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GROUPOID ALGEBRAS AS CUNTZ-PIMSNER ALGEBRAS

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Abstract

We show that if G is a second countable locally compact Hausdorff étale groupoid carrying a suitable cocycle $c\colon G \to \mathbb{Z}$, then the reduced C^*-algebra of G can be realised naturally as the Cuntz-Pimsner algebra of a correspondence over the reduced C^*-algebra of the kernel G_0 of c. If the full and reduced C^*-algebras of G_0 coincide, we deduce that the full and reduced C^*-algebras of G coincide. We obtain a six-term exact sequence describing the K-theory of $C_r^*(G)$ in terms of that of $C_r^*(G_0)$.

In this short note we provide a sufficient condition for a groupoid C^*-algebra to have a natural realisation as a Cuntz-Pimsner algebra. The advantage to knowing this is the complementary knowledge one obtains from the two descriptions.

Our main result starts with a second-countable locally compact Hausdorff étale groupoid G with a continuous cocycle into the integers which is unperforated in the sense that G is generated by $c^{-1}(1)$. We then show that the reduced groupoid C^*-algebra $C_r^*(G)$ is the Cuntz-Pimsner algebra of a natural C^*-correspondence over the reduced C^*-algebra $C_r^*(c^{-1}(0))$ of the kernel of c.

We also show that if $C^*(c^{-1}(0))$ and $C_r^*(c^{-1}(0))$ coincide, then $C^*(G)$ and $C_r^*(G)$ coincide as well. We finish by applying results of Katsura to present a six-term exact sequence relating the K-theory of $C_r^*(G)$ to that of $C_r^*(c^{-1}(0))$.

Notation 1. For the duration of the paper, we fix a second-countable locally compact Hausdorff étale groupoid G with unit space $G^{(0)}$ and a continuous cocycle $c\colon G \to \mathbb{Z}$; that is, a map satisfying $c(\gamma_1\gamma_2) = c(\gamma_1) + c(\gamma_2)$ for composable γ_1 and γ_2. We suppose that c is *unperforated* in the sense that if $c(\gamma) = n > 0$, then there exist composable $\gamma_1, \ldots, \gamma_n$ such that each $c(\gamma_i) = 1$ and $\gamma = \gamma_1 \ldots \gamma_n$. For $n \in \mathbb{Z}$ we write $G_n := c^{-1}(n)$. Recall from [6] that for $u \in G^{(0)}$, we write $G_u = s^{-1}(u)$ and $G^u = r^{-1}(u)$. The convolution multi-

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plication of functions is denoted by juxtaposition, as is multiplication in the C^*-completions.

REMARKS 2. (i) Observe that $G^{(0)} \subseteq G_0$, and since c is continuous, G_0 is a clopen subgroupoid of G.

(ii) If c is strongly surjective in the sense that $c(r^{-1}(u)) = \mathbb{Z}$ for all $u \in G^{(0)}$ [1, Definition 5.3.7], and $\gamma \in G_n$, then there exists $\alpha \in r^{-1}(r(\gamma)) \cap G_1$, and then $\gamma = \alpha(\alpha^{-1}\gamma) \in G_1 G_{n-1}$. So an induction shows that if c is strongly surjective then it is unperforated.

LEMMA 3. *Let $A_0 \subseteq C_r^*(G)$ be the completion of $\{f \in C_c(G) : \operatorname{supp}(f) \subseteq G_0\}$. Then there is an isomorphism $I_0 \colon C_r^*(G_0) \to A_0$ extending the identity map on $C_c(G_0)$.*

PROOF. Fix $u \in G^{(0)}$, let π_u^0 be the regular representation of $C^*(G_0)$ on $\mathscr{H}_u^0 := \ell^2((G_0)_u)$ and let π_u be the regular representation of $C^*(G)$ on $\mathscr{H}_u := \ell^2(G_u)$. For $a \in C_c(G_0)$, the subspace $\mathscr{H}_u^0 \subseteq \mathscr{H}_u$ is reducing for the operator $\pi_u(a)$. Let $P_0 \colon \mathscr{H}_u \to \mathscr{H}_u^0$ be the orthonormal projection. We have

$$\|\pi_u^0(a)\| = \|P_0 \pi_u(a) P_0\| \leq \|\pi_u(a)\|.$$

Taking the supremum over all u, we deduce that $\|a\|_{C_r^*(G_0)} \leq \|a\|_{C_r^*(G)}$. So there is a C^*-homomorphism $\pi \colon A_0 \to C_r^*(G_0)$ extending the identity map on $C_c(G_0)$.

The restriction map from $C_c(G)$ to $C_c(G^{(0)})$ extends to a faithful conditional expectation† $\Psi \colon C_r^*(G) \to C_0(G^{(0)})$ and a faithful conditional expectation $\Psi_0 \colon C_r^*(G_0) \to C_0(G_0^{(0)}) = C_0(G^{(0)})$. We clearly have $\pi \circ \Psi = \Psi_0 \circ \pi$, and so a standard argument (see [7, Lemma 3.13]) shows that π is faithful and hence isometric. It follows immediately that $I_0 := \pi^{-1} \colon C_r^*(G_0) \to A_0$ is an isometric embedding of $C_r^*(G_0)$ in $C_r^*(G)$ as claimed.

LEMMA 4. *Let $X(G)$ be the completion in $C_r^*(G)$ of the subspace $\{f \in C_c(G) : \operatorname{supp}(f) \subseteq G_1\}$. Then $X(G)$ is a C^*-correspondence over $C_r^*(G_0)$ with module actions $a \cdot \xi = I_0(a)\xi$ and $\xi \cdot a = \xi I_0(a)$ and inner product $\langle \cdot\,, \cdot \rangle_{C_r^*(G_0)} \colon X(G) \times X(G) \to C_r^*(G)$ given by $\langle \xi, \eta \rangle_{C_r^*(G_0)} = I_0^{-1}(\xi^* \eta)$.*

PROOF. Since c is a cocycle, we have $G_1^{-1} = G_{-1}$. That c is a cocycle also implies that if $f \in C_c(G_m)$ and $g \in C_c(G_n)$ then $fg \in C_c(G_{m+n})$

† To see this observe that for $f \in C_c(G)$, we have $f(u) = (\pi_u(f)\delta_u | \delta_u)$, giving $\|f|_{G^{(0)}}\| = \sup_u (\pi_u(f)\delta_u|\delta_u) \leq \sup_u \|\pi_u(f)\| = \|f\|_{C_r^*(G)}$, so restriction induces an idempotent map $\Psi \colon C_r^*(G) \to C_0(G^{(0)})$ of norm one. Hence [2, Theorem II.6.10.2] implies that Ψ is a conditional expectation. Given $a \in C_r^*(G) \setminus \{0\}$, [6, Proposition II.4.2] shows that $a \in C_0(G)$, so $a(\gamma) \neq 0$ for some γ. Then $\Psi(a^*a)(s(\gamma)) = \sum_{\alpha \in G_{s(\gamma)}} \overline{a}(\alpha)a(\alpha) \geq |a(\gamma)|^2 > 0$, so Ψ is faithful.

and $f^* \in C_c(G_{-m})$, and it follows that $C_c(G_1)C_c(G_0), C_c(G_0)C_c(G_1) \subseteq C_c(G_1)$ and that $C_c(G_1)^*C_c(G_1) \subseteq C_c(G_0)$. The C^*-identity and Lemma 3 show that $\langle \cdot\, , \cdot \rangle_{C_r^*(G)}$ is isometric in the sense that $\|\langle \xi, \xi \rangle_{C_r^*(G_0)}\| = \|\xi\|^2_{C_r^*(G)}$, and in particular is positive definite. Continuity therefore guarantees that the operations described determine a Hilbert-module structure on $X(G)$. The left action of $C_r^*(G_0)$ on $X(G)$ is adjointable since $\langle a \cdot \xi, \eta \rangle_{C_r^*(G)} = (a\xi)^*\eta = \xi^*(a^*\eta) = \langle \xi, a^* \cdot \eta \rangle_{C_r^*(G)}$.

NOTATION 5. We write I_1 for the inclusion map $X(G) \hookrightarrow C_r^*(G)$.

Recall from [5] that a Toeplitz representation (ψ, π) of a C^*-correspondence X over a C^*-algebra A in a C^*-algebra B consists of a homomorphism $\pi\colon A \to B$ and a linear map $\psi\colon X \to B$ such that $\pi(a)\psi(x) = \psi(a \cdot x)$, $\psi(x)\pi(a) = \psi(x \cdot a)$ and $\pi(\langle x, y \rangle_A) = \psi(x)^*\psi(y)$ for all $x, y \in X$ and $a \in A$.

LEMMA 6. *The embeddings $I_1\colon X(G) \hookrightarrow C_r^*(G)$ and $I_0\colon C_r^*(G_0) \hookrightarrow C_r^*(G)$ form a Toeplitz representation (I_1, I_0) of $X(G)$ in $C_r^*(G)$.*

PROOF. For $\xi \in X(G)$ and $a \in C_r^*(G_0)$, we have $I_0(a)I_1(\xi) = a\xi = I_1(a \cdot \xi)$ by definition of the actions on $X(G)$. For $\xi, \eta \in C_c(G_1)$, we have

$$I_0(\langle \xi, \eta \rangle_{A_0}) = I_0(\xi^*\eta) = \xi^*\eta = I_1(\xi)^*I_1(\eta).$$

So (I_1, I_0) is a Toeplitz representation as claimed.

We aim to show that (I_1, I_0) is in fact a Cuntz-Pimsner covariant representation, but we have a little work to do first.

LEMMA 7. *For each integer $n > 0$, the space*

$$\operatorname{span}\{f_1 f_2 \cdots f_n : f_i \in C_c(G_1) \text{ for all } i\}$$

is dense in $C_c(G_n)$ in both the uniform norm and the bimodule norm of Lemma 4, and we have $C_c(G_n)^* = C_c(G_{-n})$.

PROOF. Proposition 4.1 of [6] shows that the reduced norm on $C_c(G)$ dominates the uniform norm. As discussed on [6, page 82], the reduced norm is dominated by the full norm which in turn is dominated by the I-norm by [6, Proposition 1.4]. By its definition, the I-norm agrees with the uniform norm on functions supported on bisections, forcing equality of all four norms on such functions. Thus, on functions supported on open bisections $U \subseteq G_1$, the uniform norm and the bimodule norm on $C_c(G_1)$ agree. Thus it suffices to show that $\operatorname{span}\{f_1 f_2 \cdots f_n : f_i \in C_c(G_1) \text{ for all } i\}$ is dense in $C_c(G_n)$ in the uniform norm. Fix $x \neq y \in G_n$. Since $G_n = c^{-1}(n)$ is locally compact, by the Stone-Weierstrass theorem it suffices to construct functions $f_1, \ldots, f_n \in C_c(G_1)$ such that $(f_1 \cdots f_n)(x) = 1$ and $(f_1 \cdots f_n)(y) = 0$.

Since c is unperforated, there exist $x_1, \ldots, x_n \in G_1$ such that $x = x_1 \cdots x_n$. Choose a precompact open bisection $U_n \subseteq G_1$ such that $x_n \in U_n$ and such that if $s(x) \neq s(y)$ then $s(y) \notin s(U_n)$; this is possible because s is a local homeomorphism. For each $1 \leq i \leq n-1$ choose a precompact open bisection $U_i \subseteq G_1$ with $x_i \in U_i$. Then $U_1 \cdots U_n$ is a precompact open bisection containing x because multiplication in G is continuous and open. If $s(x) = s(y)$, then $y \notin U_1 \cdots U_n$ because the latter is a bisection; and if $s(x) \neq s(y)$ then $y \notin U_1 \cdots U_n$ by choice of U_n.

By Urysohn's Lemma, for each i there exists $f_i \in C_c(U_i)$ with $f_i(x_i) = 1$. Then $(f_1 \cdots f_n)(x) = 1$ by construction, and the convolution formula says that

$$\operatorname{supp}(f_1 \cdots f_n) = \operatorname{supp}(f_1) \operatorname{supp}(f_2) \cdots \operatorname{supp}(f_n) \subseteq U_1 \cdots U_n,$$

which yields $(f_1 \cdots f_n)(y) = 0$.

The involution formula $f^*(\gamma) = \overline{f}(\gamma^{-1})$ and the cocycle property $c(\gamma^{-1}) = -c(\gamma)$ show that $f \in C_c(G_n)$ if and only if $f^* \in C_c(G_{-n})$.

Corollary 8. *There is an injection* $\psi\colon \mathscr{K}(X(G)) \to C_r^*(G_0)$ *such that* $\psi(\theta_{\xi,\eta}) = \xi\eta^*$ *for all* $\xi, \eta \in C_c(G_1)$.

Proof. By the discussion on page 202 of [5] (see also [3, Section 1]), there is a homomorphism $I_1^{(1)}\colon \mathscr{K}(X(G)) \to C_r^*(G)$ such that $I_1^{(1)}(\theta_{\xi,\eta}) = I_1(\xi)I_1(\eta)^* = \xi\eta^* = I_0(\xi\eta^*)$. Since I_0 is isometric, the composition $\psi := I_0^{-1} \circ I_1^{(1)}$ is a homomorphism satisfying the desired formula, and we need only show that ψ is injective.

If $T \in \mathscr{K}(X(G))$ and $\psi(T) = 0$, then $I_1^{(1)}(T) = 0$ as well. For $\eta, \xi, \zeta \in X(G)$, we have

$$I_1^{(1)}(\theta_{\eta,\xi})I_1(\zeta) = I_1(\eta)I_1(\xi)^*I_1(\zeta) = I_1(\eta \cdot \langle \xi, \zeta \rangle_{A_0}) = I_1(\theta_{\eta,\xi}(\zeta)),$$

and linearity and continuity show that for all $S \in \mathscr{K}(X(G))$ and $\zeta \in X(G)$, we have the formula $I_1^{(1)}(S)I_1(\zeta) = I_1(S\zeta)$. In particular, $0 = I_1^{(1)}(T)I_1(\zeta) = I_1(T\zeta)$ for all $\zeta \in X(G)$. Since I_1 is isometric, we deduce that $T\zeta = 0$ for all $\zeta \in C_c(G_1)$, and so $T = 0$. So ψ is injective.

Let X be a C^*-correspondence over a C^*-algebra A, and write $\phi\colon A \to \mathscr{L}(X)$ for the homomorphism implementing the left action. Following [4], we say that a Toeplitz representation (ψ, π) of X is *Cuntz-Pimsner covariant* if the homomorphism $\psi^{(1)}$ of $\mathscr{K}(X)$ satisfying $\psi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*$ satisfies $\psi^{(1)} \circ \phi(a) = \pi(a)$ for all a in the *Katsura ideal* $\phi^{-1}(\mathscr{K}(X)) \cap \ker(\phi)^{\perp}$.

Lemma 9. *Let* $\phi = \phi_{X(G)}\colon C_r^*(G_0) \to \mathscr{L}(X(G))$ *be the homomorphism implementing the left action. Then*

(1) $\ker(\phi) = C_r^*(G_0) \cap C_0(\{g \in G_0 : s(g) \notin r(G_1)\})$;

(2) $\ker(\phi)^\perp = C_r^*(G_0) \cap C_0(\{g \in G_0 : s(g) \in r(G_1)\}) = \overline{\operatorname{span}}\{fg^* : f, g \in C_c(G_1)\}$;

(3) $\phi_{X(G)}(a) \in \mathscr{K}(X(G))$ *for all* $a \in C_r^*(G)$, *and the Katsura ideal* $J_{X(G)}$ *is*

$$J_{X(G)} = \overline{\operatorname{span}}\{fg^* : f, g \in C_c(G_1)\};$$

(4) *the Toeplitz representation* (I_1, I_0) *of Lemma 6 is Cuntz-Pimsner covariant.*

Proof. (1) Choose $a \in C_r^*(G_0)$; by [6, Proposition II.4.2], $a \in C_0(G)$. Suppose that $a \notin \ker\phi$. Then there exists $\xi \in X(G)$ such that $a\xi = \phi(a)\xi$ is nonzero, so there exist composable γ, γ' with $a(\gamma) \neq 0$ and $\xi(\gamma') \neq 0$. We have $r(\gamma') \in r(G_1)$ by definition of $X(G)$, and so $a \notin C_0(\{g \in G_0 : s(g) \notin r(G_1)\})$.

On the other hand, suppose that $a \notin C_0(\{g \in G_0 : s(g) \notin r(G_1)\})$. Choose $g \in G_0$ such that $s(g) \in r(G_1)$ and $a(g) \neq 0$. Choose $\gamma \in G_1$ with $r(\gamma) = s(g)$, fix a precompact open bisection $U \subseteq G_1$ containing γ and use Urysohn's lemma to choose $\xi \in C_c(U) \subseteq C_c(G_1)$ such that $\xi(\gamma) = 1$. Then $(\phi(a)\xi)(g\gamma) = \sum_{\alpha\beta=g\gamma} a(\alpha)\xi(\beta)$. Since ξ is supported on the bisection U, and since $\alpha\beta = g\gamma$ implies $s(\beta) = s(\gamma)$, if $\alpha\beta = g\gamma$ and $a(\alpha)\xi(\beta) \neq 0$, we have $\beta = \gamma$ and then $\alpha = g$ by cancellation. So $(\phi(a)\xi)(g\gamma) = a(g) \neq 0$ and in particular $a \notin \ker\phi$.

(2) Suppose that $a \in \ker(\phi)^\perp$. Fix $g \in G_0$ with $s(g) \notin r(G_1)$. Since $G_1 = c^{-1}(1)$ is clopen and r is a local homeomorphism, $r(G_1)$ is also clopen and so there is an open $U \subseteq G^{(0)} \setminus r(G_1)$ with $s(g) \in U$. By Urysohn, there exists $f \in C_c(U)$ with $f(s(g)) = 1$. Part (1) gives $f \in \ker(\phi)$ and so $af = 0$, which gives $0 = (af)(g) = a(g)$. So $a \in C_0(\{g \in G_0 : s(g) \in r(G_1)\})$. On the other hand, if $a \in C_0(\{g \in G_0 : s(g) \in r(G_1)\})$ and $f \in \ker(\phi)$, then the preceding paragraph shows that $s(\{\gamma : a(\gamma) \neq 0\}) \cap r(\{\gamma : f(\gamma) \neq 0\}) = \emptyset$, and so $af = 0$ showing that $a \in \ker(\phi)^\perp$. To see that $C_r^*(G_0) \cap C_0(\{g \in G_0 : s(g) \in r(G_1)\}) = \overline{\operatorname{span}}\{fg^* : f, g \in C_c(G_1)\}$, observe that the containment $\supseteq$ is immediate from the definition of multiplication in $C_c(G)$. For the reverse containment, fix $f \in C_c(\{g \in G_0 : s(g) \in r(G_1)\})$. Choose an open set U in $r(G_1)$ that contains $s(\operatorname{supp}(f))$. Cover U by finitely many sets $r(V_i)$ where each V_i is a precompact open bisection in G_1. Choose a partition of unity a_i on $s(\operatorname{supp}(f))$ subordinate to the V_i, and define functions b_i supported on the V_i by $b_i(\gamma) := \sqrt{a_i(r(\gamma))}$ for $\gamma \in V_i$. Then $f = \sum_i (fb_i)b_i^* \in \operatorname{span}\{fg^* : f, g \in C_c(G_1)\}$.

(3) We have seen that $r(G_1)$ is clopen, and since G_0 is also clopen, $s^{-1}(r(G_1)) \cap G_0$ is clopen. So for $a \in C_c(G_0)$ the pointwise products $a_1 =$

$1_{s^{-1}(r(G_1))}a$ and $a_0 = 1_{G_0 \setminus s^{-1}(r(G_1))}a$ belong to $C_c(G_0)$ and satisfy $a = a_0 + a_1$. Since $a_0 \in \ker(\phi)$, we have $\phi(a) = \phi(a_1)$. So

$$\phi(C_r^*(G_0)) = \overline{\phi(C_c(s^{-1}(r(G_1)) \cap G_0))}.$$

As observed above, if $\gamma \in s^{-1}(r(G_1)) \cap G_0$ then for any $\alpha \in G_1$ with $r(\alpha) = s(\gamma)$, we can write $\gamma = (\gamma\alpha)\alpha^{-1} \in G_1 G_1^{-1}$. Since multiplication in G is open, it follows that

$$\{UV^{-1} : U, V \subseteq G_1 \text{ are precompact open bisections}\}$$

is a base for the topology on $s^{-1}(r(G_1)) \cap G_0$, and so $C_c(s^{-1}(r(G_1)) \cap G_0) = \overline{\operatorname{span}}\{fg^* : f, g \in C_c(G_1)\}$. For $f, g \in C_c(G_1)$ and $\xi \in C_c(G_1) \subseteq X(G)$, we have $\phi(fg^*)(\xi) = fg^*\xi = \theta_{f,g}(\xi)$. So $\phi(fg^*) = \theta_{f,g} \in \mathcal{K}(X(G))$. We deduce that $\phi(C_c(s^{-1}(r(G_1)) \cap G_0)) \subseteq \mathcal{K}(X(G))$, and the result follows.

(4) Part (3) shows that $\operatorname{span}\{fg^* : f, g \in C_c(G_1)\}$ is dense in the Katsura ideal $J_{X(G)}$. For $f, g \in C_c(G_1)$ we have $I_0(fg^*) = fg^*$ because I_0 extends the inclusion $C_c(G_0) \subseteq C_c(G)$ by definition, and we have just seen that $\phi(fg^*) = \theta_{f,g}$, giving $I_1^{(1)}(\phi(fg^*)) = I_1^{(1)}(\theta_{f,g}) = I_1(f)I_1(g)^* = fg^*$ as well.

NOTATION 10. Recall that if the pair (ψ, π) is a Toeplitz representation of a C^*-correspondence X in a C^*-algebra B, then for $n \geq 2$ we write ψ_n for the continuous linear map from $X^{\otimes n}$ to B such that $\psi(x_1 \otimes \cdots \otimes x_n) = \psi(x_1) \cdots \psi(x_n)$ for all $x_i \in X$.

THEOREM 11. *Suppose that G is a second-countable locally compact Hausdorff étale groupoid, and that $c \colon G \to \mathbb{Z}$ is an unperforated continuous cocycle as in Notation 1. Let $G_0 := c^{-1}(0)$ and $G_1 = c^{-1}(1)$. Let $X(G)$ be the Hilbert-$C_r^*(G_0)$-module completion of $C_c(G_1)$ described in Lemma 4. The inclusion $I_0 \colon C_c(G_0) \to C_c(G)$ extends to an embedding $I_0 \colon C_r^*(G_0) \to C_r^*(G)$ and the inclusion $I_1 \colon C_c(G_1) \to C_c(G)$ extends to a linear map $I_1 \colon X(G) \to C_r^*(G)$. The pair (I_1, I_0) is a Cuntz-Pimsner covariant representation of $X(G)$, and the integrated form $I_1 \times I_0$ is an isomorphism of $\mathcal{O}_{X(G)}$ onto $C_r^*(G)$.*

PROOF. Lemma 3 shows that I_0 extends to an embedding of $C_r^*(G_0)$. The map I_1 extends to $X(G)$ by definition of the latter (see Lemma 4 and Notation 5). Lemma 6 says that (I_1, I_0) is a representation of $X(G)$, and Lemma 9(4) says that this representation is Cuntz-Pimsner covariant.

The grading $c \colon C_r^*(G) \to \mathbb{Z}$ induces an action β of $\mathbb{T}$ on $C^*(G)$ such that

$$\beta_z(a) = z^{c(a)}a \quad \text{for all } z \in \mathbb{T} \text{ and } a \in \bigcup_n C_c(G_n).$$

For each n, let $C_c(G_1)^{\odot n}$ denote the dense subspace of $X(G)^{\otimes n}$ spanned by elementary tensors of the form $z_1 \otimes \cdots \otimes z_n$ with each $z_i \in C_c(G_1)$. Lemma 7 implies that for $n \geq 1$, the image $I_n(C_c(G_1)^{\odot n})$ is dense in $C_c(G_n)$. This implies that $I_1 \times I_0$ is equivariant for the gauge action α on $\mathcal{O}_{X(G)}$ and β. The gauge-invariant uniqueness theorem [4, Theorem 6.4] implies that $I_1 \times I_0$ is injective. Moreover, since each $I_n(C_c(G_1)^{\odot n})^*$ is dense in $C_c(G_n)^*$, which is $C_c(G_{-n})$ by Lemma 7, $I_1 \times I_0$ has dense range, and so is surjective since it is a homomorphism between C^*-algebras.

COROLLARY 12. *Suppose that $C^*(G_0) = C_r^*(G_0)$. Then $C^*(G) = C_r^*(G)$ and so Theorem 11 describes an isomorphism $\mathcal{O}_{X(G)} \cong C^*(G)$.*

PROOF. We saw in Lemma 3 that the completion A_0 of $C_c(G_0)$ in $C_r^*(G)$ coincides with $C_r^*(G_0)$, and therefore, by hypothesis, with $C^*(G_0)$. So $\|I_0(a)\| = \|a\|_{C^*(G_0)}$ for all $a \in C_c(G_0)$. For $\xi \in C_c(G_1)$, we have

$$\|\xi^*\xi\|_{C^*(G)} \geq \|\xi^*\xi\|_{C_r^*(G)}.$$

The function $\xi^*\xi$ belongs to $C_c(G_0)$ and so $\|\xi^*\xi\|_{C_r^*(G)} = \|\xi^*\xi\|_{C_r^*(G_0)}$ by Lemma 3, and this last is equal to $\|\xi^*\xi\|_{C^*(G_0)}$ by hypothesis. Since the I-norm on $C_c(G)$ agrees with the I-norm on $C_c(G_0)$, the canonical inclusion $C_c(G_0) \hookrightarrow C_c(G)$ is I-norm bounded, and so extends to a C^*-homomorphism $C^*(G_0) \to C^*(G)$, which forces $\|\xi^*\xi\|_{C^*(G_0)} \geq \|\xi^*\xi\|_{C^*(G)}$. So we have equality throughout, giving

$$\|\xi^*\xi\|_{C^*(G)} = \|\xi^*\xi\|_{C_r^*(G_0)} = \|\langle I_1(\xi), I_1(\xi)\rangle_{C_r^*(G_0)}\|.$$

In particular, $\|\xi\|_{C^*(G)} = \|I_1(\xi)\|$. We deduce that (I_0, I_1) extends to a Cuntz-Pimsner covariant Toeplitz representation $(\tilde{I}_0, \tilde{I}_1)$ of $X(G)$ in $C^*(G)$. Let $q\colon C^*(G) \to C_r^*(G)$ denote the quotient map. Then $q \circ (\tilde{I}_1 \times \tilde{I}_0) = I_1 \times I_0$, which is injective by Theorem 11. Since $\tilde{I}_1 \times \tilde{I}_0$ is surjective, we deduce that q is also injective.

REMARK 13. If G_0 is an amenable groupoid (we don't have to specify which flavour because topological amenability and measurewise amenability coincide for second-countable locally compact Hausdorff étale groupoids [1, Theorem 3.3.7]), then [8, Proposition 9.3] shows that G is amenable as well, and then the preceding corollary follows. So the corollary has independent content only if G_0 is not amenable but nevertheless has identical full and reduced norms.

Let

$$L_0 = \big\{a \in M_2(C_c(G)) : a_{11} \in J_{X(G)}, a_{22} \in C_c(G_0), \\ a_{12} \in C_c(G_1) \text{ and } a_{21} \in C_c(G_{-1})\big\}.$$

Then the completion of L_0 in the norm induced by the reduced norm on $C_r^*(G)$ is

$$L = \begin{pmatrix} J_{X(G)} & X(G) \\ X(G)^* & C_r^*(G_0) \end{pmatrix}.$$

Part (3) of Lemma 9 shows that L is a C^*-subalgebra of $M_2(C_r^*(G))$, and that the corners $\left(\begin{smallmatrix} J_{X(G)} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & C_r^*(G_0) \end{smallmatrix}\right)$ are full. So the inclusion maps $i^{11}\colon J_{X(G)} \to L$ into the top-left corner and $i^{22}\colon C_r^*(G_0) \to L$ into the bottom-right corner determine isomorphisms $i_*^{11}\colon K_*(J_{X(G)}) \to K_*(L)$ and $i_*^{22}\colon K_*(C^*(G_0)) \to K_*(L)$. The composite $[X] := (i_*^{22})^{-1} \circ i_*^{11}\colon K_*(J_{X(G)}) \to K_*(C_r^*(G_0))$ is the map appearing in [4, Theorem 8.6] (see [4, Remark B4]). The inclusion map $\iota\colon J_{X(G)} \hookrightarrow C_r^*(G_0)$ also induces a homomorphism $\iota_*\colon K_*(J_{X(G)}) \to K_*(C_r^*(G_0))$, and Theorem 8.6 of [4] gives the following.

COROLLARY 14. *There is an exact sequence in K-theory as follows:*

$$\begin{array}{ccccc} K_0(J_{X(G)}) & \xrightarrow{\iota_* - [X]} & K_0(C_r^*(G_0)) & \xrightarrow{(I_0)_*} & K_0(C_r^*(G_0)) \\ \uparrow & & & & \downarrow \\ K_1(J_{X(G)}) & \xleftarrow{(I_0)_*} & K_1(C_r^*(G_0)) & \xleftarrow{\iota_* - [X]} & K_1(C_r^*(G)) \end{array}$$

REFERENCES

1. Anantharaman-Delaroche, C., and Renault, J., *Amenable groupoids*, Monographies de L'Enseignement Mathématique, vol. 36, L'Enseignement Mathématique, Geneva, 2000.
2. Blackadar, B., *Operator algebras: Theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006.
3. Fowler, N. J., and Raeburn, I., *The Toeplitz algebra of a Hilbert bimodule*, Indiana Univ. Math. J. 48 (1999), no. 1, 155–181.
4. Katsura, T., *On C^*-algebras associated with C^*-correspondences*, J. Funct. Anal. 217 (2004), no. 2, 366–401.
5. Pimsner, M. V., *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* **Z**, Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 189–212.
6. Renault, J., *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980.
7. Sims, A., Whitehead, B., and Whittaker, M. F., *Twisted C^*-algebras associated to finitely aligned higher-rank graphs*, Doc. Math. 19 (2014), 831–866.

8. Spielberg, J., *Groupoids and C^*-algebras for categories of paths*, Trans. Amer. Math. Soc. 366 (2014), no. 11, 5771–5819.

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