q_{\mathbf{k}}^{(\nu,\theta)}\left(\mathbf{m}+\frac{\nu}{2}-\rho\right)\Phi_{\mathbf{k}}(w).\end{aligned}$$

To prove Theorem 9.1 we will compute $D_{\nu,\theta}^{(2)}\Phi_{\mathbf{k}}(w)$.

LEMMA 9.2. *The following formulas hold.*

(i)

$$\operatorname{tr}\big(\nabla\varphi_{\mathbf{s}}(z)\big)=\sum_{j=1}^{n}\left(s_j+\frac{d}{4}(n-1)\right)\alpha_j(-\mathbf{s})\varphi_{\mathbf{s}-\varepsilon_j}(z).$$

(ii)

$$\begin{aligned}&D_{\nu,\theta}^{(2)}\varphi_{\mathbf{s}}(z)\\ &=e^{i\theta}\left(\sum_{j=1}^{n}\left(s_j-\frac{d}{4}(n-1)+\nu\right)\alpha_j(\mathbf{s})\varphi_{\mathbf{s}+\varepsilon_j}(z)+\left(\sum_{j=1}^{n}s_j\right)\varphi_{\mathbf{s}}(z)\right)\\ &-e^{-i\theta}\left(\sum_{j=1}^{n}\left(s_j+\frac{d}{4}(n-1)\right)\alpha_j(-\mathbf{s})\varphi_{\mathbf{s}-\varepsilon_j}(z)+\left(\sum_{j=1}^{n}s_j\right)\varphi_{\mathbf{s}}(z)+n\nu\varphi_{\mathbf{s}}(z)\right).\end{aligned}$$

PROOF. (i) For $t>0$ we consider the following Laplace integral:

$$\int_{\Omega} e^{-(x|y)}e^{-t\operatorname{tr} y}\varphi_{\mathbf{s}}(y)\Delta(y)^{-N/n}\,m(dy)=\Gamma_{\Omega}(\mathbf{s}+\rho)\varphi_{-\mathbf{s}}(te+x).$$

Taking the derivative with respect to t for $t=0$, one gets

$$-\int_{\Omega} e^{-(x|y)}\operatorname{tr} y\,\varphi_{\mathbf{s}}(y)\Delta(y)^{-N/n}\,m(dy)=\Gamma_{\Omega}(\mathbf{s}+\rho)\operatorname{tr}(\nabla\varphi_{-\mathbf{s}}(x)).$$

By using Pieri's formula for spherical functions,

$$\operatorname{tr} y\, \varphi_{\mathbf{s}}(y) = \sum_{j=1}^{n} \alpha_j(\mathbf{s}) \varphi_{\mathbf{s}+\varepsilon_j}(y),$$

and since

$$\begin{aligned}\sum_{j=1}^{n} \alpha_j(\mathbf{s}) \int_\Omega e^{-(x|y)} \varphi_{\mathbf{s}+\varepsilon_j}(y) \Delta(y)^{-N/n}\, m(dy) \\ = \sum_{j=1}^{n} \alpha_j(\mathbf{s}) \Gamma_\Omega(\mathbf{s}+\varepsilon_j+\rho) \varphi_{-\mathbf{s}-\varepsilon_j}(x),\end{aligned}$$

one obtains

$$\begin{aligned}\operatorname{tr}(\nabla \varphi_{-\mathbf{s}}(x)) &= -\sum_{j=1}^{n} \alpha_j(\mathbf{s}) \frac{\Gamma_\Omega(\mathbf{s}+\varepsilon_j+\rho)}{\Gamma_\Omega(\mathbf{s}+\rho)} \varphi_{-\mathbf{s}-\varepsilon_j}(x) \\ &= -\sum_{j=1}^{n} \alpha_j(\mathbf{s}) \Big(s_j - \frac{d}{4}(n-1) \Big) \varphi_{-\mathbf{s}-\varepsilon_j}(x),\end{aligned}$$

or

$$\operatorname{tr}(\nabla \varphi_{\mathbf{s}}(x)) = \sum_{j=1}^{n} \alpha_j(-\mathbf{s}) \Big(s_j + \frac{d}{4}(n-1) \Big) \varphi_{\mathbf{s}-\varepsilon_j}(x).$$

In fact the explicit formula for Γ_Ω,

$$\Gamma_\Omega(\mathbf{s}+\rho) = (2\pi)^{N-n} \prod_{j=1}^{n} \Gamma\Big(s_j - \frac{d}{4}(n-1) \Big),$$

gives

$$\frac{\Gamma_\Omega(\mathbf{s}+\varepsilon_j+\rho)}{\Gamma_\Omega(\mathbf{s}+\rho)} = \frac{\Gamma\big(s_j + 1 - \frac{d}{4}(n-1)\big)}{\Gamma\big(s_j - \frac{d}{4}(n-1)\big)} = s_j - \frac{d}{4}(n-1).$$

(ii) Recall that

$$D_\nu^{(2,-)} F(z) = -\langle z+e, \nabla F(z) \rangle - n\nu F(z).$$

From (i) we obtain

$$D_\nu^{(2,-)} \varphi_{\mathbf{s}}(z) = \sum_{j=1}^{n} \Big(s_j + \frac{d}{4}(n-1) \Big) \alpha_j(-\mathbf{s}) \varphi_{\mathbf{s}-\varepsilon_j}(z) - \Big(\sum_{j=1}^{n} s_j + n\nu \Big) \varphi_{\mathbf{s}}(z).$$

By using $D_\nu^{(2,+)} = S_\nu^{(2)} D_\nu^{(2,-)} S_\nu^{(2)}$ and $S_\nu^{(2)}\varphi_{\mathbf{s}}(z) = \varphi_{-\mathbf{s}-\nu}(z)$, we get (ii).

We continue the proof of Theorem 9.1. Let us write out (ii) of Lemma 9.2 with $\mathbf{s} = \mathbf{k} - \rho$:

$$
\begin{aligned}
&D_{\nu,k}^{(2)}\Phi_{\mathbf{k}}(w)\\
&= e^{i\theta}\left(\sum_{j=1}^{n}\left(k_j + \nu - \frac{d}{2}(j-1)\right)\alpha_j(\mathbf{k}-\rho)\Phi_{\mathbf{k}+\varepsilon_j}(w) + |\mathbf{k}|\Phi_{\mathbf{k}}(w)\right)\\
&\quad - e^{-i\theta}\left(\sum_{j=1}^{n}\left(k_j + \frac{d}{2}(n-j)\right)\alpha_j(-\mathbf{k}+\rho)\Phi_{\mathbf{k}-\varepsilon_j}(w) + (|\mathbf{k}| + n\nu)\Phi_{\mathbf{k}}(w)\right).
\end{aligned}
$$

(Observe that $\sum_{j=1}^n \rho_j = 0$.) Now, equating the coefficients of $\Phi_{\mathbf{k}}(z)$ in both sides of (9.1), we obtain the formula of Theorem 9.1 for all $\mathbf{s} = \mathbf{m} + \frac{\nu}{2} - \rho$. Since both sides are polynomial functions in $\mathbf{s}$, the equality holds for every $\mathbf{s}$.

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