A NOTE ON HOLOMORPHIC FUNCTIONS AND THE FOURIER-LAPLACE TRANSFORM

MARCUS CARLSSON and JENS WITTSTEN

Abstract

We revisit the classical problem of when a given function, which is analytic in the upper half plane $\mathbb{C}_+$, can be written as the Fourier transform of a function or distribution with support on a half axis $(-\infty, b]$, $b \in \mathbb{R}$. We derive slight improvements of the classical Paley-Wiener-Schwartz Theorem, as well as softer conditions for verifying membership in classical function spaces such as $H^p(\mathbb{C}_+)$.

1. Introduction

Let $\mathbb{C}_+$ denote the complex upper half plane $\{x + iy : y > 0\}$, $\overline{\mathbb{C}}_+$ its closure and let t denote the identity function on $\mathbb{R}$. Given a function $u \in L^1(\mathbb{R})$ with support in a half axis $(-\infty, b]$, $b \in \mathbb{R}$, we may define a function $\hat{u}$ by

$$\hat{u}(z) = \int u(s)e^{-izs}\,ds = \mathscr{F}(e^{yt}u)(x), \quad z = x + iy \in \overline{\mathbb{C}}_+, \qquad (1.1)$$

which is analytic in $\mathbb{C}_+$. In other words, the Fourier transform $\mathscr{F}u$ on $\mathbb{R}$ has a natural extension as a holomorphic function to the upper half plane $\mathbb{C}_+$. Moreover,

$$|\hat{u}(z)| \leq \int \left|u(s)e^{ys}\right|\,ds \leq \|u\|_{L^1}e^{by},$$

so the upper limit of the support of u is related to the growth of $\hat{u}(z)$ as $y \to \infty$. These simple observations lie at the heart of e.g. the theory of Laplace transforms and of H^p spaces. Similarly, if $u \in L^2$ has support on an interval $[a, b]$, then $\hat{u}$ defines an entire function (of exponential type), and its growth in the lower half plane is determined by a (or rather, the largest possible a). The Paley-Wiener Theorem [17] gives a precise converse to the above observations, and it has been generalized by L. Schwartz [19] in several ways, to include e.g. distributions in $\mathbb{R}^n$ with support in a compact set or on a half space. These are all reproduced in Hörmander [9], see Theorems 7.3.1,

7.4.2 and 7.4.3. The following books and articles [1], [4], [11], [12], [18], [21], [22] also contain various versions of the Paley-Wiener Theorem, to mention a few.

For the moment, we focus on the one-dimensional case $n = 1$. A typical theorem of the above type then reads:

> *A holomorphic function f in $\mathbb{C}_+$ can be represented as $f = \hat{u}$, where u is a distribution with support on $(-\infty, b]$ for some $b \in \mathbb{R}$, provided that f satisfies certain growth estimates.*

The conditions imposed in particular require f to be of exponential type, at least in $\mathbb{C}_+ + i\eta$ for some $\eta > 0$. We show in this paper that this is unnecessarily restrictive; it suffices to assume that f is of exponential growth on the imaginary half axis $i\mathbb{R}_+ = \{iy : y > 0\}$, behaves "nicely" near $\mathbb{R}$, and has order 2 and type 0, or less, elsewhere. The last condition means that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(z)| \leq C_\varepsilon e^{\varepsilon |z|^2}$$

for z in the domain of interest. Theorem A below is typical for the type of extensions we prove in this paper. We let $\mathrm{Hol}\,(\mathbb{C}_+)$ denote the class of functions holomorphic in $\mathbb{C}_+$, and we let $\mathscr{C}(\overline{\mathbb{C}}_+)$ denote the set of continuous functions on $\overline{\mathbb{C}}_+$. Given a function f on $\overline{\mathbb{C}}_+$ we write $f|_{\mathbb{R}}$ for its restriction to $\mathbb{R}$. We denote the space of temperate distributions on $\mathbb{R}$ by $\mathscr{S}'$.

THEOREM A. *Suppose that $f \in \mathrm{Hol}\,(\mathbb{C}_+) \cap \mathscr{C}(\overline{\mathbb{C}}_+)$ satisfies:*

(i) *f is of order/type $\leq (2, 0)$,*

(ii) *$b = \limsup_{y \to \infty} y^{-1} \ln |f(iy)|$ is finite,*

(iii) *$f|_{\mathbb{R}} \in \mathscr{S}'$.*

Then $u = \mathscr{F}^{-1}(f|_{\mathbb{R}})$ has support included in $(-\infty, b]$ (as a distribution in $\mathscr{S}'$). Moreover, this is not true for any smaller b and $f = \hat{u}$ in $\overline{\mathbb{C}}_+$.

As an application, we give a slight improvement of the classical Paley-Wiener-Schwartz Theorem in $\mathbb{R}^n$. We also consider extensions of the above result to the case when no information is available on $\mathbb{R}$, i.e. we consider weaker forms of (iii). This will force us to leave $\mathscr{S}'$ and move to a larger class of distributions, namely distributions with sub-exponential growth, which we denote by $\mathscr{M}'$. We study this class in Section 2. Section 3 contains several theorems of the same type as Theorem A, and Section 4 shows how to apply these results e.g. to deduce new characterizations of the $H^p(\mathbb{C}_+)$ spaces. The paper includes several examples to illustrate the results.

EXAMPLE 1.1. Given any $\varepsilon > 0$, consider the "chirp function" $f(z) = e^{i\varepsilon z^2}$. It is bounded on $\mathbb{R}$, and hence $f|_{\mathbb{R}} \in \mathscr{S}'$. In fact, it is well-known that

$$u(t) = \mathscr{F}^{-1}(f|_{\mathbb{R}})(t) = \sqrt{\frac{\pi}{\varepsilon}} \exp\left(-i\left(\frac{t^2}{4\varepsilon} - \frac{\pi}{4}\right)\right),$$

see e.g. Ozaktas, Kutay-Alper and Zalevsky [16], Table 2.6(7), which also can be checked directly by residue calculus. (Multiply with $e^{-\delta x^2}$ and take the limit as $\delta \to 0^+$.) f is certainly of order 2 and type ε (see Section 3 for definitions). It is also bounded on $i\mathbb{R}_+$, which implies that $\limsup_{y\to\infty} y^{-1} \ln |f(iy)| \le 0$. However, the support of u is clearly not restricted to $(\infty, 0]$, which shows that the growth restrictions imposed in Theorem A cannot be relaxed further.

2. Distributions with sub-exponential growth

We first recall some standard facts concerning distributions with one-sided support and the Fourier-Laplace transform, following Hörmander [9]. For $n \ge 1$ we let $\mathscr{C}_c^\infty(\mathbb{R}^n)$ denote the space of smooth functions on $\mathbb{R}^n$ with compact support, $\mathscr{S}(\mathbb{R}^n)$ the Schwartz class, and $\mathscr{B}(\mathbb{R}^n)$ the Fréchet space of all smooth functions with bounded derivatives of all orders. We denote the dual of $\mathscr{S}(\mathbb{R}^n)$ by $\mathscr{S}'(\mathbb{R}^n)$, the dual of $\mathscr{C}_c^\infty(\mathbb{R}^n)$ by $\mathscr{D}'(\mathbb{R}^n)$ and the distributions with compact support by $\mathscr{E}'(\mathbb{R}^n)$. We define the Fourier transform $\mathscr{F}: \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by

$$\mathscr{F}\phi(\xi) = \hat{\phi}(\xi) = \int e^{-ix\cdot\xi}\phi(x)\,dx, \quad x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n,$$

and extend the definition to $\mathscr{S}'(\mathbb{R}^n)$ by duality. Since we will mostly be treating the case $n = 1$, we will omit $\mathbb{R}^n$ from the notation and simply write e.g. $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ when there is no ambiguity. As usual we identify $\mathscr{S}'$ with a subspace of $\mathscr{D}'$. In the entire paper, t will denote the identity function on $\mathbb{R}$, so that e.g. e^t is a function. When there is no risk of confusion we will also use t as an independent variable in expressions like $\int_0^\infty e^{-t}\,dt$.

Let $u \in \mathscr{E}'$ be given. Then the Fourier-Laplace transform

$$\hat{u}(z) = \langle u, e^{-izt}\rangle, \quad z \in \mathbb{C}, \tag{2.1}$$

is an entire analytic function of z which agrees with (1.1) when $u \in L^1$ (see Hörmander [9, Theorem 7.1.14]). In accordance with Hörmander [9, Section 7.4] we introduce the set

$$\Gamma_u = \{\, y \in \mathbb{R} : e^{yt}u \in \mathscr{S}'\,\}$$

for general $u \in \mathscr{D}'$. If Γ_u has non-empty interior Γ_u° and $y \in \Gamma_u^\circ$, it turns out that $\mathscr{F}(e^{yt}u)$ is in $\mathscr{C}^\infty(\mathbb{R})$ and $(x, y) \mapsto \mathscr{F}(e^{yt}u)(x)$ is an analytic function

in the strip $\mathbb{R} \times i\Gamma_u^\circ$ (where we identify (x, y) with $z = x + iy$). Moreover, if $u \in \mathscr{E}'$ then

$$\mathscr{F}(e^{yt}u)(x) = \langle u, e^{-izt}\rangle.$$

Since we are interested in analytic extensions to the upper half plane, we will consider a class of distributions u lying intermediate to $\mathscr{S}'$ and $\mathscr{D}'$ for which Γ_u contains $\mathbb{R}_+$ when u has one-sided support, so that $e^{yt}u \in \mathscr{S}'$ for all $y > 0$. For the chosen class, the above facts are proved below using basic methods relying on the structure theorem for $\mathscr{S}'$.

We consider a concrete example: on the one hand, the distribution $u \in \mathscr{D}'$ given by the function $t \mapsto \exp(|t|^{\frac{1}{2}})1_{\mathbb{R}_-}$ is not in $\mathscr{S}'$ (where $1_{\mathbb{R}_-}$ denotes the characteristic function of $\mathbb{R}_-$), but on the other,

$$\mathscr{F}(e^{yt}u(t))(x) = \int_{-\infty}^0 e^{-i(x+iy)t+|t|^{1/2}}\, dt$$

is clearly a well defined holomorphic function in $\mathbb{C}_+$. Motivated by this, we introduce the distributions of sub-exponential growth.

DEFINITION 2.1. Set $\langle t \rangle = (1+t^2)^{1/2}$. Let $\mathscr{M}' \subset \mathscr{D}'$ be the subset consisting of all distributions u such that for each $\varepsilon > 0$ we have

$$e^{-\varepsilon\langle t\rangle}u \in \mathscr{S}'. \tag{2.2}$$

Clearly $\mathscr{S}' \subset \mathscr{M}' \subset \mathscr{D}'$. As indicated by the choice of notation, $\mathscr{M}'$ is the dual of a functions space $\mathscr{M}$ defined as follows (we refer to Trèves [22] for basic information on the topological notions involved): for $\varepsilon > 0$ let $\mathscr{S}_\varepsilon \subset \mathscr{S}$ denote the set of functions f such that $\iota_\varepsilon(f) = e^{\varepsilon\langle t\rangle}f \in \mathscr{S}$. Moreover, give $\mathscr{S}_\varepsilon$ the topology such that ι_ε is an isomorphism with $\mathscr{S}$. Finally, let $\mathscr{M} = \bigcup_{\varepsilon>0}\mathscr{S}_\varepsilon$ and give it the inductive limit topology. In this topology a sequence $(\phi_k)_{k=1}^\infty$ in $\mathscr{M}$ tends to 0 as $k \to \infty$ precisely when there exists an $\varepsilon > 0$ such that $\phi_k \in \mathscr{S}_\varepsilon$ for all k and $\phi_k \to 0$ in $\mathscr{S}_\varepsilon$ as $k \to \infty$.

PROPOSITION 2.2. *We have*

$$\mathscr{C}_c^\infty \subset \mathscr{M} \subset \mathscr{S}$$

and the canonical embeddings are continuous with dense range.

It is now easy to see that $\mathscr{M}'$ is characterized by (2.2). Indeed, u is in the dual of $\mathscr{M}$ if and only if its restriction to $\mathscr{S}_\varepsilon$ is continuous for every $\varepsilon > 0$ (see Trèves [22, Proposition 13.1]). By definition, this happens if and only if $u \circ \iota_{-\varepsilon}$ is continuous on $\mathscr{S}$, which is equivalent to (2.2). The proof of Proposition 2.2 can

be found in the appendix, where we also prove the following basic observations about $\mathcal{M}$.

PROPOSITION 2.3. $\mathcal{M}$ *is closed under differentiation. Moreover,* $\mathcal{O}_M \cdot \mathcal{M} \subset \mathcal{M}$.

Here, $\mathcal{O}_M$ is the set of multipliers on $\mathcal{S}$, i.e. $\psi \in \mathcal{O}_M$ if ψ and all its derivatives are of polynomial growth (see (2.7)). We mention that the construction of $\mathcal{M}'$ is reminiscent of the definition of the (larger) space of temperate ultra-distributions of J. Sebastião e Silva [20], as well as the distributions of exponential growth, see Hasumi [8] or Yoshinaga [24]. In contrast, Gelfand-Shilov spaces [7] are (in analogy with $\mathcal{S}$) characterized by symmetric conditions with respect to the Fourier transform, see Chung, Chung and Kim [2]. For an alternative treatment of Fourier transforms of arbitrary distributions, see Ehrenpreis [4].

We next recall a result about convergent sequences, which holds for any barreled topological vector space (TVS). Examples of such spaces are $\mathcal{C}_c^\infty$, $\mathcal{M}$ and $\mathcal{S}$, since any Fréchet space or inductive limit of such is a barreled TVS (Corollary 1-3, Trèves [22, Chapter 33]). The importance of barreled TVS stems from the fact that it is the weakest topological notion to which the Banach-Steinhaus theorem extends, and the fact below is a simple consequence of this. A proof is given in the appendix.

THEOREM 2.4. *Let* $\mathcal{V}$ *be a barreled TVS and* $\mathcal{V}'$ *its* (*topological*) *dual. Let* $(u_k)_{k=1}^\infty$ *be a sequence in* $\mathcal{V}'$ *that converges point-wise (on* $\mathcal{V}$*) and denote its limit by u. Then* $u \in \mathcal{V}'$. *Moreover, if* $(\varphi_k)_{k=1}^\infty$ *is a sequence in* $\mathcal{V}$ *with limit* φ, *then*
$$\lim_{k \to \infty} \langle u_k, \varphi_k \rangle = \langle u, \varphi \rangle.$$

2.1. Distributions with one-sided support

Given any space $\mathcal{V}' \subset \mathcal{D}'$ of distributions we will write $\mathcal{V}'_-$ for the subset of $\mathcal{V}'$ consisting of elements with left-sided support, i.e. $u \in \mathcal{V}'_-$ if $\operatorname{supp} u \subset (-\infty, b]$ for some $b \in \mathbb{R}$. The smallest b such that $\operatorname{supp} u \subset (-\infty, b]$ will then be denoted $\operatorname{usl} u$, where usl stands for *upper support limit*.

Given $z = x + iy \in \mathbb{C}_+$, the function $e^{-izt} = e^{yt-ixt}$ is not an element of $\mathcal{M}$ or $\mathcal{S}$, due to the rapid growth as $t \to \infty$. However, for $u \in \mathcal{M}'_-$ this is irrelevant since u vanishes for large t, and hence an expression like
$$\hat{u}(z) = \langle u, e^{-izt} \rangle, \quad z \in \mathbb{C}_+, \tag{2.3}$$

should make sense (compare with (2.1)). The next proposition shows that this is indeed the case. Note that the analytic function $z = x + iy \mapsto \mathcal{F}(e^{yt}u)(x)$

discussed in connection with (2.1) is denoted $\hat{u}$ by Hörmander [9, Section 7.4]. This is in agreement with (2.3), but it requires a proof, see Proposition 2.10 below.

PROPOSITION 2.5. *Let $z \in \mathbb{C}_+$ and $u \in \mathscr{M}'_-$ be given. Pick $\rho \in \mathscr{C}^\infty$ with $\rho \equiv 1$ on $(-\infty, \operatorname{usl} u + 1)$ and $\rho \equiv 0$ on $(\operatorname{usl} u + 2, \infty)$. Then $e^{-izt}\rho \in \mathscr{M}$ and*

$$\langle u, e^{-izt} \rangle := \langle u, e^{-izt}\rho \rangle \qquad (2.4)$$

is independent of ρ.

PROOF. It is easy to see that $\iota_{y/2}(e^{-izt}\rho) = e^{y\langle t \rangle/2 + yt - ixt}\rho$ lies in $\mathscr{S}$, which by definition implies that $\rho e^{-izt} \in \mathscr{M}$. Thus (2.4) is well defined, and its independence of ρ is immediate by the definition of support of a distribution.

For $z \in \mathbb{C}_+$ and $u \in \mathscr{M}'_-$, we shall henceforth write $\hat{u}$ for the function defined by (2.3) with the right-hand side interpreted by means of Proposition 2.5. Next, we show that $e^{-izt}u \in \mathscr{S}'$. Recall that $\langle t \rangle$ satisfies

$$|\partial_t^n \langle t \rangle| \le C_n \langle t \rangle^{1-n} \qquad (2.5)$$

for any $n \in \mathbb{N}$. (The function $(x, \xi) \mapsto \langle \xi \rangle$ belongs to the symbol class $S^1(\mathbb{R}_x \times \mathbb{R}_\xi)$ appearing in the pseudodifferential calculus, see Definition 18.1.1 and p. 75 in Hörmander [10].) In particular, $e^{-\varepsilon\langle t \rangle} \in \mathscr{B}$ for any $\varepsilon \ge 0$.

PROPOSITION 2.6. *Given $z \in \mathbb{C}_+$ and $u \in \mathscr{M}'_-$, we have $e^{-izt}u \in \mathscr{S}'$. Moreover, with ρ as in Proposition 2.5, we have that $e^{-izt}\rho\varphi \in \mathscr{M}$ for any $\varphi \in \mathscr{S}$ and*

$$\langle e^{-izt}u, \varphi \rangle = \langle u, e^{-izt}\rho\varphi \rangle. \qquad (2.6)$$

PROOF. Clearly $ue^{-izt} = \rho e^{-izt+y\langle t \rangle} u e^{-y\langle t \rangle}$. By definition we have $ue^{-y\langle t \rangle} \in \mathscr{S}'$ and using (2.5), it is easy to see that $\rho e^{-izt+y\langle t \rangle} \in \mathscr{B}$, so ue^{-izt} defines an element of $\mathscr{S}'$. Now take $\varphi \in \mathscr{S}$. That $e^{-izt}\rho\varphi \in \mathscr{M}$ follows by Proposition 2.3 (since $e^{-izt}\rho \in \mathscr{M}$ and $\varphi \in \mathscr{S}$). Finally, if $\varphi \in \mathscr{C}_c^\infty$ then (2.6) clearly holds. The identity for general $\varphi \in \mathscr{S}$ follows by Proposition 2.2 and a limit argument.

We end with a technical observation.

PROPOSITION 2.7. *Let $(\varepsilon_k)_{k=1}^\infty$ be a positive sequence tending to 0 as $k \to \infty$. If $u \in \mathscr{D}'$ is such that $ue^{-\varepsilon_k\langle t \rangle} \in \mathscr{S}'$ for all k, then $u \in \mathscr{M}'$. Similarly, if $u \in \mathscr{D}'_-$ and $ue^{\varepsilon_k t} \in \mathscr{S}'$ for all k, then $u \in \mathscr{M}'_-$.*

PROOF. Given $\varepsilon > 0$ pick $\varepsilon_k < \varepsilon$ and note that $e^{-\varepsilon\langle t \rangle}u = e^{-(\varepsilon-\varepsilon_k)\langle t \rangle}e^{-\varepsilon_k\langle t \rangle}u$, from which the first statement follows since $\mathscr{B} \cdot \mathscr{S}' \subset \mathscr{S}'$. For the second claim, let ρ be as in Proposition 2.5 and fix $k \in \mathbb{N}$. Then

$$ue^{-\varepsilon_k\langle t \rangle} = u\rho e^{-\varepsilon_k\langle t \rangle} = ue^{\varepsilon_k t}\rho e^{-\varepsilon_k t - \varepsilon_k\langle t \rangle}$$

which as before shows that $ue^{-\varepsilon_k\langle t\rangle} \in \mathscr{S}'$ since $\rho e^{-\varepsilon_k t - \varepsilon_k\langle t\rangle} \in \mathscr{B}$ by (2.5). By the first part of the proof we get $u \in \mathscr{M}'$ from which it immediately follows that $u \in \mathscr{M}'_-$ since usl $u < \infty$ by assumption.

2.2. Representation formulas

We say that a function $U \in \mathscr{C}(\mathbb{R})$ has polynomial growth if there are constants C and $M > 0$ such that

$$|U(t)| < C\langle t\rangle^M. \tag{2.7}$$

For $\phi \in \mathscr{C}^\infty$ we let $\phi^{(k)}$ denote the kth derivative of ϕ.

THEOREM 2.8. *Given $u \in \mathscr{S}'$ there exists a number $N \in \mathbb{N}$ and a function U with polynomial growth such that*

$$\langle u, \varphi \rangle = \int U(t)\varphi^{(N)}(t)\,dt, \quad \varphi \in \mathscr{S}, \tag{2.8}$$

and conversely, (2.8) defines an element of $\mathscr{S}'$. Also, $u \in \mathscr{S}'_-$ if and only if U can be chosen with $U(t) \equiv 0$ for large t. Moreover, if this holds and (2.8) holds for all $\varphi \in \mathscr{C}^\infty_c$, then U is unique and usl $u =$ usl U.

PROOF. The first statement is well-known, see e.g. Friedlander [6, Theorem 8.3.1]. Next, if u is given by (2.8) and $U(t) \equiv 0$ for large t, then obviously $u \in \mathscr{S}'_-$. Conversely, let $u \in \mathscr{S}'_-$ be given and let V be any function (of polynomial growth) such that $\langle u, \varphi \rangle = \int V(t)\varphi^{(N)}(t)\,dt$ for all $\varphi \in \mathscr{S}$. Note that $V^{(N)}|_{(\text{usl}\,u,\infty)} \equiv 0$ in the distributional sense. By Hörmander [9, Corollary 3.1.6] it follows that $V|_{(\text{usl}\,u,\infty)} \in \mathscr{C}^N$ and that $V^{(N)}|_{(\text{usl}\,u,\infty)} \equiv 0$ also in the classical sense. Hence V coincides with a polynomial P of degree $\leq N - 1$ on $(\text{usl}\,u, \infty)$. Clearly, $U = V - P$ satisfies all the requirements. The proof concerning uniqueness follows identically. Finally, note that the inequality usl $U \leq$ usl u is a consequence of the above argument, whereas the reverse inequality is obvious.

We remark that it is not the case that an element of $\mathscr{S}' \cap \mathscr{C}^\infty$ is necessarily of polynomial growth, as follows by considering the function $u(t) = e^t \cos(e^t)$; it is clearly not of polynomial growth, but since its primitive $U(t) = \sin(e^t)$ is bounded, u is an element of $\mathscr{S}'$.

We say that U has sub-exponential growth if for all $\varepsilon > 0$ there exists a C_ε such that

$$|U(t)| < C_\varepsilon e^{\varepsilon\langle t\rangle}.$$

If this is the case and $u \in \mathscr{D}'$ is defined via (2.8), then clearly $u \in \mathscr{M}'$. However, we remark that the converse is false, as shown by the following example.

EXAMPLE 2.9. Consider an increasing function $v \in \mathscr{C}^\infty$ with support on $\mathbb{R}_+$ such that $v \equiv 1$ on $(1, \infty)$ and set

$$V_k(t) = \frac{1}{\|v^{(k)}\|_\infty} \sum_{j=1}^\infty e^{j/k} v^{(k)}\big(e^{j/k^2}(t - j)\big).$$

Note that the jth term in the sum has support in $(j, j + 1)$, so that they are mutually disjoint. Let $I_1[w]$ denote the antiderivative of a given function w, i.e. $I_1[w](x) = \int_{-\infty}^x w(y)\, dy$ (supposing we know that the integral is convergent). Also let I_k denote the composition of I_1 with itself k times. Note that for $0 \leq N < k$ and $t > 0$,

$$I_N[V_k](t) = \frac{1}{\|v^{(k)}\|_\infty} e^{\lfloor t \rfloor (k-N)/k^2} v^{(k-N)}\big(e^{\lfloor t \rfloor / k^2}(t - \lfloor t \rfloor)\big).$$

We leave the verification of the following facts to the reader:

(i) $e^{-\langle t \rangle (k-N)/k^2} I_N[V_k](t)$ is bounded by 1. In particular, given $N \in \mathbb{N}$, the sequence $(I_N[V_k])_{k=1}^\infty$ is uniformly bounded on any finite interval.

(ii) $I_k[V_k] = \|v^{(k)}\|_\infty^{-1} \sum_{j=1}^\infty v(e^{j/k^2}(t - j))$ has polynomial growth.

By (i), $U = \sum_{k=1}^\infty k^{-2} V_k$ is well defined as a function in L^1_{loc}, and hence defines a distribution $u \in \mathscr{D}'$ via

$$\langle u, \varphi \rangle = \int U(t)\varphi(t)\, dt, \quad \varphi \in \mathscr{C}_c^\infty.$$

For any $N \in \mathbb{N}$,

$$I_N[U] = \sum_{k=1}^N \frac{1}{k^2} I_N[V_k] + \sum_{k=N+1}^\infty \frac{1}{k^2} I_N[V_k],$$

where the change of order of integration and summation is allowed by the dominated convergence theorem and (i). The first sum is of polynomial growth by (ii). By (i), each of the remaining terms $I_N[V_k]$ is bounded by the function $\exp(\langle t \rangle (k - N)/k^2)$, and the factor $(k - N)/k^2$ is largest for $k = 2N$. Summing up, we conclude that

$$0 < \limsup_{t \to \infty} e^{-\frac{1}{4N} \langle t \rangle} I_N[U](t) < \infty. \tag{2.9}$$

In particular, $I_N[U]$ is not of sub-exponential growth for any N (since this class is closed under application of I_1). By the proof of Theorem 2.8, we see

that any representation of u of the form (2.8) will look like

$$\langle u, \varphi \rangle = (-1)^N \int \big(I_N[U](t) + P(t)\big)\varphi^{(N)}(t)\, dt, \quad \varphi \in \mathscr{C}_c^\infty,$$

where P is a polynomial of degree $< N$. It follows that u cannot be represented via (2.8) using a function of sub-exponential growth. However, despite this we do have that $u \in \mathscr{M}'$. To see this, fix N and note that

$$\big\langle e^{-\frac{1}{4N}\langle t \rangle} u, \varphi \big\rangle = (-1)^N \int I_N[U](t) \frac{d^N}{dt^N}\big(e^{-\frac{1}{4N}\langle t \rangle}\varphi(t)\big)\, dt.$$

Upon expanding the derivative and recalling (2.9) and (2.5), it is easily seen that the corresponding terms have polynomial growth, and hence $e^{-\frac{1}{4N}\langle t \rangle} u \in \mathscr{S}'$ by Theorem 2.8. The argument is thus completed by applying Proposition 2.7.

2.3. Analyticity and the class $\mathscr{H}$

Let $u \in \mathscr{M}'_-$ be given. Proposition 2.6 applied with $z = iy$ implies that $\mathscr{F}(e^{yt}u)$ defines a one parameter family of temperate distributions for $y > 0$, that is, $\Gamma_u^\circ \supset (0, \infty)$. Moreover, Proposition 2.5 shows that $\hat{u}(z) = \langle u, e^{-izt}\rangle$ is a function of $z \in \mathbb{C}_+$. As mentioned we in fact have

$$\hat{u}(. + iy) = \mathscr{F}(e^{yt}u), \tag{2.10}$$

see e.g. Hörmander [9, Section 7.4]. For completeness we include a basic proof of this fact, based on the representation formulas obtained above.

PROPOSITION 2.10. *Let $u \in \mathscr{M}'_-$ be given. Then $\hat{u}$ is an analytic function in $\mathbb{C}_+$ and (2.10) holds.*

PROOF. It is easy to see that it suffices to prove the statement with u replaced by $e^{y_0 t}u$, where $y_0 > 0$ is fixed but arbitrary. Hence we may assume that $u \in \mathscr{S}'$, and in particular that Theorem 2.8 applies. With U and N as in that theorem, we have

$$\hat{u}(z) = \langle u, e^{-izt}\rangle = (-iz)^N \int_{-\infty}^{\text{usl}\, u} e^{-izt} U(t)\, dt, \quad z \in \mathbb{C}_+. \tag{2.11}$$

Now let $\varphi \in \mathscr{S}$ be arbitrary. By Proposition 2.6 we get

$$\langle \mathscr{F}(e^{yt}u), \varphi \rangle = \int_{-\infty}^{\text{usl}\, u} \big(e^{yt}\hat{\varphi}(t)\big)^{(N)} U(t)\, dt$$

$$= \int_{-\infty}^{\text{usl}\, u} \frac{d^N}{dt^N}\left(\int_{-\infty}^{\infty} e^{yt}\varphi(x)e^{-ixt}\, dx\right) U(t)\, dt.$$

By standard real analysis (see e.g. Folland [5, Theorem 2.27]), the order of integration and differentiation can be interchanged. Writing $z = x + iy$ we get

$$\langle \mathscr{F}(e^{yt}u), \varphi \rangle = \int_{-\infty}^{\mathrm{usl}\, u} \left(\int_{-\infty}^{\infty} (-iz)^N e^{-izt} \varphi(x)\, dx \right) U(t)\, dt$$

$$= \int_{-\infty}^{\infty} \varphi(x) \left((-iz)^N \int_{-\infty}^{\mathrm{usl}\, u} e^{-izt} U(t)\, dt \right) dx$$

$$= \langle \hat{u}(.\, + iy), \varphi \rangle$$

in view of (2.11), where the use of Fubini's theorem is allowed since both $(-iz)^N \varphi(x)$ and $e^{yt} U(t)$ are in $L^1(\mathbb{R})$ and e^{-ixt} is unimodular. Since $\varphi \in \mathscr{S}$ was arbitrary, the proof is complete.

DEFINITION 2.11. We let $\mathscr{H}$ denote the class of functions $f \in \mathrm{Hol}\,(\mathbb{C}_+)$ of the form $f = \hat{u}$ for some $u \in \mathscr{M}'_-$.

The following proposition is an immediate consequence of Proposition 2.3.

PROPOSITION 2.12. $\mathscr{H}$ is closed under differentiation and multiplication by polynomials.

EXAMPLE 2.13. Let $u \in \mathscr{M}'_-$ be given by (2.8), with $N = 0$ and

$$U(t) = \lfloor e^{-t} \rfloor e^t.$$

Note that $\mathrm{usl}\, u = \mathrm{usl}\, U = 0$. A short calculation then gives

$$\hat{u}(z) = \frac{\zeta(1 - iz)}{1 - iz},$$

where ζ denotes the Riemann ζ function. In particular, Proposition 2.12 yields that $\mathbb{C}_+ \ni z \mapsto \zeta(1 - iz)$ is in $\mathscr{H}$.

Once it has been established that a given function $f \in \mathrm{Hol}\,(\mathbb{C}_+)$ is in $\mathscr{H}$, Theorem 2.8 can be used to recover various structural properties of $f = \hat{u}$, as follows: fix $y_0 > 0$ and find U_0 and N_0 (depending on y_0) by applying Theorem 2.8 to $e^{y_0 t/2} u$. For $z \in \mathbb{C}_+$ we then have

$$\hat{u}(z + iy_0/2) = \langle e^{y_0 t/2} u, e^{-izt} \rangle = (-iz)^{N_0} \int_{-\infty}^{\mathrm{usl}\, u} U_0(t) e^{-izt}\, dt,$$

which gives

$$\hat{u}(z + iy_0) = (y_0/2 - iz)^{N_0} \int_{-\infty}^{\mathrm{usl}\, u} U_0(t) e^{y_0 t/2} e^{-izt}\, dt.$$

Since $|e^{-izt}|$ is bounded by $e^{\operatorname{usl} u \operatorname{Im} z}$ on $(-\infty, \operatorname{usl} u]$ and $U_0 e^{y_0 t/2} \in L^1(-\infty, \operatorname{usl} u)$ (as U_0 has polynomial growth) this can be used to establish e.g. the estimate

$$|f(z)| \le C(1 + |z|)^{N_0} e^{\operatorname{usl} u \operatorname{Im} z}, \quad \operatorname{Im} z \ge y_0, \tag{2.12}$$

where C is a constant depending on y_0. In particular, f is of exponential type in $iy_0 + \mathbb{C}_+$ and $f(. + iy_0) \in \mathscr{S}'$ for all $y_0 > 0$. In the case when $u \in \mathscr{S}'_-$, C's dependence on y_0 has been further studied, see e.g. Vladimirov [23, Section II.12]. See also Shambayati and Zielezny [21, Theorem 2].

Given $u \in \mathscr{M}'_-$ and $\varphi \in \mathscr{S}$, we will now discuss the function

$$z \mapsto \langle \hat{u}(. + z), \varphi \rangle = \int \hat{u}(t + x + iy)\varphi(t)\, dt,$$

which is well defined by (2.12). Although $\hat{u}$ does not necessarily exist as a function on $\mathbb{R}$, the above one does if $\hat{\varphi} \in \mathscr{M}$. We collect this and a few other observations in the following proposition, compare with Hörmander [9, Lemma 7.4.1].

PROPOSITION 2.14. *Given $u \in \mathscr{S}'_-$ and $\varphi \in \mathscr{S}$ we have*

$$\langle \hat{u}(. + z), \varphi \rangle = \langle u, e^{-izt}\hat{\varphi} \rangle. \tag{2.13}$$

Moreover, the right-hand side is continuous on $\overline{\mathbb{C}}_+$ with values $\langle u, e^{-ixt}\hat{\varphi} \rangle$ on $\mathbb{R}$. The same holds if $u \in \mathscr{M}'_-$ and $\hat{\varphi} \in \mathscr{M}$. Finally, with $u \in \mathscr{M}'_-$, $\varphi \in \mathscr{S}$ and $y > 0$, we have

$$\frac{d^k}{dy^k}\langle \hat{u}(. + iy), \varphi \rangle = \langle \partial_y^k \hat{u}(. + iy), \varphi \rangle.$$

PROOF. We only prove (2.13) for the case when $u \in \mathscr{M}'_-$ and $\hat{\varphi} \in \mathscr{M}$; the other case can be treated similarly. Moreover, by translation invariance it suffices to consider the case $x = 0$. Note that $\varphi \in \mathscr{S}$ by Proposition 2.2, so Proposition 2.10 then implies that

$$\langle \hat{u}(. + iy), \varphi \rangle = \langle \mathscr{F}(e^{yt}u), \varphi \rangle = \langle u, e^{yt}\hat{\varphi} \rangle,$$

as desired. Moreover, by Proposition 2.3 we have that $e^{-ix_0 t}\hat{\varphi} \in \mathscr{M}$ for each $x_0 \in \mathbb{R}$, so the right-hand side of (2.13) also makes sense on $\mathbb{R}$. To see that $\langle u, e^{-izt}\hat{\varphi} \rangle$ converges to $\langle u, e^{-ix_0 t}\hat{\varphi} \rangle$ as $z \to x_0$ in $\mathbb{C}_+$, let ρ be as in Proposition 2.5 and note that it suffices to show that $\rho e^{-izt}\hat{\varphi}$ converges to $\rho e^{-ix_0 t}\hat{\varphi}$ in $\mathscr{M}$. Since $\hat{\varphi} \in \mathscr{M}$ there is an $\varepsilon > 0$ such that $\hat{\varphi} \in \mathscr{S}_\varepsilon$, which implies that

$\rho(e^{-izt} - e^{-ix_0t})\hat{\varphi} \in \mathscr{S}_\varepsilon$ for all $z \in \mathbb{C}_+$ (compare with the proof of Proposition 2.6). Moreover, for any fixed $m, n \in \mathbb{N}$ we have

$$\lim_{\mathbb{C}_+ \ni z \to x_0} \sup_t \left| \langle t \rangle^m \frac{\partial^n}{\partial t^n} \left(\left(\rho(t)e^{-izt} - \rho(t)e^{-ix_0t} \right) e^{\varepsilon\langle t \rangle} \hat{\varphi}(t) \right) \right| = 0.$$

Indeed, for each $0 \le k \le n$, the difference $\partial_t^k \left(\rho e^{-izt} - \rho e^{-ix_0t} \right)$ is uniformly bounded in t independently of z in a neighborhood of x_0, and converges uniformly on compacts to 0 when $z \to x_0$. Together with the Leibniz formula this is easily seen to yield the claim since $e^{\varepsilon\langle t \rangle}\hat{\varphi} \in \mathscr{S}$. In view of the comment preceding Proposition 2.2, this shows that $\rho e^{-izt}\hat{\varphi} \to \rho e^{-ix_0t}\hat{\varphi}$ in $\mathcal{M}$ as $z \to x_0$ in $\mathbb{C}_+$.

We turn to the claim concerning differentiability. Set $f(z) = \hat{u}(z)$ and note that by (2.12) and standard real analysis (see Folland [5], Theorem 2.27) we have

$$\frac{d}{dy} \int_{-\infty}^{\infty} \varphi(t) f(t + iy) \, dt = \int_{-\infty}^{\infty} \varphi(t) \partial_y f(t + iy) \, dt.$$

Hence, the desired conclusion follows by Proposition 2.12 and iteration.

3. Representation of functions in Hol $(\mathbb{C}_+)$ and Hol $(\mathbb{C})$ via Fourier-Laplace transforms

As explained in Section 2.3, each $u \in \mathcal{M}'_-$ naturally gives rise to a function in Hol $(\mathbb{C}_+)$ via an extension of the Fourier transform. In this section we give various converse statements. We first recall some standard terminology on the growth of analytic functions. Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in$ Hol (Ω) be given. The function f is said to be of finite order if it satisfies a growth restriction of the form

$$|f(z)| \le Ce^{\tau|z|^o}, \quad C, \tau, o \in \mathbb{R}_+.$$

Moreover, the *order* of f is defined as the infimum of all possible o's above. If f has order o, note that the infimum of all possible τ's is given by

$$\limsup_{z \to \infty} \frac{\ln|f(z)|}{|z|^o}.$$

This is called the *type* of f (which may be infinite). We introduce the lexicographical order on pairs of order/type, i.e.

$$(o_1, \tau_1) \le (o_2, \tau_2)$$

signifies that either $o_1 < o_2$ or $o_1 = o_2$ and $\tau_1 \le \tau_2$. Holomorphic functions of order/type $< (1, \infty)$ are said to be of *exponential type*, and the infimum of all

possible τ's is called the exponential type of f. This terminology is slightly confusing; as long as f has order 1, then the exponential type and the type coincide, but functions of order < 1 may well have a positive "type", although the "exponential type" clearly is zero. We need the following version of the Phragmén-Lindelöf principle, see e.g. Levin [13], Chapter I, Theorem 22.

THEOREM 3.1. *Given* $0 < \omega < 2$ *consider the domain*

$$\Omega = \{ re^{i\theta} : r > 0, |\theta - \theta_0| < \omega\pi/2 \}.$$

Let $f \in \text{Hol}(\Omega) \cap \mathscr{C}(\overline{\Omega})$ *have order/type* $\leq (\omega^{-1}, 0)$ *and satisfy* $\sup_{z \in \partial\Omega} |f(z)| = M$. *Then* $|f(z)| \leq M$ *for all* $z \in \Omega$.

3.1. Functions on $\overline{\mathbb{C}}_+$

PROPOSITION 3.2. *Suppose that* $f \in \text{Hol}(\mathbb{C}_+) \cap \mathscr{C}(\overline{\mathbb{C}}_+)$ *satisfies:*

(i) f *is of order/type* $\leq (2, 0)$,

(ii) $b = \limsup_{y \to \infty} y^{-1} \ln |f(iy)|$ *is finite,*

(iii) $f|_{\mathbb{R}} \in \mathscr{S}'$.

Then $u = \mathscr{F}^{-1}(f|_{\mathbb{R}})$ *satisfies* $\text{usl } u = b$ *and* $f = \hat{u}$ *in* $\mathbb{C}_+$.

Note that Proposition 3.2 contains Theorem A from the introduction. Moreover, Example 1.1 shows that condition (i) is optimal in the sense that if it is relaxed to $(2, \varepsilon)$ for any $\varepsilon > 0$, the theorem is false. When discussing the inverse Fourier transform of a function defined on $\overline{\mathbb{C}}_+$, we shall in the remainder of the paper sometimes permit us to write $\mathscr{F}^{-1}(f)$ instead of $\mathscr{F}^{-1}(f|_{\mathbb{R}})$ when there is no ambiguity.

The proof will be based on the following lemmas.

LEMMA 3.3. *Let* $f \in \text{Hol}(\mathbb{C}_+) \cap \mathscr{C}(\overline{\mathbb{C}}_+)$ *be such that* $f|_{\mathbb{R}} \in \mathscr{S}'$. *Set* $u = \mathscr{F}^{-1}(f|_{\mathbb{R}})$ *and suppose that* $u \in \mathscr{S}'_-$. *Then* $f(z) = \hat{u}(z)$ *for all* $z \in \mathbb{C}_+$.

PROOF. Let $\varphi \in \mathscr{C}_c^\infty$ be arbitrary. It is sufficient to prove that

$$\int f(z + t)\varphi(t)\,dt = \int \hat{u}(z + t)\varphi(t)\,dt, \quad z \in \mathbb{C}_+. \tag{3.1}$$

By Section 2.3 we have that both sides are analytic in $\mathbb{C}_+$. Since continuous functions are uniformly continuous on compact sets, we see that the left-hand side is continuous on $\overline{\mathbb{C}}_+$. By Proposition 2.14 the same holds for the right-hand side, the boundary values of which are given by

$$\langle u, e^{-ixt}\hat{\varphi} \rangle = \langle u, \mathscr{F}(\varphi(. - x)) \rangle = \int f(t)\varphi(t - x)\,dt = \int f(x + t)\varphi(t)\,dt.$$

Hence the two functions coincide on $\mathbb{R}$, so (3.1) follows by Privalov's theorem (see Koosis [12], Chapter III, §D.3 for the theorem on the disc and Chapter VI for how to transfer the theorem to $\mathbb{C}_+$).

LEMMA 3.4. *Let $f \in \mathrm{Hol}\,(\mathbb{C}_+)$ be of order/type $\leq (2,0)$ and suppose that*

$$\limsup_{y \to \infty} y^{-1} \ln |f(iy)| \leq 0.$$

Given any $\varepsilon > 0$ we then have

$$\limsup_{y \to \infty} y^{-1} \ln \sup_{|x| < \varepsilon} |f(x + iy)| \leq 2\varepsilon. \tag{3.2}$$

PROOF. We only prove (3.2) with the supremum taken over $0 < x < \varepsilon$, the other case being handled in a similar fashion. Let $\eta > 0$, $y_0 > 0$ be arbitrary and consider $e^{i\eta z} f(z + iy_0)$. If we establish (3.2) for this function, it is straightforward to check that (3.2) holds for the original f as well, only with 2ε replaced with $2\varepsilon + \eta$. Since η and y_0 are arbitrary, we may thus assume that f is bounded on $i\mathbb{R}_+$, continuous and of order/type $\leq (2, 0)$ in $\overline{\mathbb{C}}_+$, to begin with. Consider the angle $\Omega = \{z : \pi/4 \leq \arg(z) \leq \pi/2\}$ and the function $f(z)/e^{-iz^2}$. For $z \in i\mathbb{R}_+$ we have $|f(z)/e^{-iz^2}| = |f(z)|$. On the other ray, i.e. $z \in (1 + i)\mathbb{R}_+$, we have

$$\left| f(z)/e^{-iz^2} \right| = |f(z)|e^{-2xy} = |f(z)|e^{-|z|^2},$$

which is bounded since f is of order/type $\leq (2, 0)$. The order of $f(z)/e^{-iz^2}$ is clearly ≤ 2 which is much less than 4. In the notation of Theorem 3.1 we may therefore apply that result to the case $\omega = 1/4$ and $\theta_0 = 3\pi/8$ and conclude that $f(z)/e^{-iz^2}$ is bounded in Ω. Thus

$$|f(z)| \leq C|e^{-iz^2}| = C|e^{-i(x^2 + 2ixy - y^2)}| = Ce^{2xy}.$$

This easily yields the desired statement.

PROOF OF PROPOSITION 3.2. Given $0 < \varepsilon < 1$ pick any $\alpha \in \mathscr{C}_c^\infty$ with $\mathrm{supp}\,\alpha \subset [-\varepsilon, \varepsilon]$ that does not vanish identically, and consider the regularization

$$(f * \alpha)(z) = \int f(z - t)\alpha(t)\,dt.$$

This defines a function in $\mathrm{Hol}\,(\mathbb{C}_+)$ which is of order/type $\leq (2, 0)$ since f is, and moreover

$$|(f * \alpha)(x)| \leq C(1 + |x|)^m \tag{3.3}$$

for some constants $m \in \mathbb{N}$ and $C > 0$. Indeed, we have $|t| \leq \varepsilon + |x| < 1 + |x|$ when $x - t \in \operatorname{supp} \alpha$. Since $f|_{\mathbb{R}}$ is continuous with respect to the topology of $\mathscr{S}$, we can find constants $m, n \in \mathbb{N}$ and $C_{m,n} > 0$ such that

$$|(f * \alpha)(x)| \leq C_{m,n} \sum_{k=0}^{n} \sup_{t} |t^m \partial_t^k \alpha(x - t)|,$$

which gives (3.3). Moreover, for $z \in i\mathbb{R}_+$ we have the estimate

$$(f * \alpha)(iy) \leq \|\alpha\|_{L^1} \sup_{|t| < \varepsilon} |f(iy - t)|.$$

Using assumption (ii) and Lemma 3.4 it is then straightforward to check that

$$\limsup_{y \to \infty} y^{-1} \ln |e^{ib(iy)}(f * \alpha)(iy)| \leq 2\varepsilon.$$

Now, choose any $\beta \in \mathscr{C}_c^\infty$ with support contained in $(-1, 0]$ and consider the function

$$g(z) = e^{i(b+3\varepsilon)z}(f * \alpha)(z)\hat{\beta}(z). \tag{3.4}$$

By construction, g is bounded on both $\mathbb{R}$ and $i\mathbb{R}_+$, so by Theorem 3.1 it is actually bounded in $\mathbb{C}_+$. Pick any $\gamma \in \mathscr{C}_c^\infty$ with support contained in $(-1, 0]$ and note that $\hat{\gamma}g \in H^2 H^\infty \subset H^2$, which by standard Hardy space theory (see Theorem 3.7 below) implies that

$$\operatorname{supp} \mathscr{F}^{-1}(\hat{\gamma}g) \subset \overline{\mathbb{R}}_-,$$

where $\mathscr{F}^{-1}(\hat{\gamma}g)$ should be understood as the inverse Fourier transform of $\hat{\gamma}g$ restricted to $\mathbb{R}$. In other words, $\operatorname{supp}(\gamma * \mathscr{F}^{-1}g) \subset \overline{\mathbb{R}}_-$ which gives $\operatorname{supp} \mathscr{F}^{-1}(g) \subset \overline{\mathbb{R}}_-$ since we can pick any $\gamma \in \mathscr{C}^\infty$ with support in $(-1, 0]$. Inserting (3.4) we get

$$\operatorname{supp}(\mathscr{F}^{-1}(f * \alpha) * \beta)(b + 3\varepsilon + .) \subset \overline{\mathbb{R}}_-$$

which implies that $\operatorname{supp} \mathscr{F}^{-1}(f * \alpha) * \beta \subset (-\infty, b + 3\varepsilon]$ and hence

$$\operatorname{supp} \mathscr{F}^{-1}(f * \alpha) \subset (-\infty, b + 3\varepsilon],$$

since we can pick any $\beta \in \mathscr{C}^\infty$ with support in $(-1, 0]$. Next, note that

$$\mathscr{F}^{-1}(f * \alpha) = 2\pi \mathscr{F}^{-1}(\alpha)\mathscr{F}^{-1}(f) = \hat{\alpha}(-.)\mathscr{F}^{-1}(f).$$

Since $\alpha \in \mathscr{C}_c^\infty$, it follows that $\operatorname{supp} \hat{\alpha} = \mathbb{R}$ since $\hat{\alpha}$ is the restriction to $\mathbb{R}$ of an entire analytic function, and can therefore only have isolated zeros. Hence,

$\mathrm{supp}\,\mathscr{F}^{-1}(f*\alpha) = \mathrm{supp}\,\mathscr{F}^{-1}(f)$. Since $\varepsilon > 0$ was arbitrary, we finally conclude that

$$\mathrm{supp}\,\mathscr{F}^{-1}(f) \subset (-\infty, b].$$

Then $u = \mathscr{F}^{-1}(f)$ belongs to $\mathscr{S}'$ by assumption (iii), so by Lemma 3.3 we conclude that $f(z) = \hat{u}(z)$ for all $z \in \mathbb{C}_+$.

It remains to prove the statements concerning the upper support limit $\mathrm{usl}\,u$. We already have $\mathrm{usl}\,u \leq b$, and the converse inequality follows by (2.12).

3.2. Entire functions

We can now easily provide a slight generalization of the Paley-Wiener-Schwartz Theorem in $\mathbb{C}$. Let $\mathrm{ch}(\Omega)$ denote the *closed convex hull* of a set Ω.

THEOREM 3.5. *Let f be an entire function of order/type $\leq (2, 0)$. Moreover, suppose that $f|_{\mathbb{R}} \in \mathscr{S}'$ and that*

$$\limsup_{y\to\infty} y^{-1}\ln|f(iy)| = b, \quad \limsup_{y\to-\infty} y^{-1}\ln|f(iy)| = a$$

where a, b are finite. Then $\mathrm{ch}(\mathrm{supp}\,\mathscr{F}^{-1}(f|_{\mathbb{R}})) = [a, b]$.

Conversely, let $u \in \mathscr{D}'$ be such that $\mathrm{ch}(\mathrm{supp}\,u) = [a, b]$. Then $f(z) = \langle u, e^{-izt}\rangle$ is of exponential type $\max(|a|, |b|)$ and there are $C, N > 0$ such that

$$|f(z)| \leq C(1 + |z|)^N e^{\max(|a|, |b|)\,\mathrm{Im}\,z}.$$

PROOF. Let f be as in the first part of the theorem. That $\mathrm{ch}(\mathrm{supp}\,\mathscr{F}^{-1}(f)) \subset (-\infty, b]$ is immediate by Proposition 3.2. The other inclusion follows by applying Proposition 3.2 to $z \mapsto \overline{f(\bar{z})}$, since

$$\mathscr{F}^{-1}(\overline{f(\bar{t})})(\xi) = \mathscr{F}^{-1}(\overline{f(t)})(\xi)$$
$$= \overline{\mathscr{F}^{-1}(f(t))(-\xi)} = \overline{\mathscr{F}^{-1}(f|_{\mathbb{R}})(-\xi)}. \quad (3.5)$$

The converse part is an ingredient in the traditional Paley-Wiener-Schwartz Theorem. To prove it, use (2.12) and the flip trick (3.5).

We remark that as a consequence of the theorem we see that any f satisfying the conditions in the first part will automatically be of exponential type. Hence there are no functions with order/type between $(1, \infty)$ and $(2, 0)$ that satisfy the other conditions. The improvement over earlier results in the same category lies in that one does not need to assume f to be of exponential type from the outset. In particular the theorem can be used to prove that a given function

f, for which there is limited information, is indeed of exponential type. This feature is shared by Theorems 3.8 and 3.9 below.

The next example shows that the growth restrictions imposed in Theorem 3.5 cannot be relaxed further, in the sense that $(2, 0)$ is the weakest growth condition which, together with the other conditions, yields the conclusion of the theorem.

EXAMPLE 3.6. Given $\tau > 0$, consider the entire function

$$f(z) = \frac{e^{i\tau z^2} - 1}{z}.$$

It is easy to see that it has order 2 and type τ. Moreover, $f|_{\mathbb{R}} \in L^2$ and

$$\limsup_{y \to \pm\infty} y^{-1} \ln |f(iy)| = 0.$$

However, the conclusion of the Paley-Wiener Theorem is clearly false, since otherwise $\mathscr{F}^{-1}(f|_{\mathbb{R}})$ would be a function in L^2 with support in $\{0\}$.

3.3. Functions on $\mathbb{C}_+$

The setting of Proposition 3.2 is a bit restrictive, since it assumes a priori information about f on $\mathbb{R}$. In most applications, this is not the case. Consider for example the well-known characterization of the Hardy space $H^2(\mathbb{C}_+)$, see e.g. Theorems 11.2 and 11.9 in Duren [3].

THEOREM 3.7. *If $f \in \mathrm{Hol}\,(\mathbb{C}_+)$ satisfies*

$$\sup_{y>0} \| f(. + iy) \|_{L^2(\mathbb{R})} < \infty, \tag{3.6}$$

then $f(. + iy)$ converges in $L^2(\mathbb{R})$ as $y \to 0^+$. Moreover, if we denote the limit by F and set $u = \mathscr{F}^{-1}(F)$, then $\mathrm{supp}\, u \subset \overline{\mathbb{R}}_-$ and

$$f(z) = \hat{u}(z) = \int_{-\infty}^{0} u(t)e^{-izt}\, dt, \quad z \in \mathbb{C}_+.$$

In the same spirit, we now provide a generalization of Proposition 3.2. As an example of potential applications, we show in Section 4 how the assumption (3.6) in Theorem 3.7 can be relaxed.

THEOREM 3.8. *Suppose that $f \in \mathrm{Hol}\,(\mathbb{C}_+)$ satisfies:*

(i) *f restricted to $\{ \mathrm{Im}\, z > y_0 \}$ is of order/type $\leq (2, 0)$ for every $y_0 > 0$,*

(ii) *$b = \limsup_{y\to\infty} y^{-1} \ln |f(iy)|$ is finite,*

(iii) *for some $Y > 0$ we have $f|_{\mathbb{R}+iy} \in \mathscr{S}'$ for all $0 < y \leq Y$.*

Then there exists a distribution $u \in \mathscr{M}'_-$ with $\mathrm{usl}\, u = b$ such that $f = \hat{u}$. In particular, $f \in \mathscr{H}$.

Note that by (2.12) we actually have that f restricted to $\{\, \mathrm{Im}\, z > y_0 \,\}$ is of exponential type; in fact, it has order/type $\leq (1, b)$ independent of y_0. Also note that (i) for $y_0 = 0$ may fail for functions $f \in \mathscr{H}$, that is, they might be unbounded near $\mathbb{R}$ and thus not of order/type $\leq (2, 0)$ in all of $\mathbb{C}_+$. Consider e.g. $u = 1_{\mathbb{R}_-}$, whose Fourier-Laplace transform is

$$\hat{u}(z) = \int_{-\infty}^{0} e^{-izs} ds = \frac{1}{iz}, \quad z \in \mathbb{C}_+.$$

We also stress that this identity is only valid for $\mathrm{Im}\, z > 0$; for $z = x \in \mathbb{R}$ we have $\hat{u} = \mathrm{p.v.}\, (\frac{1}{ix}) - \pi \delta_0$, where δ_0 is the Dirac measure at 0. We will return to this example in Section 4.

PROOF. Set $u_Y = \mathscr{F}^{-1}(f(. + iY))$ and $u = e^{-Yt} u_Y$. We claim that u has the desired properties. To see this, let $0 < y_0 < Y$ be fixed and apply Proposition 3.2 to the function $z \mapsto f(z + iy_0)$, to get a distribution $u_{y_0} \in \mathscr{S}'_-$ with $\mathrm{usl}\, u_{y_0} = b$ and

$$\hat{u}_{y_0}(z) = f(iy_0 + z).$$

By Proposition 2.10 we then have for all $h > 0$

$$\mathscr{F}(e^{ht} u_{y_0})(x) = \hat{u}_{y_0}(x + ih) = f(x + i(y_0 + h)), \tag{3.7}$$

where the first identity is to be interpreted in $\mathscr{S}'$. If $h \leq Y - y_0$ then Proposition 3.2 also gives that $f(. + i(y_0 + h)) = \mathscr{F}(u_{y_0+h})$ as an identity in $\mathscr{S}'$, and thus

$$e^{ht} u_{y_0} = u_{y_0+h} \quad \text{in } \mathscr{S}'.$$

In particular, for $h = Y - y_0$ we have, upon multiplication by e^{-ht}, that $u_{y_0} = e^{(y_0-Y)t} u_Y = e^{y_0 t} u$ as an identity in $\mathscr{D}'$. Since $\mathrm{usl}\, u_{y_0} = b$ the same clearly holds for u. Moreover,

$$e^{y_0 t} u = u_{y_0} \in \mathscr{S}' \tag{3.8}$$

by assumption (iii), so in view of Proposition 2.7 we conclude that $u \in \mathscr{M}'_-$ since $y_0 > 0$ was arbitrary. Finally, given $z = x + iy$ pick any $y_0 < \min(y, Y)$ and note that by Proposition 2.10 and (3.8) we have

$$\hat{u}(z) = \langle u, e^{-izt} \rangle = \langle u_{y_0}, e^{-i(z-iy_0)t} \rangle = \hat{u}_{y_0}(z - iy_0) = f(z),$$

where the last identity follows by (3.7). The proof is complete.

3.4. The Paley-Wiener-Schwartz Theorem in $\mathbb{C}^n$

THEOREM 3.9. *Let $f \in \mathrm{Hol}\,(\mathbb{C}^n)$ be an entire function of order/type $\leq (2, 0)$ such that $f|_{\mathbb{R}^n} \in \mathscr{S}'$. Let $v_1, \ldots, v_n$ be a basis of $\mathbb{R}^n$ and, given $j \in \{1, \ldots, n\}$, let $\mathscr{V}_j \subset \mathbb{R}^n$ be the span of $\{v_k\}_{k \neq j}$. Suppose there exists a number $B > 0$ such that for each j and $R > 0$ we have*

$$\lim_{y \to \pm\infty} \sup_{\{x \in \mathscr{V}_j : |x| < R\}} y^{-1}\ln|f(iyv_j + x)| \leq B.$$

Then $\mathscr{F}(f|_{\mathbb{R}^n}) \in \mathscr{E}'$.

Note that once it is established that f is the Fourier transform of a compactly supported distribution, the classical Paley-Wiener-Schwartz Theorem already gives a precise relation between the actual growth and the support. We refer the reader e.g. to Hörmander [9, Theorem 7.3.1] or Trèves [22, Theorem 29.1]. For the proof we need the following standard lemmas, which are just slight variations of Theorems 1.3.2 and 4.1.2 in Hörmander [9].

LEMMA 3.10. *Let $\gamma \in \mathscr{S}$ be such that $\int \gamma = 1$, and set $\gamma_k(x) = k\gamma(kx)$, $k \in \mathbb{N}$. Then, given any $\varphi \in \mathscr{S}$, we have $\lim_{k \to \infty} \gamma_k * \varphi = \varphi$ in the topology of $\mathscr{S}$.*

LEMMA 3.11. *Let $\varphi, \psi \in \mathscr{S}$ and $u \in \mathscr{S}'$ be given. Then*

$$\langle u, \psi * \varphi \rangle = \int \langle u, \varphi(.-x) \rangle \psi(x)\, dx.$$

PROOF OF THEOREM 3.9.   By a change of variables we may assume that $\{v_j\}_{j=1}^n$ is the canonical basis of $\mathbb{R}^n$. Given any $\psi' \in \mathscr{C}_c^\infty(\mathbb{R}^{n-1})$, let $z \in \mathbb{C}$ and $x' \in \mathbb{R}^{n-1}$ be independent variables and consider

$$g(z) = \int_{\mathbb{R}^{n-1}} f(z, x')\psi'(x')\, dx'.$$

By Theorem 3.5 it follows that $\mathscr{F}(g|_{\mathbb{R}})$ is a distribution with support in $[-B, B]$. Thus, given any $\psi_1 \in \mathscr{C}_c^\infty$ with $\mathrm{supp}\,\psi_1 \subset \mathbb{R} \setminus [-B, B]$ we have

$$\begin{aligned}
0 = \int_{\mathbb{R}} g(x_1)\hat{\psi}_1(x_1)\, dx_1 &= \int_{\mathbb{R}^n} f(x_1, x')\hat{\psi}_1(x_1)\psi'(x')\, dx \\
&= \langle \hat{f}, \psi_1 \otimes \mathscr{F}^{-1}(\psi') \rangle.
\end{aligned} \tag{3.9}$$

Now suppose that

$$\mathrm{supp}\,\hat{f} \not\subset [-B, B] \times \mathbb{R}^{n-1}, \tag{3.10}$$

where $\hat{f}$ denotes $\mathscr{F}(f|_{\mathbb{R}^n})$. Choose a $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ such that $\langle \hat{f}, \varphi \rangle \neq 0$ and

$$\operatorname{supp} \varphi \cap ([-B, B] \times \mathbb{R}^{n-1}) = \emptyset. \tag{3.11}$$

Let $\gamma \in \mathscr{S}(\mathbb{R}^n)$ be of the form $\gamma(x_1, x') = \phi_1(x_1)\mathscr{F}^{-1}(\phi')(x')$, where $\phi_1 \in \mathscr{C}_c^\infty(\mathbb{R})$ and $\phi' \in \mathscr{C}_c^\infty(\mathbb{R}^{n-1})$ are chosen such that $\operatorname{supp}\phi_1 \subset [-1, 1]$ and $\int \gamma = 1$. We now construct $(\gamma_k)_{k=1}^\infty$ as in Lemma 3.10. By Lemmas 3.10 and 3.11 we then have

$$0 \neq \langle \hat{f}, \varphi \rangle = \lim_{k \to \infty} \langle \hat{f}, \varphi * \gamma_k \rangle = \lim_{k \to \infty} \int \langle \hat{f}, \gamma_k(. - x) \rangle \varphi(x)\,dx. \tag{3.12}$$

However, for each fixed $x = (x_1, x') \in \mathbb{R}^n$, $\gamma_k(. - x)$ can be written on the form $\psi_1 \otimes \mathscr{F}^{-1}(\psi')$ where $\psi_1 = \phi_1(k(. - x_1))$. This function has support

$$\operatorname{supp}\psi_1 = \{x_1 + s/k : s \in \operatorname{supp}\phi_1\} \subset [x_1 - 1/k, x_1 + 1/k].$$

By (3.11) it is clear that K can be chosen such that $\operatorname{supp}\psi_1 \subset \mathbb{R} \setminus [-B, B]$ for all $k \geq K$ and all $x \in \operatorname{supp}\varphi$. By (3.9) it follows that $\langle \hat{f}, \gamma_k(. - x) \rangle = 0$ for all such x and k. But this is impossible by (3.12), and hence we conclude that (3.10) is false. The same argument can of course be carried out with respect to any of the other coordinate axes, yielding

$$\operatorname{supp}\hat{f} \subset [-B, B]^n,$$

and the proof is complete.

4. Applications and examples

We recall from Definition 2.11 that $\mathscr{H}$ denotes the class of all functions of the form $z \mapsto \hat{u}(z)$ for some $u \in \mathscr{M}'_-$. We show in this section that functions in $\mathscr{H}$ can be quite ill-behaved near $\mathbb{R}$, whereas they are quite stable away from it.

As a first example, let us consider $f(z) = z^{-1}$, which is clearly in $\mathscr{H}$ by Theorem 3.8. Moreover, by the same result there is a $u \in \mathscr{M}'_-$ such that $\hat{u}(z) = z^{-1}$ and $\operatorname{usl} u = 0$. This may seem to be a contradiction at first, since it is well-known that $\mathscr{F}(\frac{i}{2}\operatorname{sgn}(t))(x) = x^{-1}$, where $x \in \mathbb{R}$ and sgn denotes the sign function, which clearly is not supported on $\mathbb{R}_-$. However, with $u(t) = -i\mathbb{1}_{\mathbb{R}_-}(t)$ we also have $\hat{u}(z) = z^{-1}$ in $\mathbb{C}_+$, as expected by Theorem 3.8. The slight confusion arises from the fact that

$$\frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\pi P_{iy},$$

where P_{iy} is the Poisson kernel. Thus, as a family of distributions acting in the x variable, $y \mapsto f(x + iy)$ does not converge to the function $f(x)$ as

$y \to 0^+$, but to the distribution $f(x) - i\pi\delta_0$. Compare with Theorem 3.1.12 and Example 3.1.13 in Hörmander [9]. This shows that the growth near $\mathbb{R}$ of an $f \in \mathscr{H}$ can be quite strong. However, as long as $u \in \mathscr{S}'_-$, there are $C, m, n \in \mathbb{N}$ such that $\hat{u}(x + iy) \le C\langle x \rangle^n y^{-m}$ for $0 < y < 1$, see e.g. the presentation by Vladimirov [23, Section II.12] or Shambayati and Zielezny [21, Theorem 2]. Functions in $\mathscr{H}$ can have much more erratic behavior near $\mathbb{R}$, consider e.g. $e^{i/z}$ (which is in $\mathscr{H}$ by Theorem 3.8).

As an illustration of the possible applications of the material in Section 3, we now show how it can be used to obtain softer conditions for checking membership in classical Hardy spaces.

PROPOSITION 4.1. *Let* $1 \le p \le \infty$ *be given. Suppose that* $f \in \mathrm{Hol}\,(\mathbb{C}_+)$ *satisfies:*

  (i) *f restricted to* $\{\,\mathrm{Im}\, z > y_0\,\}$ *is of order/type* $\le (2, 0)$ *for every* $y_0 > 0$,

  (ii) $\limsup_{y \to \infty} y^{-1} \ln |f(iy)| \le 0$,

 (iii) *there are* Y *and* C *such that* $\|f|_{\mathbb{R}+iy}\|_{L^p} \le C$ *for all* $0 < y < Y$.

Then $f \in H^p(\mathbb{C}_+)$.

PROOF. By Theorem 3.8 we immediately get that $f \in \mathscr{H}$. We need to show that $\|f\|_{H^p} = \sup_{y>0} \|f|_{\mathbb{R}+iy}\|_{L^p} < \infty$. It clearly suffices to show this for the function $z \mapsto f(iY/2 + z)$, and hence we can simply assume that f is holomorphic in a neighborhood of $\mathbb{C}_+$, $f|_{\mathbb{R}} \in L^p$ and that f has polynomial growth in $\overline{\mathbb{C}}_+$ by (2.12). It follows that we can fix an $N \in \mathbb{N}$ such that $f_\varepsilon(z) = (1 - i\varepsilon z)^{-N} f(z)$ belongs to $H^\infty(\mathbb{C}_+)$ for all $\varepsilon > 0$. This means that we can express $f_\varepsilon(. + iy)$ as a Poisson integral (see e.g. Duren [3, Theorem 11.2]), i.e.

$$f_\varepsilon(x + iy) = \int f_\varepsilon(x - t) P_{iy}(t)\, dt. \qquad (4.1)$$

If we now assume that $p < \infty$, then it is clear that $\lim_{\varepsilon \to 0^+} f_\varepsilon|_{\mathbb{R}} = f|_{\mathbb{R}}$ in L^p, so upon taking this limit on both sides in (4.1), we see that this equation is valid also for $f = f_0$. This immediately gives that $f \in H^p$ and $\|f\|_{H^p} \le C$ by Young's inequality. When $p = \infty$ the same argument applies, although $\lim_{\varepsilon \to 0^+} f_\varepsilon = f$ only holds in the weak-* topology of L^∞.

Finally, we remark that $\mathscr{H}$ contains functions f that extend analytically across $\mathbb{R}$, while at the same time $f(. + iy) \notin \mathscr{S}'$ for values $y < 0$. More precisely, we can pick an entire function f such that $f|_{\mathbb{C}_+} \in H^\infty(\mathbb{C}_+)$, but $f|_{-iy+\mathbb{R}} \notin \mathscr{S}'$ for some $y < 0$. To see an example of this, consider the function $f(z) = \int_0^\infty t^{-t} e^{zt}\, dt$. In Newman [15] it is shown that this is an entire function with the curious property of being unbounded in the strip $-\pi/2 < \mathrm{Im}\, z < \pi/2$,

whereas it is bounded elsewhere. In fact, it is easy to see that it grows extremely rapidly on $\mathbb{R}$; the integrand increases until the point $t = e^{x-1}$ so we can estimate

$$f(x) \geq \int_{e^{x-2}}^{e^{x-1}} e^{(x-\ln e^{x-2})e^{x-2}} \, dt = (e^{-1} - e^{-2})e^{x+2e^{x-2}}.$$

Summing up, we have that $f(z + iy) \in \mathscr{H}$ for all $y > \pi/2$, but at some value of y between $\pi/2$ and 0 it ceases to belong to $\mathscr{H}$.

Appendix A

Throughout the appendix we will repeatedly use that $e^{\varepsilon\langle t\rangle}$ is an *order function* in the sense of Martinez [14, p. 11], i.e. one which satisfies $(d/dt)^k e^{\varepsilon\langle t\rangle} = \mathcal{O}(e^{\varepsilon\langle t\rangle})$ for any $k \in \mathbb{N}$, uniformly on $\mathbb{R}$, see (2.5). As before we write $\varphi^{(k)}$ for the kth derivative of $\varphi \in \mathscr{C}^k$. (All function spaces are assumed to be defined on $\mathbb{R}$ unless explicitly stated otherwise.)

LEMMA A.1. *The topology of $\mathscr{S}_\varepsilon$ is given by the semi-norms*

$$p_{\varepsilon,m,n}(\varphi) = \sup_t \left| \langle t\rangle^m e^{\varepsilon\langle t\rangle} \varphi^{(n)}(t) \right|, \quad m, n \in \mathbb{N}.$$

PROOF. Note that the semi-norms $\{p_{0,m,n}\}_{m,n}$ give the topology of $\mathscr{S}$. By definition, $\mathscr{S}_\varepsilon$ has the topology such that the map $\iota_\varepsilon : \mathscr{S}_\varepsilon \to \mathscr{S}$ given by $\iota(\varphi) = e^{\varepsilon\langle t\rangle} \varphi$ is an isomorphism, which means that the topology of $\mathscr{S}_\varepsilon$ is given by the semi-norms

$$\varphi \mapsto p_{0,m,n}(e^{\varepsilon\langle t\rangle} \varphi) = \sup_t \left| \langle t\rangle^m \frac{d^n}{dt^n} \left(e^{\varepsilon\langle t\rangle} \varphi(t) \right) \right|, \quad m, n \in \mathbb{N}.$$

The proof is complete upon showing that a fixed element in this class of semi-norms can be bounded by a finite sum of the semi-norms in $\{p_{\varepsilon,m,n}\}_{m,n}$, and vice versa. That this is the case follows by routine estimates which we omit.

PROOF OF PROPOSITION 2.2. We first show that the identity map I is continuous from $\mathscr{C}_c^\infty$ into $\mathscr{M}$. Recall that the topology of $\mathscr{C}_c^\infty$ is given e.g. by the inductive limit of $\mathscr{C}_c^\infty([-N, N])$, $N \in \mathbb{N}$. By Trèves [22, Proposition 13.1] we only need to show that I restricted to this subspace is continuous. Since $\mathscr{C}_c^\infty([-N, N]) \subset \mathscr{S}_\varepsilon$ for any $\varepsilon > 0$, and since the restriction of the inductive limit topology on $\mathscr{M}$ to $\mathscr{S}_\varepsilon$ is the same as the original topology (Trèves [22, Lemma 13.1]), it suffices to show that $I : \mathscr{C}_c^\infty([-N, N]) \to \mathscr{S}_\varepsilon$ is continuous for any fixed ε. For all $\varphi \in \mathscr{C}_c^\infty([-N, N])$ we clearly have

$$p_{\varepsilon,m,n}(\varphi) = \sup_t \left| \langle t\rangle^m e^{\varepsilon\langle t\rangle} \varphi^{(n)}(t) \right| \leq e^{\varepsilon\langle N\rangle} p_{0,m,n}(\varphi)$$

and hence we are done by Lemma A.1 and the fact that the latter semi-norms define the topology of $\mathscr{C}_c^\infty([-N, N])$, see Example II in Trèves [22, Chapter 13]. Similarly, to show that $I : \mathscr{M} \to \mathscr{S}$ is continuous it suffices according to Trèves [22, Proposition 13.1] to show that $I : \mathscr{S}_\varepsilon \to \mathscr{S}$ is continuous, which is immediate by Lemma A.1 and the obvious inequality $p_{0,m,n}(\varphi) \leq p_{\varepsilon,m,n}(\varphi)$.

Finally, we turn to the statements concerning density. It is well-known that $\mathscr{C}_c^\infty$ is dense in $\mathscr{S}$ (see Hörmander [9, Lemma 7.1.8]). It immediately follows that $\mathscr{C}_c^\infty$ is dense in each $\mathscr{S}_\varepsilon$, $\varepsilon > 0$, since $\iota_{-\varepsilon}$ is an isomorphism from $\mathscr{S}$ to $\mathscr{S}_\varepsilon$ and $\iota_{-\varepsilon}(\mathscr{C}_c^\infty) \subset \mathscr{C}_c^\infty$. To prove the corresponding statement concerning $\mathscr{M}$, let M be an open set in $\mathscr{M}$. Then $M \cap \mathscr{S}_\varepsilon$ is open in $\mathscr{S}_\varepsilon$ and non-void for sufficiently small ε, by definition of the inductive limit topology, and hence $M \cap \mathscr{C}_c^\infty \neq \emptyset$, as desired.

PROOF OF PROPOSITION 2.3.   The first statement is an immediate consequence of Lemma A.1, whereas the second follows by the definition of $\mathscr{M}$ and the fact that $\mathscr{O}_M \cdot \mathscr{S} \subset \mathscr{S}$.

PROOF OF PROPOSITION 2.4.   By the corollary to Theorem 33.1 in Trèves [22] (a general version of the Banach-Steinhaus Theorem), u_k converges uniformly on compacts to u and u is continuous, i.e. $u \in \mathscr{V}'$. Write

$$\langle u, \varphi \rangle - \langle u_k, \varphi_k \rangle = \langle u, \varphi - \varphi_k \rangle + \langle u - u_k, \varphi_k \rangle.$$

By the continuity of u, the first bracket goes to zero as $k \to \infty$, and the same is true for the second as well since $\{\varphi_j\}_{j=1}^\infty$ is a compact set.

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MATHEMATICS
FACULTY OF SCIENCE
LUND UNIVERSITY
P.O. BOX 118
SE-221 00 LUND
SWEDEN
E-mail: marcus.carlsson@math.lu.se

GRADUATE SCHOOL OF HUMAN AND
ENVIRONMENTAL STUDIES
KYOTO UNIVERSITY
YOSHIDA NIHONMATSU-CHO
SAKYO-KU
KYOTO 606-8501
JAPAN
E-mail: jensw@maths.lth.se