MATH. SCAND. 120 (2017), 145–160

MILD SINGULAR POTENTIALS AS EFFECTIVE LAPLACIANS IN NARROW STRIPS

CÉSAR R. DE OLIVEIRA and ALESSANDRA A. VERRI

Abstract

We propose to obtain information on one-dimensional Schrödinger operators on bounded intervals by approaching them as effective operators of the Laplacian in thin planar strips. Here we develop this idea to get spectral knowledge of some mild singular potentials with Dirichlet boundary conditions. Special preparations, including a result on perturbations of quadratic forms, are included as well.

1. Introduction

In this paper we combine two questions. The first one is about the spectrum of the Laplacian operator in a narrow neighborhood of a plane curve. In particular, we are interested in the spectral properties of the Dirichlet Laplacian operator when the width of this strip tends to zero. There are several reasons why this study is attractive [11], [12]. In this work we consider ad hoc curved strips from rather different constructions. The other question is about the spectral characteristics of one-dimensional Schrödinger operators with some mild singularities. Specifically, we are interested in the class of potentials $V(s) = Cs^{-2m}$, $0 < m \leq 1/2$; see (3) below, whose interpretation is of a quantum particle in a box (i.e., the interval $(0, 1)$) under such potentials. Although this kind of Sturm-Liouville problem is quite traditional [19], [13], [21], we believe that our approach adds some interesting points. We mention motivating papers dealing with singular potentials [4], [8], [20], and also the well-known one-dimensional Coulomb problem [16], [7], [17] (see also [10]), even though the latter is not on a compact interval and presents different characteristics.

At first, these two questions may seem unconnected, but we relate them by reversing the usual order of obtaining information on the Laplacian in thin planar regions from one-dimensional effective operators; here we try to get spectral information for particular one-dimensional operators from the Dirichlet Laplacian in particular thin regions. In this process we needed a new perturbation of forms that is presented in Theorem 1.3.

Received 24 November 2014.
DOI: https://doi.org/10.7146/math.scand.a-25510

We start talking about a particular situation which motivated our strategy. Consider a plane curve $r\colon (0,1) \to \mathbb{R}^2$ of class C^2 parameterized by its arc length s. Denote by $N(s)$ the normal vector of r at the point $r(s)$. Fix $d > 0$. For each $\varepsilon > 0$, we can build a curved strip

$$\Lambda_\varepsilon := \{\vec{\mathbf{x}} \in \mathbb{R}^2 : \vec{\mathbf{x}} = r(s) + \varepsilon y N(s),\ y \in (0,d)\},$$

of width $\varepsilon d > 0$. The constant $d > 0$ is taken so that the strip doesn't have self intersection. Denote by $k(s)$ the curvature of r at the point $r(s)$ and suppose that $k > 0$ and $k \in L^\infty(0,1)$. Let $-\Delta_{\Lambda_\varepsilon}$ be the Dirichlet Laplacian in Λ_ε. After a standard renormalization (see Section 2), it is possible to show that

$$-\Delta_{\Lambda_\varepsilon} - \frac{\pi^2}{d^2\varepsilon^2}\mathbf{1} \longrightarrow -\frac{d^2}{ds^2} - \frac{k^2(s)}{4}, \quad \varepsilon \to 0, \tag{1}$$

in the norm resolvent sense. Note that the limit operator on the right-hand side of (1) presents a potential which "inherits" geometric characteristics of Λ_ε. For the norm convergence in (1) we mention [2], [5], [12] (and [6] when the width is not constant, resulting in different effective potentials and a useful technique).

In this paper, we formulate a rather different view of the problem. We build strips of width εd and the reference curve r_ε also depends on the small parameter $\varepsilon > 0$, as described in the sequel. Let $0 < m \le 1$ and take a positive number a so that $am < 1$. For each $\varepsilon > 0$, write

$$k_\varepsilon(s) := \frac{\varepsilon^{am(1-m)}}{C_1 s^m + C_2}, \quad C_1, C_2 > 0,\ \ s \in (0,\infty). \tag{2}$$

It is known that there exists a differentiable planar curve $r_\varepsilon(s)$, parameterized by its arc length s, whose curvature is given by (2). If $N_\varepsilon(s)$ denotes the normal vector to r_ε at the point $r_\varepsilon(s)$, consider the sequence of regions

$$\Omega_\varepsilon := \{\vec{\mathbf{x}} \in \mathbb{R}^2 : \vec{\mathbf{x}} = r_\varepsilon(s/\varepsilon^a) + \varepsilon^a y N_\varepsilon(s/\varepsilon^a),\ s \in (0,1),\ y \in (0,d)\}.$$

Note that in the definition of Ω_ε we have $s \in (0,1)$ and the scale s/ε^a. We are interested in the limit, as $\varepsilon \to 0$, of the Laplacian $-\Delta_{\Omega_\varepsilon}$ restricted to Ω_ε with Dirichlet condition at the boundary $\partial\Omega_\varepsilon$.

Now, consider the one-dimensional self-adjoint operator

$$T_m = -\frac{d}{ds^2} - \frac{1}{4C_1^2 s^{2m}}, \quad \operatorname{dom} T_m = \mathscr{H}_0^1(0,1) \cap \mathscr{H}^2(0,1), \tag{3}$$

and the closed subspace

$$\mathscr{L} := \left\{ w(s)(\sqrt{2/d}\,)\sin(\pi y/d) : w \in L^2(0,1) \right\}$$

of $\mathrm{L}^2((0,1)\times(0,d))$ (here $(\sqrt{2/d}\,)\sin(\pi y/d)$ is the eigenfunction associated with the first eigenvalue π^2/d^2 of the Dirichlet Laplacian in $(0,d)$; for details of this particularity see Section 3). The following is one of the main results of this work.

THEOREM 1.1. *Let T_m be as in* (3), $0 < m \le 1/2$ *and a be a positive number so that $am < 1$. Put $\delta := am^2 > 0$. Then, for all $\varepsilon > 0$ small enough, there exist $K > 0$, independent of ε, and a unitary operator U_ε such that*

$$\left\| \varepsilon^{2am} U_\varepsilon \left(-\Delta_{\Omega_\varepsilon} - \frac{\pi^2}{d\,\varepsilon^2}\mathbf{1} \right)^{-1} U_\varepsilon^{-1} - T_m^{-1} \oplus 0 \right\| \le K\varepsilon^\delta,$$

where 0 *is the null operator on the subspace* $\mathscr{L}^\perp$.

This result has an important consequence:

THEOREM 1.2. *Let T_m be as in* (3). *If $0 < m \le 1/2$ and $C_1 > 1$, then T_m has purely discrete spectrum. Furthermore, if one denotes by $\lambda_j(\varepsilon)$ and λ_j the eigenvalues of $-\Delta_{\Omega_\varepsilon}$ and T_m, respectively, then, for each $j \in \mathbb{N}$,*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{2am}} \left(\lambda_j(\varepsilon) - \frac{\pi^2}{\varepsilon^2 d^2} \right) = \lambda_j.$$

Theorem 1.1 will be a consequence of a set of results, that is, Proposition 3.1, Corollary 3.2 and Proposition 3.3 of Section 3, and Theorem 1.3 below. Now we would like to anticipate an inequality in Corollary 3.2 which basically states that, for $\varepsilon > 0$ small enough,

$$\left\| \varepsilon^{2am} U_\varepsilon \left(-\Delta_{\Omega_\varepsilon} - \frac{\pi^2}{d\,\varepsilon^2}\mathbf{1} \right)^{-1} U_\varepsilon^{-1} - T_{m,\varepsilon}^{-1} \oplus 0 \right\| \le K\,\varepsilon^\delta, \tag{4}$$

where $T_{m,\varepsilon}$ is the one-dimensional self-adjoint operator given by

$$T_{m,\varepsilon} := -\frac{d^2}{ds^2} - \frac{1}{4(C_1^2 s^m + C_2\varepsilon^{am^2})^2}, \quad \operatorname{dom} T_{m,\varepsilon} = \mathscr{H}_0^1(0,1) \cap \mathscr{H}^2(0,1). \tag{5}$$

By taking into account (4) and (5), our problem is reduced to the study of the sequence of operators (5) as $\varepsilon \to 0$. This is one of the main steps of the proof of Theorem 1.1. For this, we make use of the following theorem which is also one of our main contributions in this work and may be of independent interest.

THEOREM 1.3. *Let $H_0 \ge 0$ be a self-adjoint operator in a Hilbert space $\mathscr{H}$. Let V be such that*

$$H := H_0 + V, \quad \operatorname{dom} H = \operatorname{dom} H_0,$$

is self-adjoint and $H \geq \beta$*, for some* $\beta \in \mathbb{R}$*.*

Let $\{V_\varepsilon\}_{\varepsilon>0}$ *be a family of operators so that, for each* $\varepsilon > 0$*,*

$$H_\varepsilon := H_0 + V_\varepsilon, \quad \operatorname{dom} H_\varepsilon = \operatorname{dom} H_0,$$

is self-adjoint and $H_\varepsilon \geq \beta$*. Denote by* $b_0(\psi)$ *the quadratic form associated with* H_0*. Suppose that there are* $0 < \alpha < 1$ *and* $0 < \nu, \nu_\varepsilon, \alpha_\varepsilon$*, so that, for all* $\psi \in \operatorname{dom} b_0$*,*

$$|\langle V\psi, \psi\rangle| \leq \alpha b_0(\psi) + \nu\|\psi\|^2, \quad |\langle (V_\varepsilon - V)\psi, \psi\rangle| \leq \alpha_\varepsilon b_0(\psi) + \nu_\varepsilon\|\psi\|^2,$$

with $\lim_{\varepsilon\to 0}\alpha_\varepsilon = \lim_{\varepsilon\to 0}\nu_\varepsilon = 0$*.*

Then, there exists $K > 0$ *such that, for* $\varepsilon > 0$ *small enough,*

$$\|H_\varepsilon^{-1} - H^{-1}\| \leq K(\alpha_\varepsilon + \nu_\varepsilon).$$

One of the main tools in the proof of this theorem is Theorem 3 in [1], which relates approximation of quadratic forms with norm resolvent convergence of the associated operators.

Theorem 1.3 includes the following situation. Let $H_0 := -d^2/ds^2$, $\operatorname{dom} H_0 = \mathscr{H}^2(0,1) \cap \mathscr{H}_0^1(0,1)$. Consider the class of potentials

$$V(s) := \frac{\gamma}{C_1^2 s^{2m}}, \qquad V_\varepsilon(s) := \frac{\gamma}{(C_1 s^m + C_2\varepsilon^b)^2}, \quad \varepsilon > 0,$$

with $\gamma \in \mathbb{R}$, $b > 0$, $C_1, C_2 > 0$, $0 < m \leq 1/2$ and $4|\gamma|/C_1^2 < 1$, and the self-adjoint operators

$$H := H_0 + V(s), \quad H_\varepsilon := H_0 + V_\varepsilon(s), \quad \operatorname{dom} H = \operatorname{dom} H_\varepsilon = \operatorname{dom} H_0.$$

By Theorem 1.3, for each fixed m, the sequence H_ε converges to H in the norm resolvent sense, as $\varepsilon \to 0$ (for details see Appendix B). In particular, this result includes the sequence in (5).

REMARK 1.4. The "dimensional reduction" in the proofs holds for $m \leq 1$, but to apply Theorem 1.3 (for concluding our main results) we need the restriction $m \leq 1/2$. It would be interesting to extend Theorem 1.3 in order to include larger values of m, but at this point another new idea seems necessary.

REMARK 1.5. For $m < 1/2$ the end points $\{0, 1\}$ are both regular for the Sturm-Liouville operator T, so its spectrum is known to be discrete in this case [20]. Here we have a different proof. The case $m = 1/2$ is not regular at zero and we are not aware of related spectral studies.

This work is organized in the following way. In Section 2 we detail the construction of the regions Ω_ε, and discuss the necessary renormalization related to Dirichlet Laplacian restricted to Ω_ε. In Section 3 we prove Theorems 1.1 and 1.2. Section 4 is dedicated to proofs of auxiliary results stated in Section 3. The proof of Theorem 1.3 is presented in Appendix B. In the proofs, different constants will be indicated by the same symbol K and we simplify $T_m = T$ for all valid m.

2. Configuration space and quadratic forms

For each $0 < \varepsilon \leq 1$, consider the function $k_\varepsilon \colon (0, +\infty) \to \mathbb{R}$ given by (2), with

$$C_1 > 1, \quad C_2 > 0, \quad 0 < m \leq 1, \quad a > 0 \quad \text{with } am < 1. \tag{6}$$

There exists a differentiable curve $r_\varepsilon \colon (0, +\infty) \to \mathbb{R}^2$, parametrized by its arc length s, $r_\varepsilon(s) = (r_{\varepsilon 1}(s), r_{\varepsilon 2}(s))$, fully determined (except for its position and orientation in the plane) by the curvature function k_ε [18].

We denote by $T_\varepsilon(s) := (r'_{\varepsilon 1}(s), r'_{\varepsilon 2}(s))$ the unit tangent vector to r_ε at the point $r_\varepsilon(s)$. The function $N_\varepsilon(s) := (-r'_{\varepsilon 2}(s), r'_{\varepsilon 1}(s))$ defines a unit normal vector field to r_ε and the pair $(T_\varepsilon, N_\varepsilon)$ gives a Frenet frame. Note the Frenet-Serret formulas

$$T'_\varepsilon(s) = k_\varepsilon(s) N_\varepsilon(s), \quad N'_\varepsilon(s) = -k_\varepsilon(s) T_\varepsilon(s),$$

with $|k_\varepsilon(s)| = \|T'_\varepsilon(s)\|$, for all $s \in (0, +\infty)$.

Let $d > 0$, $I = (0, d)$ and $\Omega := (0, 1) \times I$ be a straight strip of width $d > 0$. For each $0 < \varepsilon \leq 1$, consider a bounded curved strip Ω_ε, based on the reference curve r_ε, via the map $\mathscr{F}_\varepsilon$, where $\Omega_\varepsilon := \mathscr{F}_\varepsilon(\Omega)$ and

$$\mathscr{F}_\varepsilon \colon \Omega \to \Omega_\varepsilon, \quad \mathscr{F}_\varepsilon \colon (s, y) \mapsto r_\varepsilon(s/\varepsilon^{am}) + \varepsilon y N_\varepsilon(s/\varepsilon^{am}). \tag{7}$$

We take $d > 0$ small enough so that the strip Ω_1 (and consequently Ω_ε, for all $0 < \varepsilon < 1$) is not self-intersecting.

We study the Laplacian $-\Delta_{\Omega_\varepsilon}$ in Ω_ε, and with Dirichlet condition at the boundary $\partial\Omega_\varepsilon$. We initially consider the corresponding family of quadratic forms

$$b_\varepsilon(\psi) = \int_{\Omega_\varepsilon} |\nabla\psi|^2 \, ds \, dy, \quad \operatorname{dom} b_\varepsilon = \mathscr{H}_0^1(\Omega_\varepsilon), \tag{8}$$

and we are interested in the limit of the sequence $b_\varepsilon(\psi)$ as $\varepsilon \to 0$. Recall that π^2/d^2 is the lowest eigenvalue of the negative Laplacian with Dirichlet boundary conditions in $(0, d)$, and $1/(\sqrt{2d})\ \sin(\pi y/d) \geq 0$ is the corresponding eigenfunction of this restricted problem. This function is directly related to transverse oscillations in Ω_ε, i.e., when d is replaced by εd.

The standard renormalization is to remove the diverging energy $\pi^2/(d\varepsilon)^2$ from the quadratic forms (8). Also, for technical reasons, we fix $A > 0$ so that $4AC_2^2 > 1$ and add the constant potential $A\varepsilon^{2am(1-m)}$ to such quadratic forms, which ensures that b_ε becomes positive. Thus, we pass to study the sequence

$$\tilde{b}_\varepsilon(\psi) = \int_{\Omega_\varepsilon} \left(|\nabla\psi|^2 - \frac{\pi^2}{d^2\varepsilon^2}|\psi|^2 + A\varepsilon^{2am(1-m)}|\psi|^2 \right) \mathrm{d}s\, \mathrm{d}y,$$

$\operatorname{dom} \tilde{b}_\varepsilon = \mathscr{H}_0^1(\Omega_\varepsilon)$.

As usual, we perform a change of variables so that the integration region in the definition of $\tilde{b}_\varepsilon$, and consequently the form domains, become independent of $0 < \varepsilon \leq 1$. We make this change by using (7) and pass to work in the fixed region Ω, for all $0 < \varepsilon \leq 1$, and get a non-trivial Riemannian metric $G = G_\varepsilon$ which is induced by $\mathscr{F}_\varepsilon$, i.e.,

$$G = (G_{ij}), \quad G_{ij} = \langle e_i, e_j \rangle = G_{ji}, \qquad 1 \leq i, j \leq 2,$$

with $e_1 = \partial\mathscr{F}_\varepsilon/\partial s$ and $e_2 = \partial\mathscr{F}_\varepsilon/\partial y$.

Some calculations show that in the Frenet frame

$$J := \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1^\varepsilon(s, y) & \Gamma_2^\varepsilon(s, y) \\ -\varepsilon r'_{\varepsilon 2}(s/\varepsilon^{am}) & \varepsilon r'_{\varepsilon 1}(s/\varepsilon^{am}) \end{pmatrix},$$

with

$$\Gamma_1^\varepsilon(s, y) := (1/\varepsilon^{am}) r'_{\varepsilon 1}(s/\varepsilon^{am}) - \varepsilon^{1-am} y r''_{\varepsilon 2}(s/\varepsilon^{am}),$$
$$\Gamma_2^\varepsilon(s, y) := (1/\varepsilon^{am}) r'_{\varepsilon 2}(s/\varepsilon^{am}) + \varepsilon^{1-am} y r''_{\varepsilon 1}(s/\varepsilon^{am}).$$

The inverse matrix of J is given by

$$J^{-1} = (\varepsilon^{1-am}\beta_\varepsilon)^{-1} \begin{pmatrix} \varepsilon r'_{\varepsilon 1}(s/\varepsilon^{am}) & -\Gamma_2^\varepsilon(s, y) \\ \varepsilon r'_{\varepsilon 2}(s/\varepsilon^{am}) & \Gamma_1^\varepsilon(s, y) \end{pmatrix},$$

where

$$\beta_\varepsilon(s, y) := 1 - \varepsilon y k_\varepsilon(s/\varepsilon^{am}).$$

Note that $JJ^t = G$ and $\det J = |\det G|^{1/2} = \varepsilon^{1-am}\beta_\varepsilon$. Since k is a bounded function, for ε small enough β_ε does not vanish in Ω. Thus, $\beta_\varepsilon > 0$ and $\mathscr{F}_\varepsilon$ is a local diffeomorphism. By requiring that $\mathscr{F}_\varepsilon$ is injective, we get a global diffeomorphism.

Introducing the notation

$$\|\psi\|_G^2 := \int_\Omega |\psi(s,y)|^2 \beta_\varepsilon(s,y)\,\mathrm{d}s\,\mathrm{d}y,$$

and the unitary transformation

$$\begin{aligned} \mathscr{V}_\varepsilon \colon \mathrm{L}^2(\Omega_\varepsilon) &\to \mathrm{L}^2(\Omega, \beta_\varepsilon(s,y)\,\mathrm{d}s\,\mathrm{d}y) \\ \psi &\mapsto \varepsilon^{(1-am)/2}\psi \circ \mathscr{F}_\varepsilon, \end{aligned}$$

we obtain the sequence of quadratic forms

$$c_\varepsilon(\mathscr{V}_\varepsilon\psi) := \left\| J^{-1}\nabla(\mathscr{V}_\varepsilon\psi)\right\|_G^2 - \frac{\pi^2}{d^2}\frac{1}{\varepsilon^2}\left\|\mathscr{V}_\varepsilon\psi\right\|_G^2 + A\varepsilon^{2am(1-m)}\left\|\mathscr{V}_\varepsilon\psi\right\|_G^2,$$

$\operatorname{dom} c_\varepsilon = \mathscr{H}_0^1(\Omega)$.

However, we still denote $\mathscr{V}_\varepsilon\psi$ by ψ. Thus,

$$c_\varepsilon(\psi) = \left\| J^{-1}\nabla\psi\right\|_G^2 - \frac{\pi^2}{d^2}\frac{1}{\varepsilon^2}\left\|\psi\right\|_G^2 + A\varepsilon^{2am(1-m)}\left\|\psi\right\|_G^2, \quad \operatorname{dom} c_\varepsilon = \mathscr{H}_0^1(\Omega).$$

In details, we obtain

$$c_\varepsilon(\psi) = \int_\Omega \left[\frac{\varepsilon^{2am}}{\beta_\varepsilon}|\psi'|^2 + \frac{\beta_\varepsilon}{\varepsilon^2}\left(|\psi_y|^2 - \frac{\pi^2}{d^2}|\psi|^2\right) + A\varepsilon^{2am(1-m)}\beta_\varepsilon|\psi|^2\right]\mathrm{d}s\,\mathrm{d}y, \tag{9}$$

$\operatorname{dom} c_\varepsilon = \mathscr{H}_0^1(\Omega)$ as a subspace of $\mathrm{L}^2(\Omega, \beta_\varepsilon\,\mathrm{d}s\,\mathrm{d}y)$. The measure $\beta_\varepsilon\,\mathrm{d}s\,\mathrm{d}y$ comes from the above Riemannian metric. Thus, we introduce a new change of variables

$$\begin{aligned} W_\varepsilon \colon \mathrm{L}^2(\Omega) &\to \mathrm{L}^2(\Omega, \beta_\varepsilon(s,y)\,\mathrm{d}s\,\mathrm{d}y) \\ \psi &\mapsto \psi/\beta_\varepsilon^{1/2}, \end{aligned}$$

so that the quadratic form (9) becomes

$$\begin{aligned} d_\varepsilon(\psi) = &\int_\Omega \frac{\varepsilon^{2am}}{\beta_\varepsilon^2}\left|\psi' - \frac{\psi}{2}\frac{\beta_\varepsilon'}{\beta_\varepsilon}\right|^2 \mathrm{d}s\,\mathrm{d}y + \int_\Omega \frac{1}{\varepsilon^2}\left(|\psi_y|^2 - \frac{\pi^2}{d^2}|\psi|^2\right)\mathrm{d}s\,\mathrm{d}y \\ &- \int_\Omega \frac{1}{4\beta_\varepsilon^2}\frac{\varepsilon^{2am}}{(C_1 s^m + C_2\varepsilon^{am^2})^2}|\psi|^2 \mathrm{d}s\,\mathrm{d}y + \int_\Omega A\varepsilon^{2am(1-m)}|\psi|^2 \mathrm{d}s\,\mathrm{d}y. \end{aligned}$$

Now, $\operatorname{dom} d_\varepsilon = \mathscr{H}_0^1(\Omega)$ is a subspace of the Hilbert space $\mathrm{L}^2(\Omega)$.

Ahead it will be convenient to consider the quadratic form

$$\begin{aligned} \tilde{d}_\varepsilon(\psi) := &\int_\Omega \varepsilon^{2am}|\psi'|^2 + \frac{1}{\varepsilon^2}\left(|\psi_y|^2 - \frac{\pi^2}{d^2}|\psi|^2\right)\mathrm{d}s\,\mathrm{d}y \\ &- \int_\Omega \frac{1}{4}\frac{\varepsilon^{2am}}{(C_1 s^m + C_2\varepsilon^{am^2})^2}|\psi|^2 \mathrm{d}s\,\mathrm{d}y + \int_\Omega A\varepsilon^{2am(1-m)}|\psi|^2 \mathrm{d}s\,\mathrm{d}y, \end{aligned}$$

$\operatorname{dom} \tilde{d}_\varepsilon = \mathscr{H}_0^1(\Omega)$. Denote by D_ε and $\tilde{D}_\varepsilon$ the self-adjoint operators associated with $d_\varepsilon(\psi)$ and $\tilde{d}_\varepsilon(\psi)$, respectively. The proof of Theorem 2.1 appears in Appendix A.

THEOREM 2.1. *In addition to conditions* (6)*, take a number* $a > 0$ *so that* $\delta = 1 + am(1 - 3m) > 0$*. There exists* $K > 0$ *such that, for* $\varepsilon > 0$ *small enough,*

$$\|D_\varepsilon^{-1} - \tilde{D}_\varepsilon^{-1}\| \le K\varepsilon^\delta.$$

Finally, we pass to study the sequence

$$\begin{aligned}\ell_\varepsilon(\psi) &:= 1/\varepsilon^{2am}\tilde{d}_\varepsilon(\psi)\\ &= \int_\Omega |\psi'|^2 + \frac{1}{\varepsilon^{2+2am}}\left(|\psi_y|^2 - \frac{\pi^2}{d^2}|\psi|^2\right)\mathrm{d}s\,\mathrm{d}y\\ &\quad - \int_\Omega \frac{1}{4}\frac{1}{(C_1 s^m + C_2\varepsilon^{am^2})^2}|\psi|^2\mathrm{d}s\,\mathrm{d}y + \int_\Omega \frac{A}{\varepsilon^{2am^2}}|\psi|^2\mathrm{d}s\,\mathrm{d}y,\end{aligned}$$

with $\operatorname{dom} \ell_\varepsilon = \mathscr{H}_0^1(\Omega)$. Note that the condition $4AC_2^2 > 1$ implies that $\ell_\varepsilon(\psi) > 0$, for all $\psi \in \operatorname{dom} \ell_\varepsilon$. Denote by L_ε the positive self-adjoint operator associated with $\ell_\varepsilon(\psi)$.

3. Proofs of main results

In this section we prove Theorems 1.1 and 1.2 stated in the introduction. We use some auxiliary results whose proofs will be postponed to Section 4.

Define $u_0(y) := (\sqrt{2/d}\,)\sin(\pi y/d)$. Recall that $u_0(y)$ is the eigenfunction associated with the first eigenvalue of the negative Dirichlet Laplacian in $\mathscr{H}^2(0,d) \cap \mathscr{H}_0^1(0,d)$. Now, consider the closed subspace

$$\mathscr{L} = \{\, w(s)u_0(y) : w \in \mathrm{L}^2(0,1)\,\} \subset \mathrm{L}^2(\Omega),$$

already mentioned in the introduction, and the sequence of one-dimensional quadratic forms

$$t_\varepsilon(w) := \int_0^1 \left(|w'|^2 - \frac{|w|^2}{4(C_1 s^m + C_2\varepsilon^{am^2})^2} + \frac{A}{\varepsilon^{2am^2}}|w|^2\right)\mathrm{d}s\,\mathrm{d}y,$$

with $\operatorname{dom} t_\varepsilon = \mathscr{H}_0^1(0,1)$. We denote by T_ε the self-adjoint operator associated with $t_\varepsilon(w)$.

PROPOSITION 3.1. *There exists* $K > 0$ *so that, for* $\varepsilon > 0$ *small enough,*

$$\left\|L_\varepsilon^{-1} - (T_\varepsilon)^{-1} \oplus 0\right\|_{\mathrm{L}^2(\Omega)} \le K\varepsilon^{2(1+am)},$$

where 0 *is the null operator on the subspace* $\mathscr{L}^\perp$.

COROLLARY 3.2. *In addition to conditions* (6), *take $a > 0$ so that $\delta = 2+2am(1-2m) > 0$. There exists $K > 0$ such that, for $\varepsilon > 0$ small enough,*

$$\left\| \left(L_\varepsilon - \left(\frac{A}{\varepsilon^{2am^2}} + i\right)\mathbf{1}\right)^{-1} - \left[\left(T_\varepsilon - \left(\frac{A}{\varepsilon^{2am^2}} + i\right)\mathbf{1}\right)^{-1} \oplus 0\right]\right\| \le K\varepsilon^{\delta},$$

where 0 is the null operator on the subspace $\mathscr{L}^{\perp}$.

Now, define the quadratic form

$$t(w) := \int_0^1 \left(|w'|^2 - \frac{1}{4C_1^2 s^{2m}}|w|^2\right) \mathrm{d}s\, \mathrm{d}y,$$

$\operatorname{dom} t = \mathscr{H}_0^1(0, 1)$, and denote by T its associated self-adjoint operator.

PROPOSITION 3.3. *Take $0 < m \le 1/2$. There exists a number $K > 0$ so that, for $\varepsilon > 0$ small enough,*

$$\left\| \left(T_\varepsilon - \left(\frac{A}{\varepsilon^{2am^2}} + i\right)\mathbf{1}\right)^{-1} - (T - i\mathbf{1})^{-1}\right\| \le K\varepsilon^{am^2}.$$

Proposition 3.3 is a consequence of Theorem 1.3 stated in the introduction of this work. For more details, see Example B.1 in Appendix B.

PROOF OF THEOREM 1.1. By combining Corollary 3.2 and Proposition 3.3, there exists $K > 0$ so that, for $\varepsilon > 0$ small enough,

$$\left\| \left(L_\varepsilon - \left(\frac{A}{\varepsilon^{2am^2}} + i\right)\mathbf{1}\right)^{-1} - (T - i\mathbf{1})^{-1} \oplus 0\right\| \le K\varepsilon^{am^2}. \tag{10}$$

For simplicity of notation, let $\zeta = A\varepsilon^{2am(1-m)} - i\varepsilon^{2am}$. By employing Theorem 2.1, it is possible to show that, for $\varepsilon > 0$ small enough,

$$\left\|(D_\varepsilon - \zeta\mathbf{1})^{-1} - (\tilde{D}_\varepsilon - \zeta\mathbf{1})^{-1}\right\| \le K\varepsilon^{\delta}. \tag{11}$$

The proof of this inequality it is very similar to proof of Proposition 3.2 (see Section 4 ahead), and so it will be omitted here.

Now, recall the unitary operators $\mathscr{V}_\varepsilon$ and W_ε defined in Section 2 and note that

$$D_\varepsilon - \zeta\mathbf{1} = W_\varepsilon^{-1}\mathscr{V}_\varepsilon(-\Delta_{\Omega_\varepsilon} - \zeta\mathbf{1})\mathscr{V}_\varepsilon^{-1}W_\varepsilon.$$

We also have

$$\tilde{D}_\varepsilon - \zeta\mathbf{1} = \varepsilon^{2am}\left(L_\varepsilon - \left(\frac{A}{\varepsilon^{2am^2}} + i\right)\mathbf{1}\right).$$

Defining $U_\varepsilon = W_\varepsilon^{-1}\mathscr{V}_\varepsilon$ and combining this characterizations with (10) and (11), Theorem 1.1 follows.

PROOF OF THEOREM 1.2. Denote by $\lambda_j(\varepsilon)$ and λ_j the eigenvalues of $-\Delta_{\Omega_\varepsilon}$ and T, respectively. By Theorem 4.10, page 291 in [9], and Theorem 1.1 we have

$$\begin{aligned}
&\left|\varepsilon^{2am}\left(\lambda_j(\varepsilon) - \frac{\pi^2}{\varepsilon^2 d^2}\right)^{-1} - \lambda_j^{-1}\right| \\
&\quad\le \left\|\varepsilon^{2am} U_\varepsilon\left(-\Delta_{\Omega_\varepsilon} - \frac{\pi^2}{d\,\varepsilon^2}\mathbf{1}\right)^{-1} U_\varepsilon^{-1} - T^{-1}\oplus 0\right\| \\
&\quad\le K\,\varepsilon^{am^2}.
\end{aligned}$$

Thus,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{2am}}\left(\lambda_j(\varepsilon) - \frac{\pi^2}{\varepsilon^2 d^2}\right) = \lambda_j.$$

REMARK 3.4. If $0 < m \le 1$, by Theorem 7.9 in [3], $(T_\varepsilon - A\varepsilon^{2am^2}\mathbf{1})$ converges to T in the strong resolvent sense, as $\varepsilon \to 0$. Thus, as a weaker version of Theorem 1.1, this implies that

$$\varepsilon^{2am} U_\varepsilon\left(-\Delta_{\Omega_\varepsilon} - \frac{\pi^2}{d\,\varepsilon^2}\mathbf{1}\right)^{-1} U_\varepsilon^{-1} \longrightarrow T^{-1}\oplus 0,$$

where 0 is the null operator on $\mathscr{L}^\perp$, in the strong sense, as $\varepsilon \to 0$.

4. Proofs of auxiliary results

Given the subspace $\mathscr{L}$ defined in previous sections, consider the orthogonal decomposition

$$\mathrm{L}^2(\Omega) = \mathscr{L}\oplus\mathscr{L}^\perp.$$

Thus, if $\psi \in \mathrm{L}^2(\Omega)$ we can write $\psi = wu_0 + \eta$ with $w \in \mathrm{L}^2(0,1)$ and $\eta \in \mathscr{L}^\perp$. Note that $wu_0 \in \mathscr{H}_0^1(\Omega)$ if $w \in \mathscr{H}_0^1(0,1)$. Correspondingly, for $\psi \in \mathscr{H}_0^1(\Omega)$, write

$$\psi(s,y) = w(s)u_0(y) + \eta(s,y) \tag{12}$$

with $w \in \mathscr{H}_0^1(0,1)$ and $\eta \in \mathscr{H}_0^1(\Omega)\cap\mathscr{L}^\perp$.

PROOF OF PROPOSITION 3.1. We begin with some observations. If $\eta \in \mathscr{H}_0^1(\Omega)\cap\mathscr{L}^\perp$,

$$\int_0^d u_0(y)\eta(s,y)\mathrm{d}y = 0 \quad\text{and}\quad \int_0^d u_0(y)\eta'(s,y)\mathrm{d}y = 0, \quad \text{a.e.}[s]. \tag{13}$$

An integration by parts shows that

$$\int_0^d u_{0y}(y)\eta_y(s,y)\mathrm{d}y = 0, \quad \text{a.e.}[s]. \tag{14}$$

Note also that

$$\int_\Omega |\eta_y|^2 \mathrm{d}s\,\mathrm{d}y \geq \frac{4\pi^2}{d^2}\int_\Omega |\eta|^2 \mathrm{d}s\,\mathrm{d}y,$$

where $4\pi^2/d^2$ is the second eigenvalue of the negative Dirichlet Laplacian in $(0,d)$. Thus,

$$\int_\Omega \left(|\eta_y|^2 - \frac{\pi^2}{d^2}|\eta|^2\right)\mathrm{d}s\,\mathrm{d}y \geq \frac{3\pi^2}{d^2}\int_\Omega |\eta|^2 \mathrm{d}s\,\mathrm{d}y. \tag{15}$$

Now, denote by $\ell_\varepsilon(\psi_1,\psi_2)$ the sesquilinear form associated with the quadratic form $\ell_\varepsilon(\psi)$. For $\psi \in \mathscr{H}_0^1(\Omega)$ and the decomposition in (12),

$$\ell_\varepsilon(\psi) = \ell_\varepsilon(wu_0) + \ell_\varepsilon(\eta) + 2\ell_\varepsilon(wu_0,\eta).$$

We are going to check that there are $C_0 > 0$ and functions $0 \leq Q(\varepsilon)$, $0 \leq P(\varepsilon)$ and $C(\varepsilon)$ so that $\ell_\varepsilon(wu_0)$, $\ell_\varepsilon(\eta)$ and $\ell_\varepsilon(wu_0,\eta)$ satisfy the following conditions:

$$\ell_\varepsilon(wu_0) \geq C(\varepsilon)\|wu_0\|^2_{\mathrm{L}^2(\Omega)}, \quad \forall w \in \mathscr{H}_0^1(0,1), \;\; C(\varepsilon) \geq C_0 > 0; \tag{16}$$

$$\ell_\varepsilon(\eta) \geq P(\varepsilon)\|\eta\|^2_{\mathrm{L}^2(\Omega)}, \quad \forall \eta \in \mathscr{H}_0^1(\Omega)\cap\mathscr{L}^\perp; \tag{17}$$

$$|\ell_\varepsilon(wu_0,\eta)|^2 \leq Q(\varepsilon)^2\ell_\varepsilon(wu_0)\ell_\varepsilon(\eta), \quad \forall \psi \in \mathscr{H}_0^1(\Omega); \tag{18}$$

and with

$$P(\varepsilon)\to\infty, \quad C(\varepsilon) = O(P(\varepsilon)), \quad Q(\varepsilon)\to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{19}$$

Thus, Proposition 3.1 in [6] guarantees that, for $\varepsilon > 0$ small enough,

$$\left\|L_\varepsilon^{-1} - (T_\varepsilon)^{-1}\oplus 0\right\|_{\mathrm{L}^2(\Omega)} \leq P(\varepsilon)^{-1} + KQ(\varepsilon)C(\varepsilon)^{-1},$$

for some $K > 0$.

Now we check that (16), (17) and (18) are satisfied. Due to the conditions on A, we have

$$\begin{aligned}\ell_\varepsilon(wu_0) &= \int_0^1 \left(|w'|^2 - \frac{|w|^2}{4(C_1 s^m + C_2\varepsilon^{am})^2} + \frac{A}{\varepsilon^{2am^2}}|w|^2\right)\mathrm{d}s\,\mathrm{d}y \\ &\geq C_0\int_0^1 |w|^2 \mathrm{d}s\,\mathrm{d}y,\end{aligned}$$

for some $C_0 > 0$. Thus, (16) holds true. The condition (17) follows by (15); in fact,

$$\ell_\varepsilon(\eta) \geq \frac{3\pi^2}{d^2}\frac{1}{\varepsilon^{2+2am}}\int_\Omega |\eta|^2 \mathrm{d}s\,\mathrm{d}y,$$

and just take $P(\varepsilon) = 3\pi^2/d^2\varepsilon^{2+2am}$.

Finally, due to (13) and (14), we have

$$b_\varepsilon(wu_0,\eta) = \int_\Omega \left(w' u_0 \eta' + \frac{w u_{0y}\eta_y}{\varepsilon^{2+2am}} - \frac{\pi^2}{d^2}\frac{w u_0 \eta}{\varepsilon^{2+2am}} \right) \mathrm{d}s\,\mathrm{d}y$$
$$- \int_\Omega \frac{w u_0 \eta}{4(C_1 s^m + C_2 \varepsilon^{am^2})^2}\,\mathrm{d}s\,\mathrm{d}y + \int_\Omega \frac{A}{\varepsilon^{2am^2}} w u_0 \eta\,\mathrm{d}s\,\mathrm{d}y = 0.$$

We take $Q(\varepsilon) = 0$ and so (18) and (19) are satisfied. This completes the proof of the proposition.

PROOF OF COROLLARY 3.2. If S and T are linear operators and z, z_0 are common elements of the resolvent sets of both S and T, then

$$R_z(T) - R_z(S) = (\mathbf{1} + (z - z_0)R_z(T))[R_{z_0}(T) - R_{z_0}(S)](\mathbf{1} + (z - z_0)R_z(S)). \tag{20}$$

This identity was dubbed the *third resolvent identity* in [14]. For simplicity, we write $\xi = (A/\varepsilon^{2am^2} + i)$. By (20),

$$(L_\varepsilon - \xi\mathbf{1})^{-1} - \left[(T_\varepsilon - \xi\mathbf{1})^{-1} \oplus 0\right]$$
$$= \left[\mathbf{1} + \xi(L_\varepsilon - \xi\mathbf{1})^{-1}\right]\left[L_\varepsilon^{-1} - T_\varepsilon^{-1} \oplus 0\right]\left[\mathbf{1} + \xi((T_\varepsilon - \xi\mathbf{1})^{-1} \oplus 0)\right].$$

Thus, since

$$\left\|(L_\varepsilon - \xi\mathbf{1})^{-1}\right\| \leq 1 \quad \text{and} \quad \left\|(T_\varepsilon - \xi\mathbf{1})^{-1} \oplus 0\right\| \leq 1,$$

taking into account Proposition 3.1, it is found that

$$\left\|(L_\varepsilon - \xi\mathbf{1})^{-1} - (T_\varepsilon - \xi\mathbf{1})^{-1} \oplus 0\right\|$$
$$\leq \left(1 + \sqrt{A^2/\varepsilon^{4am^2} + 1}\right) K\varepsilon^{2+2am}\left(1 + \sqrt{A^2/\varepsilon^{4am^2} + 1}\right)$$
$$\leq K\,\varepsilon^{2+2am(1-2m)},$$

for some $K > 0$. This completes the proof of the corollary.

Appendix A

PROOF OF THEOREM 2.1. By replacing the global multiplicative factor $\beta_\varepsilon(s,y) = 1 - \varepsilon y k_\varepsilon(s/\varepsilon^{am})$ by 1 in the first and third integrals in the expression for $d_\varepsilon(\psi)$ (see Section 2), we get the quadratic form

$$\hat{d}_\varepsilon(\psi) := \int_\Omega \varepsilon^{2am}\left|\psi' - \frac{\psi}{2}\frac{\beta_\varepsilon'}{\beta_\varepsilon}\right|^2 \mathrm{d}s\,\mathrm{d}y + \int_\Omega \frac{1}{\varepsilon^2}\Big(|\psi_y|^2 - \frac{\pi^2}{d^2}|\psi|^2\Big)\,\mathrm{d}s\,\mathrm{d}y$$
$$- \int_\Omega \frac{\varepsilon^{2am}}{4}\frac{|\psi|^2}{(C_1 s^m + C_2\varepsilon^{am^2})^2}|\psi|^2 \mathrm{d}s\,\mathrm{d}y + \int_\Omega A\,\varepsilon^{2am(1-m)}|\psi|^2 \mathrm{d}s\,\mathrm{d}y,$$

$\operatorname{dom}\hat{d}_\varepsilon = \mathscr{H}_0^1(\Omega)$. Denote by $\hat{D}_\varepsilon$ the self-adjoint operator associated with it. We claim that there exists $K > 0$ so that, for $\varepsilon > 0$ small enough,

$$\|\hat{D}_\varepsilon^{-1} - D_\varepsilon^{-1}\| \le K\,\varepsilon^{1+am-3am^2}. \tag{21}$$

The main point of this approximation is that $\beta_\varepsilon(s,y) \to 1$ uniformly as $\varepsilon \to 0$. Its proof is quite similar to proof of Theorem 3.1 in [15] and will not be presented here.

Now, recall the quadratic form $\tilde{d}_\varepsilon(\psi)$. Note that

$$|\tilde{d}_\varepsilon(\psi) - \hat{d}_\varepsilon(\psi)| = \left|\int_\Omega \varepsilon^{2am}\left[\left(\psi' - \frac{\psi}{2}\frac{\beta_\varepsilon'}{\beta_\varepsilon}\right)^2 - (\psi')^2\right]\mathrm{d}s\,\mathrm{d}y\right|$$
$$\le \varepsilon^{2am}\int_\Omega \frac{|\psi|^2}{4}\left(\frac{\beta_\varepsilon'}{\beta_\varepsilon}\right)^2 \mathrm{d}s\,\mathrm{d}y$$
$$+ \varepsilon^{2am}\left(\int_\Omega |\psi|^2\left(\frac{\beta_\varepsilon'}{\beta_\varepsilon}\right)^2 \mathrm{d}s\,\mathrm{d}y\right)^{1/2}\left(\int_\Omega |\psi'|^2 \mathrm{d}s\,\mathrm{d}y\right)^{1/2}.$$

Some calculations show that there exists $K > 0$ so that

$$\|\beta_\varepsilon'/\beta_\varepsilon\|_\infty \le K\varepsilon^{1+am-2am^2}(1/s^{1-m}), \quad \forall s \in (0,1).$$

Thus, by the one-dimensional Hardy inequality,

$$\int_\Omega |\psi|^2\left(\frac{\beta_\varepsilon'}{\beta_\varepsilon}\right)^2 \mathrm{d}s\,\mathrm{d}y \le K\varepsilon^{2+2am-4am^2}\int_\Omega |\psi'|^2 \mathrm{d}s\,\mathrm{d}y,$$

and so

$$|\tilde{d}_\varepsilon(\psi) - \hat{d}_\varepsilon(\psi)| \le K\varepsilon^{1+am-2am^2}\tilde{d}_\varepsilon(\psi).$$

Theorem 3 in [1] implies

$$\|\hat{D}_\varepsilon^{-1} - \tilde{D}_\varepsilon^{-1}\| \le K\varepsilon^{1+am-2am^2}. \tag{22}$$

The theorem follows by combining (21) with (22).

Appendix B

PROOF OF THEOREM 1.3. First, we consider the case $\nu = \nu_\varepsilon = 0$, for all $\varepsilon > 0$, and, without loss of generality, we suppose that $\beta > 0$. Thus, $0 \in \rho(H)$ and $0 \in \rho(H_\varepsilon)$, for all $\varepsilon > 0$.

We denote by $h_\varepsilon(\psi)$ and $h(\psi)$ the quadratic forms associated with H_ε and H, respectively. For $\psi \in \operatorname{dom} H_0$,

$$|h_\varepsilon(\psi) - h(\psi)| = |\langle (V_\varepsilon - V)\psi, \psi\rangle| \le \alpha_\varepsilon\, h_0(\psi) = \alpha_\varepsilon\, |\langle H_0\psi, \psi\rangle|.$$

In what follows, we use the relation $H_0(H_0+V)^{-1} = \mathbf{1} - V(H_0+V)^{-1}$. For $\psi \in \operatorname{dom} H_0$, write $\psi = (H_0+V)^{-1}\phi$ and $B(\phi) := |\langle \phi, (H_0+V)^{-1}\phi\rangle|$. Thus,

$$\begin{aligned}
&|\langle H_0\psi, \psi\rangle| \\
&\quad = |\langle H_0(H_0+V)^{-1}\phi, (H_0+V)^{-1}\phi\rangle| \\
&\quad = |\langle \phi - V(H_0+V)^{-1}\phi, (H_0+V)^{-1}\phi\rangle| \\
&\quad \le \big(|\langle \phi, (H_0+V)^{-1}\phi\rangle| + |\langle V(H_0+V)^{-1}\phi, (H_0+V)^{-1}\phi\rangle|\big) \\
&\quad \le \big(|\langle \phi, (H_0+V)^{-1}\phi\rangle| + \alpha|\langle H_0(H_0+V)^{-1}\phi, (H_0+V)^{-1}\phi\rangle|\big) \\
&\quad \le |\langle \phi, (H_0+V)^{-1}\phi\rangle| \\
&\qquad\quad + \alpha\big(|\langle \phi, (H_0+V)^{-1}\phi\rangle| + |\langle V(H_0+V)^{-1}\phi, (H_0+V)^{-1}\phi\rangle|\big) \\
&\quad = B(\phi) + \alpha\big(B(\phi) + |\langle V(H_0+V)^{-1}\phi, (H_0+V)^{-1}\phi\rangle|\big) \\
&\quad \le B(\phi) + \alpha\big(B(\phi) + \alpha|\langle H_0(H_0+V)^{-1}\phi, (H_0+V)^{-1}\phi\rangle|\big) \\
&\quad = B(\phi)\big(1 + \alpha + (\alpha^2/B(\phi))|\langle H_0(H_0+V)^{-1}\phi, (H_0+V)^{-1}\phi\rangle|\big).
\end{aligned}$$

Write $A(\phi) := |\langle H_0(H_0+V)^{-1}\phi, (H_0+V)^{-1}\phi\rangle|$. Following these steps inductively, we have, for $j \in \mathbb{N}$,

$$\begin{aligned}
|\langle H_0\psi, \psi\rangle| &\le B(\phi)\big(1 + \alpha + \alpha^2 + \ldots + \alpha^{j-1} + (\alpha^j/B(\phi))A(\phi)\big) \\
&\le B(\phi)\sum_{j=0}^{\infty} \alpha^j = B(\phi)\,(1-\alpha)^{-1},
\end{aligned}$$

for all $\psi \in \operatorname{dom} H_0$, and then

$$|h_\varepsilon(\psi) - h(\psi)| \le \alpha_\varepsilon(1-\alpha)^{-1}|\langle \phi, (H_0+V)^{-1}\phi\rangle| = \alpha_\varepsilon(1-\alpha)^{-1}h(\psi),$$

for all $\psi \in \operatorname{dom} H_0$. By Theorem 3 in [1], this case is proven.

The proof of the general case is similar to the above one combined with the proof of Theorem 3 in [1]. So, it will be omitted here.

EXAMPLE B.1. Let $\gamma \in \mathbb{R}$, $b > 0$, $C_1, C_2 > 0$ and $0 < m \le 1/2$. Suppose that $4|\gamma|/C_1^2 < 1$ and set

$$V(s) := \frac{\gamma}{C_1^2 s^{2m}}, \quad V_\varepsilon(s) := \frac{\gamma}{(C_1 s^m + C_2 \varepsilon^b)^2}, \qquad s \in (0,1).$$

Consider the positive self-adjoint operator $H_0 w = -w''$, $\operatorname{dom} H_0 = \mathscr{H}^2(0,1) \cap \mathscr{H}_0^1(0,1)$. The quadratic form associated with H_0 is

$$h_0(w) = \int_0^1 |w'|^2 \,\mathrm{d}s, \quad \operatorname{dom} h_0 = \mathscr{H}_0^1(0,1).$$

Now, for each $\varepsilon > 0$, consider the sequence of self-adjoint operators

$$H_\varepsilon = H_0 + V_\varepsilon(s), \quad H = H_0 + V(s), \quad \operatorname{dom} H = \operatorname{dom} H_\varepsilon = \operatorname{dom} H_0.$$

Since $4|\gamma|/C_1^2 < 1$, Hardy's inequality guarantees that $H \ge 0$ and $H_\varepsilon \ge 0$, for all $\varepsilon > 0$. Note that

$$|\langle V(s)w, w\rangle| = \frac{|\gamma|}{C_1^2}\int_0^1 \frac{|w|^2}{s^{2m}}\mathrm{d}s \le \frac{|\gamma|}{C_1^2}\int_0^1 \frac{|w|^2}{s^2}\,\mathrm{d}s \le \frac{4|\gamma|}{C_1^2}\int_0^1 |w'|^2\mathrm{d}s,$$

for all $w \in \operatorname{dom} h_0$. Now, if $C_3 := \max\{2C_2/C_1^3, C_2^2/C_1^4\}$, we have

$$\begin{aligned}|\langle (V_\varepsilon - V)(s)w, w\rangle| &\le |\gamma| \int_0^1 \frac{2C_1C_2\varepsilon^b s^m + C_2^2\varepsilon^{2b}}{C_1^4 s^{4m}}|w|^2\mathrm{d}s \\ &\le 4|\gamma| C_3 \varepsilon^b \int_0^1 |w'|^2\mathrm{d}s.\end{aligned}$$

By Theorem 1.3, there exists a number $K > 0$ so that, for $\varepsilon > 0$ small enough,

$$\|H_\varepsilon^{-1} - H^{-1}\| \le K\varepsilon^b.$$

REFERENCES

1. Bedoya, R., de Oliveira, C. R., and Verri, A. A., *Complex* Γ*-convergence and magnetic Dirichlet Laplacian in bounded thin tubes*, J. Spectr. Theory 4 (2014), no. 3, 621–642.
2. Duclos, P., and Exner, P., *Curvature-induced bound states in quantum waveguides in two and three dimensions*, Rev. Math. Phys. 7 (1995), no. 1, 73–102.
3. Faris, W. G., *Self-adjoint operators*, Lecture Notes in Mathematics, Vol. 433, Springer-Verlag, Berlin-New York, 1975.
4. Frank, W. M., Land, D. J., and Spector, R. M., *Singular potentials*, Rev. Modern Phys. 43 (1971), no. 1, 36–98.

5. Freitas, P. and Krejčiřík, D., *Location of the nodal set for thin curved tubes*, Indiana Univ. Math. J. 57 (2008), no. 1, 343–375.
6. Friedlander, L., and Solomyak, M., *On the spectrum of the Dirichlet Laplacian in a narrow infinite strip*, Spectral theory of differential operators, Amer. Math. Soc. Transl. Ser. 2, vol. 225, Amer. Math. Soc., Providence, RI, 2008, pp. 103–116.
7. Gesztesy, F., *On the one-dimensional Coulomb Hamiltonian*, J. Phys. A 13 (1980), no. 3, 867–875.
8. Gesztesy, F. and Zinchenko, M., *On spectral theory for Schrödinger operators with strongly singular potentials*, Math. Nachr. 279 (2006), no. 9-10, 1041–1082.
9. Kato, T., *Perturbation theory for linear operators*, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 132, Springer-Verlag, Berlin-New York, 1976.
10. Klaus, M., *Removing cut-offs from one-dimensional Schrödinger operators*, J. Phys. A 13 (1980), no. 9, L295–L298.
11. Krejčiřík, D., and Kříz, J., *On the spectrum of curved planar waveguides*, Publ. Res. Inst. Math. Sci. 41 (2005), no. 3, 757–791.
12. Krejčiřík, D., and Šediváková, H., *The effective Hamiltonian in curved quantum waveguides under mild regularity assumptions*, Rev. Math. Phys. 24 (2012), no. 7, 1250018, 39 p.
13. Naĭmark, M. A., *Linear differential operators. Part II: Linear differential operators in Hilbert space*, Frederick Ungar Publishing Co., New York, 1968.
14. de Oliveira, C. R., *Intermediate spectral theory and quantum dynamics*, Progress in Mathematical Physics, vol. 54, Birkhäuser Verlag, Basel, 2009.
15. de Oliveira, C. R., and Verri, A. A., *On the spectrum and weakly effective operator for Dirichlet Laplacian in thin deformed tubes*, J. Math. Anal. Appl. 381 (2011), no. 1, 454–468.
16. de Oliveira, C. R., and Verri, A. A., *Mathematical predominance of Dirichlet condition for the one-dimensional Coulomb potential*, J. Math. Phys. 53 (2012), no. 5, 052104, 20 p.
17. Oseguera, U., and de Llano, M., *Two singular potentials: the space-splitting effect*, J. Math. Phys. 34 (1993), no. 10, 4575–4589.
18. Pressley, A., *Elementary differential geometry*, second ed., Springer Undergraduate Mathematics Series, Springer-Verlag London, Ltd., London, 2010.
19. Weidmann, J., *Spectral theory of ordinary differential operators*, Lecture Notes in Mathematics, vol. 1258, Springer-Verlag, Berlin, 1987.
20. Weidmann, J., *Spectral theory of Sturm-Liouville operators approximation by regular problems*, Sturm-Liouville theory, Birkhäuser, Basel, 2005, pp. 75–98.
21. Zettl, A., *Sturm-Liouville theory*, Mathematical Surveys and Monographs, vol. 121, American Mathematical Society, Providence, RI, 2005.

DEPARTAMENTO DE MATEMÁTICA – UFSCAR
SÃO CARLOS
SP, 13560-970
BRAZIL
E-mail: oliveira@dm.ufscar.br
alessandraverri@dm.ufscar.br