MATH. SCAND. 120 (2017), 129–144

EXTENSIONS OF EUCLIDEAN OPERATOR RADIUS INEQUALITIES

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Abstract

To extend the Euclidean operator radius, we define w_p for an n-tuple of operators $(T_1, \dots, T_n)$ in $\mathbb{B}(\mathscr{H})$ by $w_p(T_1, \dots, T_n) := \sup_{\|x\|=1}(\sum_{i=1}^n |\langle T_i x, x\rangle|^p)^{1/p}$ for $p \ge 1$. We generalize some inequalities including the Euclidean operator radius of two operators to those involving w_p. Further, we obtain some lower and upper bounds for w_p. Our main result states that if f and g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$, then

$$w_p^{rp}(A_1^* T_1 B_1, \dots, A_n^* T_n B_n) \le \frac{n^{r-1}}{2}\left\| \sum_{i=1}^n [B_i^* f^2(|T_i|) B_i]^{rp} + [A_i^* g^2(|T_i^*|) A_i]^{rp} \right\|,$$

for all $p \ge 1$, $r \ge 1$ and operators in $\mathbb{B}(\mathscr{H})$.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. The numerical radius of $A \in \mathbb{B}(\mathscr{H})$ is defined by

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. Namely, we have

$$\tfrac{1}{2}\|A\| \le w(A) \le \|A\|,$$

for each $A \in \mathbb{B}(\mathscr{H})$. It is also known that if $A \in \mathbb{B}(\mathscr{H})$ is self-adjoint, then $w(A) = \|A\|$. An important inequality for $w(A)$ is the power inequality stating that $w(A^n) \le w^n(A)$ for $n = 1, 2, \dots$. There are many inequalities involving the numerical radius; see [3], [5], [4], [10], [11] and references therein.

The Euclidean operator radius of an n-tuple $(T_1, \dots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)} := \mathbb{B}(\mathscr{H}) \times \cdots \times \mathbb{B}(\mathscr{H})$ is defined in [9] by

$$w_e(T_1, \dots, T_n) := \sup_{\|x\|=1}\left(\sum_{i=1}^n |\langle T_i x, x\rangle|^2\right)^{1/2}.$$

Received 15 July 2014.
DOI: https://doi.org/10.7146/math.scand.a-25509

The particular cases $n = 1$ and $n = 2$ are the numerical radius and the Euclidean operator radius. Some interesting properties of this radius are presented in [9]. For example, it is established that

$$\frac{1}{2\sqrt{n}}\left\|\sum_{i=1}^{n} T_i T_i^*\right\|^{1/2} \leq w_e(T_1, \ldots, T_n) \leq \left\|\sum_{i=1}^{n} T_i T_i^*\right\|^{1/2}. \qquad (1.1)$$

We also observe that if $A = B + \mathrm{i}C$ is the Cartesian decomposition of A, then

$$w_e^2(B, C) = \sup_{\|x\|=1} \{|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2\} = \sup_{\|x\|=1} |\langle Ax, x\rangle|^2 = w^2(A).$$

By the above inequality and $A^*A + AA^* = 2(B^2 + C^2)$, we have

$$\tfrac{1}{16}\|A^*A + AA^*\| \leq w^2(A) \leq \tfrac{1}{2}\|A^*A + AA^*\|.$$

We define w_p for n-tuples of operators $(T_1, \ldots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$, for $p \geq 1$, by

$$w_p(T_1, \ldots, T_n) := \sup_{\|x\|=1}\left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p\right)^{1/p}.$$

It follows from Minkowski's inequality for two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$, namely,

$$\left(|a_1 + b_1|^p + |a_2 + b_2|^p\right)^{1/p} \leq \left(|a_1|^p + |a_2|^p\right)^{1/p} + \left(|b_1|^p + |b_2|^p\right)^{1/p},$$

for $p \geq 1$, that w_p is a norm.

Moreover, w_p $(p \geq 1)$ for n-tuples of operators $(T_1, \ldots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$ satisfies the following properties:

(i) $w_p(T_1, \ldots, T_n) = 0 \Leftrightarrow T_1 = \cdots = T_n = 0$;

(ii) $w_p(\lambda T_1, \ldots, \lambda T_n) = |\lambda| w_p(T_1, \ldots, T_n)$ for all $\lambda \in \mathbb{C}$;

(iii) $w_p(T_1 + T_1', \ldots, T_n + T_n') \leq w_p(T_1, \ldots, T_n) + w_p(T_1', \ldots, T_n')$ for $(T_1', \ldots, T_n') \in \mathbb{B}(\mathscr{H})^{(n)}$;

(iv) $w_p(X^*T_1X, \ldots, X_n^*X) \leq \|X\|^2 w_p(T_1, \ldots, T_n)$ for $X \in \mathbb{B}(\mathscr{H})$.

Dragomir [1] obtained some inequalities for the Euclidean operator radius $w_e(B, C) = \sup_{\|x\|=1}(|\langle Bx, x\rangle|^2 + |\langle Cx, x\rangle|^2)^{1/2}$ of two bounded linear operators in a Hilbert space. In section 2 of this paper, we extend some of his results including inequalities for the Euclidean operator radius of linear operators to w_p $(p \geq 1)$. In addition, we apply some known inequalities for getting new inequalities for w_p in two operators.

In section 3, we prove inequalities for w_p on n-tuples of operators. Some of our result in this section, generalize some inequalities in section 2. Further, we find some lower and upper bounds for w_p.

2. Inequalities for w_p for two operators

To prove our generalized numerical radius inequalities, we need several known lemmas. The first lemma is a simple result of the classical Jensen inequality and a generalized mixed Cauchy-Schwarz inequality [7], [2], [6].

LEMMA 2.1. *For $a, b \geq 0$, $0 \leq \alpha \leq 1$ and $r \neq 0$:*

(a) $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq [\alpha a^r + (1-\alpha)b^r]^{1/r}$ *for* $r \geq 1$*;*

(b) *if* $A \in \mathbb{B}(\mathscr{H})$*, then* $|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha}x, x\rangle\langle |A^*|^{2(1-\alpha)}y, y\rangle$ *for all* $x, y \in \mathscr{H}$*, where* $|A| = (A^*A)^{1/2}$*;*

(c) *let* $A \in \mathbb{B}(\mathscr{H})$*, and* f *and* g *be non-negative continuous functions on* $[0, \infty)$ *satisfying* $f(t)g(t) = t$ *for all* $t \in [0, \infty)$*. Then*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\|,$$

for all $x, y \in \mathscr{H}$*.*

LEMMA 2.2 (McCarthy inequality [8]). *Let* $A \in \mathbb{B}(\mathscr{H})$*,* $A \geq 0$*, and let* $x \in \mathscr{H}$ *be any unit vector. Then*

(a) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$*, for* $r \geq 1$*;*

(b) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$*, for* $0 < r \leq 1$.

The inequalities of the following lemma were obtained for the first time by Clarkson [7].

LEMMA 2.3. *Let* X *be a normed space and* $x, y \in X$*. Then for all* $p \geq 2$ *with* $1/p + 1/q = 1$*,*

(a) $2(\|x\|^p + \|y\|^p)^{q-1} \leq \|x+y\|^q + \|x-y\|^q$*;*

(b) $2(\|x\|^p + \|y\|^p) \leq \|x+y\|^p + \|x-y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p)$*;*

(c) $\|x+y\|^p + \|x-y\|^p \leq 2(\|x\|^q + \|y\|^q)^{p-1}$.

If $1 < p \leq 2$*, then the converse inequalities hold.*

Making the transformations $x \to (x+y)/2$ and $y \to (x-y)/2$ we observe that inequalities (a) and (c) in Lemma 2.3 are equivalent and so are the first and the second inequalities of (b).

First of all, we obtain a relation between w_p and w_e for $p \geq 1$.

PROPOSITION 2.4. *Let* $B, C \in \mathbb{B}(\mathscr{H})$*. Then*

$$w_p(B, C) \leq w_q(B, C) \leq 2^{1/q-1/p} w_p(B, C),$$

for $p \geq q \geq 1$. *In particular,*

$$w_p(B, C) \leq w_e(B, C) \leq 2^{1/2-1/p} w_p(B, C), \tag{2.1}$$

for $p \geq 2$, *and*

$$2^{1/2-1/p} w_p(B, C) \leq w_e(B, C) \leq w_p(B, C),$$

for $1 \leq p \leq 2$.

PROOF. An application of Jensen's inequality says that for $a, b > 0$ and $p \geq q > 0$, we have

$$(a^p + b^p)^{1/p} \leq (a^q + b^q)^{1/q}.$$

Let $x \in \mathscr{H}$ be a unit vector. Choosing $a = |\langle Bx, x\rangle|$ and $b = |\langle Cx, x\rangle|$, we have

$$\big(|\langle Bx, x\rangle|^p + |\langle Cx, x\rangle|^p\big)^{1/p} \leq \big(|\langle Bx, x\rangle|^q + |\langle Cx, x\rangle|^q\big)^{1/q}.$$

Now the first inequality follows by taking the supremum over all unit vectors in $\mathscr{H}$. A simple consequence of the classical Jensen inequality concerning the convexity or the concavity of certain power functions says that for $a, b \geq 0$, $0 \leq \alpha \leq 1$ and $p \geq q$, we have

$$(\alpha a^q + (1-\alpha)b^q)^{1/q} \leq (\alpha a^p + (1-\alpha)b^p)^{1/p}.$$

For $\alpha = 1/2$, we get

$$(a^q + b^q)^{1/q} \leq 2^{1/q-1/p}(a^p + b^p)^{1/p}.$$

Again, let $x \in \mathscr{H}$ be a unit vector. Choosing $a = |\langle Bx, x\rangle|$ and $b = |\langle Cx, x\rangle|$ we get

$$\big(|\langle Bx, x\rangle|^q + |\langle Cx, x\rangle|^q\big)^{1/q} \leq 2^{1/q-1/p}\big(|\langle Bx, x\rangle|^p + |\langle Cx, x\rangle|^p\big)^{1/p}.$$

Now the second inequality follows by taking the supremum over all unit vectors in $\mathscr{H}$.

On making use of inequality (2.1) we find a lower bound for w_p $(p \geq 2)$.

COROLLARY 2.5. *If* $B, C \in \mathbb{B}(\mathscr{H})$, *then for* $p \geq 2$

$$w_p(B, C) \geq 2^{1/p-2}\|BB^* + CC^*\|^{1/2}.$$

PROOF. According to inequalities (1.1) and (2.1) we can write

$$w_e(B, C) \geq \frac{1}{2\sqrt{2}} \|BB^* + CC^*\|^{1/2}$$

and

$$w_p(B, C) \geq 2^{1/p-1/2} w_e(B, C),$$

respectively. We therefore get the desired inequality.

The next result is concerned with some lower bounds for w_p. The conclusion has several inequalities as special cases. Our result will be generalized to n-tuples of operators in the next section.

PROPOSITION 2.6. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then for $p \geq 1$*

$$w_p(B, C) \geq 2^{1/p-1} \max\{w(B + C), w(B - C)\}. \tag{2.2}$$

This inequality is sharp.

PROOF. We use the convexity of function $f(t) = t^p$ $(p \geq 1)$ as follows:

$$\begin{aligned}(|\langle Bx, x\rangle|^p + |\langle Cx, x\rangle|^p)^{1/p} &\geq 2^{1/p-1}(|\langle Bx, x\rangle| + |\langle Cx, x\rangle|)\\ &\geq 2^{1/p-1}|\langle Bx, x\rangle \pm \langle Cx, x\rangle|\\ &= 2^{1/p-1}|\langle (B \pm C)x, x\rangle|.\end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ yields that

$$w_p(B, C) \geq 2^{1/p-1} w(B \pm C).$$

For sharpness one can obtain the same quantity $2^{1/p} w(B)$ on both sides of the inequality by putting $B = C$.

COROLLARY 2.7. *If $A = B + \mathrm{i}C$ is the Cartesian decomposition of A, then for all $p \geq 1$*

$$w_p(B, C) \geq 2^{1/p-1} \max\{\|B + C\|, \|B - C\|\}$$

and for $p \geq 2$

$$w(A) \geq 2^{1/p-2} \max\{\|(1 - \mathrm{i})A + (1 + \mathrm{i})A^*\|, \|(1 + \mathrm{i})A + (1 - \mathrm{i})A^*\|\}.$$

PROOF. Obviously by inequality (2.2) we have the first inequality. For the second, it is enough to use $w_e(B, C) = w(A)$ and inequality (2.1).

COROLLARY 2.8. *If $B, C \in \mathbb{B}(\mathscr{H})$, then for $p \geq 1$*

$$w_p(B, C) \geq 2^{1/p-1} \max\{w(B), w(C)\}. \tag{2.3}$$

In addition, if $A = B + \mathrm{i}C$ is the Cartesian decomposition of A, then for $p \geq 2$

$$w(A) \geq 2^{1/p-2} \max\{\|A + A^*\|, \|A - A^*\|\}.$$

PROOF. By inequality (2.2) and properties of the numerical radius, we have

$$2w_p(B, C) \geq 2^{1/p-1}(w(B + C) + w(B - C)) \geq 2^{1/p-1} w(B + C + B - C).$$

So

$$w_p(B, C) \geq 2^{1/p-1} w(B).$$

By symmetry we conclude that

$$w_p(B, C) \geq 2^{1/p-1} \max\{w(B), w(C)\}.$$

The second inequality follows easily from inequality (2.1).

Now we apply part (b) of Lemma 2.3 to find some lower and upper bounds for w_p ($p > 1$).

PROPOSITION 2.9. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then for all $p \geq 2$,*

(i) $2^{1/p-1} w_p(B + C, B - C) \leq w_p(B, C) \leq 2^{-1/p} w_p(B + C, B - C)$;

(ii) $2^{1/p-1}\big(w^p(B + C) + w^p(B - C)\big)^{1/p} \leq w_p(B, C) \leq 2^{-1/p}\big(w^p(B + C) + w^p(B - C)\big)^{1/p}$.

If $1 < p \leq 2$, then these inequalities hold in the opposite direction.

PROOF. Let $x \in \mathscr{H}$ be a unit vector. Part (b) of Lemma 2.3 implies that for any $p \geq 2$

$$2^{1-p}(|a + b|^p + |a - b|^p) \leq |a|^p + |b|^p \leq \tfrac{1}{2}(|a + b|^p + |a - b|^p).$$

Inserting $a = |\langle Bx, x\rangle|$ and $b = |\langle Cx, x\rangle|$ in the above inequalities we obtain the desired results.

REMARK 2.10. In inequality (2.3), if we take $B + C$ and $B - C$ instead of B and C, then for $p \geq 1$

$$w_p(B + C, B - C) \geq 2^{1/p-1} \max\{w(B + C), w(B - C)\}.$$

By employing the first inequality of part (i) of Proposition 2.9, we get

$$w_p(B, C) \geq 2^{2/p-2} \max\{w(B + C), w(B - C)\}$$

for $p \geq 2$.

Taking $B+C$ and $B-C$ instead of B and C in the second inequality of part (ii) of Proposition 2.9, we reach

$$w_p(B+C, B-C) \leq 2^{1-1/p}(w^p(B)+w^p(C))^{1/p},$$

for all $p \geq 2$.

Now by applying the second inequality of part (i) of Proposition 2.9, we infer for $p \geq 2$ that

$$w_p(B, C) \leq 2^{1-2/p}(w^p(B)+w^p(C))^{1/p}.$$

Thus for all $p \geq 2$

$$2^{2/p-2}\max\{w(B+C), w(B-C)\} \leq w_p(B, C) \leq 2^{1-2/p}(w^p(B)+w^p(C))^{1/p}.$$

Moreover, if B and C are self-adjoint, then

$$2^{2/p-2}\max\{\|B+C\|, \|B-C\|\} \leq w_p(B, C) \leq 2^{1-2/p}(\|B\|^p+\|C\|^p)^{1/p}.$$

In the following result we find another lower bound for w_p ($p \geq 1$).

THEOREM 2.11. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then for $p \geq 1$*

$$w_p(B, C) \geq 2^{1/p-1}w^{1/2}(B^2+C^2).$$

PROOF. It follows from (2.2) that

$$2^{2/p-2}w^2(B \pm C) \leq w_p^2(B, C).$$

Hence

$$\begin{aligned} 2w_p^2(B, C) &\geq 2^{2/p-2}\big[w^2(B+C)+w^2(B-C)\big] \\ &\geq 2^{2/p-2}\big[w((B+C)^2)+w((B-C)^2)\big] \\ &\geq 2^{2/p-2}\big[w((B+C)^2+(B-C)^2)\big] = 2^{2/p-1}w(B^2+C^2). \end{aligned}$$

It follows that

$$w_p(B, C) \geq 2^{1/p-1}w^{1/2}(B^2+C^2).$$

COROLLARY 2.12. *If $A = B + \mathrm{i}C$ is the Cartesian decomposition of A, then for any $p \geq 2$*

$$w_p(B, C) \geq 2^{1/p-1}\|B^2+C^2\|^{1/2}$$

and

$$w(A) \geq 2^{1/p-3/2}\|A^*A+AA^*\|^{1/2}.$$

PROOF. The first inequality is obvious. For the second note that $A^*A + AA^* = 2(B^2 + C^2)$. Thus by using inequality (2.1) the proof is complete.

COROLLARY 2.13. *If $B, C \in \mathbb{B}(\mathscr{H})$, then for $p \geq 2$*

$$w_p(B, C) \geq 2^{2/p-3/2} w^{1/2}(B^2 + C^2).$$

PROOF. By choosing $B+C$ and $B-C$ instead of B and C in Theorem 2.11 and employing the first inequality of part (i) of Proposition 2.9 we conclude that the desired inequality.

The following result providing another bound for w_p $(p > 1)$ may be stated as follows:

PROPOSITION 2.14. *Let $B, C \in \mathbb{B}(\mathscr{H})$. Then*

$$w_p(B, C) \leq w_q\big(\tfrac{1}{2}(B + C), \tfrac{1}{2}(B - C)\big)$$

for any $p \geq 2$ with $1/p + 1/q = 1$. If $1 < p \leq 2$, then the reverse inequality holds.

PROOF. Let $x \in \mathscr{H}$ be a unit vector. Part (a) of Lemma 2.3 implies that

$$|a|^p + |b|^p \leq 2^{1/(1-q)}(|a + b|^q + |a - b|^q)^{1/(q-1)}.$$

So

$$(|a|^p + |b|^p)^{1/p} \leq 2^{1/(p(1-q))}(|a + b|^q + |a - b|^q)^{1/(p(q-1))}.$$

Now inserting $a = \langle Bx, x\rangle$ and $b = \langle Cx, x\rangle$ in the above inequality we conclude that

$$\big(|\langle Bx, x\rangle|^p + |\langle Cx, x\rangle|^p\big)^{1/p} \leq \big(\big|\big\langle \tfrac{1}{2}(B + C)x, x\big\rangle\big|^q + \big|\big\langle \tfrac{1}{2}(B - C)x, x\big\rangle\big|^q\big)^{1/q}. \tag{2.4}$$

By taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ we deduce that

$$w_p(B, C) \leq w_q\big(\tfrac{1}{2}(B + C), \tfrac{1}{2}(B - C)\big),$$

for any $p \geq 2$ with $1/p + 1/q = 1$.

COROLLARY 2.15. *Inequality* (2.4) *implies that*

$$w_p(B, C) \leq \Big(w^q\big(\tfrac{1}{2}(B + C)\big) + w^q\big(\tfrac{1}{2}(B - C)\big)\Big)^{1/q},$$

for any $p \geq 2$ with $1/p + 1/q = 1$. Further, if B and C are self-adjoint, then

$$w_p(B, C) \leq \tfrac{1}{2}(\|B + C\|^q + \|B - C\|^q)^{1/q}.$$

If $1 < p \leq 2$, then the converse inequalities hold.

COROLLARY 2.16. *If $B, C \in \mathbb{B}(\mathscr{H})$, then*

$$w_q\big(\tfrac{1}{2}(B+C), \tfrac{1}{2}(B-C)\big) \le 2^{1/p} w_p\big(\tfrac{1}{2}(B+C), \tfrac{1}{2}(B-C)\big),$$

for all $1 < p \le 2$ with $1/p + 1/q = 1$. If $p \ge 2$, then the above inequality is valid in the opposite direction.

PROOF. By Proposition 2.14 we have

$$w_q\big(\tfrac{1}{2}(B+C), \tfrac{1}{2}(B-C)\big) \le w_p(B, C)$$

for all $1 < p \le 2$ with $1/p + 1/q = 1$. Proposition 2.9 implies that

$$w_p(B, C) \le 2^{1/p-1} w_p(B+C, B-C) = 2^{1/p} w_p\big(\tfrac{1}{2}(B+C), \tfrac{1}{2}(B-C)\big).$$

We therefore get the desired inequality.

3. Inequalities of w_p for n-tuples of operators

In this section, we will obtain some numerical radius inequalities for n-tuples of operators. Some generalizations of inequalities from the previous section are also established. According to the definition of the numerical radius, we immediately get the following double inequality for $p \ge 1$

$$w_p(T_1, \dots, T_n) \le \left(\sum_{i=1}^{n} w^p(T_i)\right)^{1/p} \le \sum_{i=1}^{n} w(T_i).$$

An application of Holder's inequality gives the next result, which is a generalization of inequality (2.2).

THEOREM 3.1. *Let $(T_1, \dots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$ and fix $0 \le \alpha_i \le 1$, $i = 1, \dots, n$, with $\sum_{i=1}^{n} \alpha_i = 1$. Then*

$$w_p(T_1, \dots, T_n) \ge w\big(\alpha_1^{1-1/p} T_1 \pm \alpha_2^{1-1/p} T_2 \pm \cdots \pm \alpha_n^{1-1/p} T_n\big),$$

for any $p > 1$.

PROOF. In the Euclidean space $\mathbb{R}^n$ with the standard inner product, Holder's inequality

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}$$

holds, where p and q are in the open interval $(1, \infty)$ with $1/p + 1/q = 1$ and $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$. For $(y_1, \dots, y_n) = (\alpha_1^{1-1/p}, \dots, \alpha_n^{1-1/p})$ we have

$$\sum_{i=1}^{n} |\alpha_i^{1-1/p} x_i| \le \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} \Big(\sum_{i=1}^{n} |\alpha_i^{1-1/p}|^q\Big)^{1/q}.$$

Thus
$$\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} \geq \sum_{i=1}^{n}\left|\alpha_i^{1-1/p}x_i\right|.$$

Choosing $x_i = |\langle T_i x, x\rangle|$, $i = 1, \dots, n$, we get

$$\begin{aligned}\left(\sum_{i=1}^{n}|\langle T_i x, x\rangle|^p\right)^{1/p} \\ &\geq \sum_{i=1}^{n}\left|\left\langle\alpha_i^{1-1/p}T_i x, x\right\rangle\right| \\ &\geq \left|\left\langle\alpha_1^{1-1/p}T_1 x, x\right\rangle \pm \left\langle\alpha_2^{1-1/p}T_2 x, x\right\rangle \pm \dots \pm \left\langle\alpha_n^{1-1/p}T_n x, x\right\rangle\right| \\ &= \left|\left\langle\left(\alpha_1^{1-1/p}T_1 \pm \alpha_2^{1-1/p}T_2 \pm \dots \pm \alpha_n^{1-1/p}T_n\right)x, x\right\rangle\right|.\end{aligned}$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$.

Now we give another upper bound for the powers of w_p. This result has several inequalities as special cases, which considerably generalize the second inequality of (1.1).

THEOREM 3.2. *Let* $(T_1, \dots, T_n), (A_1, \dots, A_n), (B_1, \dots, B_n) \in \mathbb{B}(\mathscr{H})^{(n)}$ *and let* f *and* g *be non-negative continuous functions on* $[0, \infty)$ *satisfying* $f(t)g(t) = t$ *for all* $t \in [0, \infty)$. *Then*

$$\begin{aligned}w_p^{rp}(A_1^*T_1B_1, \dots, A_n^*T_nB_n) \\ &\leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}[B_i^*f^2(|T_i|)B_i]^{rp} + [A_i^*g^2(|T_i^*|)A_i]^{rp}\right\|,\end{aligned}$$

for $p \geq 1$ *and* $r \geq 1$.

PROOF. Let $x \in \mathscr{H}$ be a unit vector. Then

$$\begin{aligned}\sum_{i=1}^{n}|\langle A_i^*T_iB_ix, x\rangle|^p \\ &= \sum_{i=1}^{n}|\langle T_iB_ix, A_ix\rangle|^p \\ &\leq \sum_{i=1}^{n}\|f(|T_i|)B_ix\|^p\|g(|T_i^*|)A_ix\|^p \qquad \text{(by Lemma 2.1(c))} \\ &= \sum_{i=1}^{n}\left\langle f(|T_i|)B_ix, f(|T_i|)B_ix\right\rangle^{p/2}\left\langle g(|T_i^*|)A_ix, g(|T_i^*|)A_ix\right\rangle^{p/2}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \langle B_i^* f^2(|T_i|)B_i x, x\rangle^{p/2} \langle A_i^* g^2(|T_i^*|)A_i x, x\rangle^{p/2} \\
&\le \sum_{i=1}^{n} \big\langle (B_i^* f^2(|T_i|)B_i)^p x, x\big\rangle^{1/2} \big\langle (A_i^* g^2(|T_i^*|)A_i)^p x, x\big\rangle^{1/2} \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.2(a))} \\
&\le \sum_{i=1}^{n} \Big(\tfrac{1}{2} \Big(\big\langle (B_i^* f^2(|T_i|)B_i)^p x, x\big\rangle^r + \big\langle (A_i^* g^2(|T_i^*|)A_i)^p x, x\big\rangle^r \Big)\Big)^{1/r} \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.1(a))} \\
&\le \sum_{i=1}^{n} \Big(\tfrac{1}{2} \big\langle \big[(B_i^* f^2(|T_i|)B_i)^{rp} + (A_i^* g^2(|T_i^*|)A_i)^{rp} \big] x, x\big\rangle \Big)^{1/r} \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.2(a))} \\
&\le n^{1-1/r} \left(\frac{1}{2} \left\langle \left(\sum_{i=1}^{n} (B_i^* f^2(|T_i|)B_i)^{rp} + (A_i^* g^2(|T_i^*|)A_i)^{rp} \right) x, x \right\rangle \right)^{1/r} \\
&\qquad\qquad \text{(by the concavity of the function } f(t) = t^{1/r}\text{).}
\end{aligned}
$$

Thus

$$
\left(\sum_{i=1}^{n} |\langle A_i^* T_i B_i x, x\rangle|^p \right)^r \le \frac{n^{r-1}}{2} \left\langle \left(\sum_{i=1}^{n} (B_i^* f^2(|T_i|)B_i)^{rp} + (A_i^* g^2(|T_i^*|)A_i)^{rp} \right) x, x \right\rangle.
$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$.

Choosing $A = B = I$, we get:

COROLLARY 3.3. *Let $(T_1, \ldots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$ and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$
w_p^{rp}(T_1, \ldots, T_n) \le \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} f^{2rp}(|T_i|) + g^{2rp}(|T_i^*|) \right\|,
$$

for $p \ge 1$ and $r \ge 1$.

Letting $f(t) = g(t) = t^{1/2}$, we get:

COROLLARY 3.4. *Let $(T_1, \ldots, T_n)$, $(A_1, \ldots, A_n)$, $(B_1, \ldots, B_n)$ be in $\mathbb{B}(\mathscr{H})^{(n)}$. Then*

$$
w_p^{rp}(A_1^* T_1 B_1, \ldots, A_n^* T_n B_n) \le \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} (B_i^* |T_i| B_i)^{rp} + (A_i^* |T_i^*| A_i)^{rp} \right\|,
$$

for $p \geq 1$ *and* $r \geq 1$.

COROLLARY 3.5. *Let* $(A_1, \dots, A_n), (B_1, \dots, B_n) \in \mathbb{B}(\mathscr{H})^{(n)}$. *Then*

$$w_p^{rp}(A_1^* B_1, \dots, A_n^* B_n) \leq \frac{n^{r-1}}{2} \left\| \sum_{i=1}^{n} |B_i|^{2rp} + |A_i|^{2rp} \right\|,$$

for $p \geq 1$ *and* $r \geq 1$.

COROLLARY 3.6. *Let* $(T_1, \dots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$. *Then*

$$w_p^p(T_1, \dots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} |T_i|^{2\alpha p} + |T_i^*|^{2(1-\alpha)p} \right\|,$$

for $0 \leq \alpha \leq 1$ *and* $p \geq 1$. *In particular,*

$$w_p^p(T_1, \dots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} |T_i|^p + |T_i^*|^p \right\|.$$

COROLLARY 3.7. *Let* $B, C \in \mathbb{B}(\mathscr{H})$. *Then*

$$w_p^p(B, C) \leq \tfrac{1}{2} \left\| |B|^{2\alpha p} + |B^*|^{2(1-\alpha)p} + |C|^{2\alpha p} + |C^*|^{2(1-\alpha)p} \right\|$$

for $0 \leq \alpha \leq 1$ *and* $p \geq 1$. *In particular,*

$$w_p^p(B, C) \leq \tfrac{1}{2} \left\| |B|^p + |B^*|^p + |C|^p + |C^*|^p \right\|.$$

The next results are related to some different upper bounds for w_p for n-tuples of operators, which give several inequalities as special cases.

PROPOSITION 3.8. *Let* $(T_1, \dots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$. *Then*

$$w_p(T_1, \dots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)})^p \right\|^{1/p},$$

for $0 \leq \alpha \leq 1$ *and* $p \geq 1$.

PROOF. By using the arithmetic-geometric mean, for any unit vector $x \in \mathscr{H}$

we have

$$\begin{aligned}
\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p
&\le \sum_{i=1}^{n} \big(\langle |T_i|^{2\alpha} x, x\rangle^{1/2} \langle |T_i^*|^{2(1-\alpha)} x, x\rangle^{1/2}\big)^p && \text{(by Lemma 2.1(b))}\\
&\le \frac{1}{2^p} \sum_{i=1}^{n} \big(\langle |T_i|^{2\alpha} x, x\rangle + \langle |T_i^*|^{2(1-\alpha)} x, x\rangle\big)^p\\
&= \frac{1}{2^p} \sum_{i=1}^{n} \langle (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)}) x, x\rangle^p\\
&\le \frac{1}{2^p} \sum_{i=1}^{n} \langle (|T_i|^{2\alpha} + |T_i^*|^{2(1-\alpha)})^p x, x\rangle && \text{(by Lemma 2.2(a)).}
\end{aligned}$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$.

PROPOSITION 3.9. *Let* $(T_1, \dots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$. *Then*

$$w_p(T_1, \dots, T_n) \le \left\| \sum_{i=1}^{n} \alpha |T_i|^p + (1-\alpha)|T_i^*|^p \right\|^{1/p},$$

for $0 \le \alpha \le 1$ *and* $p \ge 2$.

PROOF. For every unit vector $x \in \mathscr{H}$, we have

$$\begin{aligned}
\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p
&= \sum_{i=1}^{n} \big(|\langle T_i x, x\rangle|^2\big)^{p/2}\\
&\le \sum_{i=1}^{n} \big(\langle |T_i|^{2\alpha} x, x\rangle \langle |T_i^*|^{2(1-\alpha)} x, x\rangle\big)^{p/2} && \text{(by Lemma 2.1(b))}\\
&\le \sum_{i=1}^{n} \langle |T_i|^{\alpha p} x, x\rangle \langle |T_i^*|^{(1-\alpha)p} x, x\rangle && \text{(by Lemma 2.2(a))}\\
&\le \sum_{i=1}^{n} \langle |T_i|^{p} x, x\rangle^{\alpha} \langle |T_i^*|^{p} x, x\rangle^{(1-\alpha)} && \text{(by Lemma 2.2(b))}\\
&\le \sum_{i=1}^{n} \big(\alpha \langle |T_i|^{p} x, x\rangle + (1-\alpha) \langle |T_i^*|^{p} x, x\rangle\big) && \text{(by Lemma 2.1(a))}
\end{aligned}$$

$$\leq \sum_{i=1}^{n} \big\langle \big(\alpha |T_i|^p + (1-\alpha)|T_i^*|^p\big)x, x\big\rangle$$
$$= \left\langle \left(\sum_{i=1}^{n} (\alpha |T_i|^p + (1-\alpha)|T_i^*|^p) \right) x, x \right\rangle.$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$.

REMARK 3.10. As special cases,
(1) For $\alpha = 1/2$, we have

$$w_p^p(T_1, \ldots, T_n) \leq \frac{1}{2} \left\| \sum_{i=1}^{n} |T_i|^p + |T_i^*|^p \right\|.$$

(2) For $B, C \in \mathbb{B}(\mathscr{H})$, $0 \leq \alpha \leq 1$ and $p \geq 1$ we have

$$w_p^p(B, C) \leq \big\| \alpha |B|^p + (1-\alpha)|B^*|^p + \alpha |C|^p + (1-\alpha)|C^*|^p \big\|.$$

In particular,

$$w_p^p(B, C) \leq \tfrac{1}{2} \big\| |B|^p + |B^*|^p + |C|^p + |C^*|^p \big\|.$$

The next result reads as follows.

PROPOSITION 3.11. *Let* $(T_1, \ldots, T_n) \in \mathbb{B}(\mathscr{H})^{(n)}$, $0 \leq \alpha \leq 1$, $r \geq 1$ *and* $p \geq 1$. *Then*

$$w_p(T_1, \ldots, T_n) \leq \left(\sum_{i=1}^{n} \big\| \alpha |T_i|^{2r} + (1-\alpha)|T_i^*|^{2r} \big\|^{p/(2r)} \right)^{1/p}.$$

PROOF. Let $x \in \mathscr{H}$ be a unit vector. Then

$$\begin{aligned}
&\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p \\
&\quad = \sum_{i=1}^{n} \big(|\langle T_i x, x\rangle|^2 \big)^{p/2} \\
&\quad \leq \sum_{i=1}^{n} \big(\langle |T_i|^{2\alpha} x, x\rangle \langle |T_i^*|^{2(1-\alpha)} x, x\rangle \big)^{p/2} && \text{(by Lemma 2.1(b))} \\
&\quad \leq \sum_{i=1}^{n} \big(\langle |T_i|^2 x, x\rangle^{\alpha} \langle |T_i^*|^2 x, x\rangle^{(1-\alpha)} \big)^{p/2} && \text{(by Lemma 2.2(b))}
\end{aligned}$$

$$\leq \sum_{i=1}^{n} \big(\alpha \langle |T_i|^2 x, x\rangle^r + (1-\alpha)\langle |T_i^*|^2 x, x\rangle^r\big)^{p/(2r)} \quad \text{(by Lemma 2.1(a))}$$

$$\leq \sum_{i=1}^{n} \big(\alpha \langle |T_i|^{2r} x, x\rangle + (1-\alpha)\langle |T_i^*|^{2r} x, x\rangle\big)^{p/(2r)} \quad \text{(by Lemma 2.2(a))}$$

$$\leq \sum_{i=1}^{n} \big\langle (\alpha |T_i|^{2r} + (1-\alpha)|T_i^*|^{2r})x, x\big\rangle^{p/(2r)}.$$

Now the result follows by taking the supremum over all unit vectors in $\mathscr{H}$.

REMARK 3.12. Some special cases can be stated as follows:
(1) For $\alpha = 1/2$, we have

$$w_p(T_1, \dots, T_n) \leq \left(\frac{1}{2^{p/(2r)}} \sum_{i=1}^{n} \big\| |T_i|^{2r} + |T_i^*|^{2r} \big\|^{p/(2r)} \right)^{1/p}.$$

(2) For $B, C \in \mathbb{B}(\mathscr{H})$, $0 \leq \alpha \leq 1$ and $p \geq 1$ we have

$$w_p(B, C) \leq \Big(\big\| \alpha |B|^{2r} + (1-\alpha)|B^*|^{2r} \big\|^{p/(2r)} + \big\| \alpha |C|^{2r} + (1-\alpha)|C^*|^{2r} \big\|^{p/(2r)} \Big)^{1/p}.$$

In particular,

$$w_p(B, C) \leq \frac{1}{2^{1/(2r)}} \Big(\big\| |B|^{2r} + |B^*|^{2r} \big\|^{p/(2r)} + \big\| |C|^{2r} + |C^*|^{2r} \big\|^{p/(2r)} \Big)^{1/p}.$$

ACKNOWLEDGEMENTS. The third author is supported by the Science College Research Center at Qassim University, project number 2638.

The corresponding author (M. S. Moslehian) would like to thank the Tusi Math. Research Group (TMRG).

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