MATH. SCAND. 120 (2017), 195–210

FINITE-RANK BRATTELI-VERSHIK DIAGRAMS ARE EXPANSIVE – A NEW PROOF

SIRI-MALÉN HØYNES

Abstract

Downarowicz and Maass (Ergod. Th. and Dynam. Sys. 28 (2008), 739–747) proved that the Cantor minimal system associated to a properly ordered Bratteli diagram of finite rank is either an odometer system or an expansive system. We give a new proof of this truly remarkable result which we think is more transparent and easier to understand. We also address the question (Question 1) raised by Downarowicz and Maass and we find a better (i.e. lower) bound. In fact, we conjecture that the bound we have found is optimal.

1. Introduction

The aim of this paper is to give a new proof of the following result.

THEOREM 1.1. *Let $(V, E, \geq)$ be a properly ordered Bratteli diagram, and let (V, E) be of finite rank. Then the associated Bratteli-Vershik system $(X_{(V,E)}, T_{(V,E)})$ is either an odometer system or an expansive system.*

REMARK 1.2. It is well known that an expansive Cantor minimal system is (conjugate to) a minimal subshift on a finite alphabet (cf. Proposition 2.9). Theorem 1.1 implies that if $(V, E \geq)$ is a properly ordered Bratteli diagram of finite rank *and* (V, E) has the ERS-property (cf. Section 2), then $(X_{(V,E)}, T_{(V,E)})$ is either an odometer or a Toeplitz flow [5].

In our judgement the proof given of Theorem 1.1 in [1] is not easy to follow, so we feel that a more transparent proof – thus hopefully making it more accessible – is in order for such an important and, frankly speaking, rather surprising result. We also address the question (Question 1) that is raised in [1] about finding a better (i.e. lower) bound than the one they give in their "Infection Lemma", and we do indeed find a significantly lower bound (cf. Corollary 4.3), which we conjecture is optimal.

We will adopt some of the definitions and terminology from [1], but in contrast to [1] we interpret the definitions directly in terms of the properly ordered Bratteli diagrams in question. We feel this makes it much easier to

Received 25 November 2014.
DOI: https://doi.org/10.7146/math.scand.a-25613

grasp the contents of the various definitions. (Cf. also Remark 4.4.) We strongly emphasize that our proof is very much inspired and motivated by the proof in [1].

2. Bratteli diagrams and Bratteli-Vershik systems

General references for this section are [3] and [2, Section 3]. A Bratteli diagram (V, E) consists of a set of vertices $V = \bigsqcup_{n=0}^{\infty} V_n$ and a set of edges $E = \bigsqcup_{n=1}^{\infty} E_n$, where the V_n's and the E_n's are finite disjoint sets and $V_0 = \{v_0\}$ is a one-point set. The edges in E_n connect vertices in V_{n-1} with vertices in V_n. If e connects $v \in V_{n-1}$ with $u \in V_n$, we write $s(e) = v$ and $r(e) = u$, where $s \colon E_n \to V_{n-1}$ and $r \colon E_n \to V_n$ are the source and range maps, respectively. We will assume that $s^{-1}(v) \neq \varnothing$ for all $v \in V$ and that $r^{-1}(v) \neq \varnothing$ for all $v \in V \setminus V_0$. A Bratteli diagram can be given a diagrammatic presentation with V_n the vertices at level n and E_n the edges between V_{n-1} and V_n. If $|V_{n-1}| = t_{n-1}$ and $|V_n| = t_n$, then the edge set E_n is described by a $t_n \times t_{n-1}$ incidence matrix $M_n = (m_{ij}^n)$, where m_{ij}^n is the number of edges connecting $v_i^n \in V_n$ with $v_j^{n-1} \in V_{n-1}$ (see Figure 1). If the row sums are constant for every M_n, then we say that (V, E) has the ERS ("equal row sum") property.

$$M_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 5 & 2 \\ 4 & 1 \\ 1 & 1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

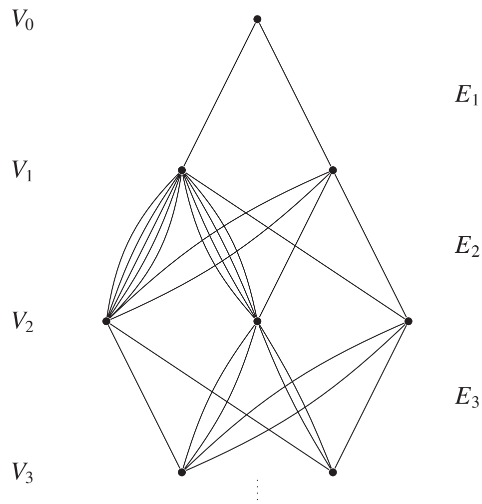

FIGURE 1. An example of a Bratteli diagram.

Let $k, \ell \in \mathbb{Z}^+$ with $k < \ell$ and let $E_{k+1} \circ E_k \circ \cdots \circ E_\ell$ denote all the paths from V_k to V_ℓ. Specifically, $E_{k+1} \circ E_k \circ \cdots \circ E_\ell = \{(e_{k+1}, \ldots, e_\ell) \mid e_i \in E_i, i = k+1, \ldots, \ell;\ r(e_i) = s(e_{i+1}), i = k+1, \ldots, \ell - 1\}$. We define $r((e_{k+1}, \ldots, e_\ell)) = r(e_\ell)$ and $s((e_{k+1}, \ldots, e_\ell)) = s(e_{k+1})$. Notice that the corresponding incidence matrix is the product $M_\ell M_{\ell-1} \cdots M_{k+1}$ of the individual incidence matrices.

DEFINITION 2.1. Given a Bratteli diagram (V, E) and a sequence $0 = m_0 < m_1 < m_2 < \cdots$ in $\mathbb{Z}^+$, we define the *telescoping* of (V, E) to $\{m_n\}$ as (V', E'), where $V'_n = V_{m_n}$ and $E'_n = E_{m_{n-1}+1} \circ \cdots \circ E_{m_n}$, and the source and the range maps are as above.

DEFINITION 2.2. The Bratteli diagram (V, E) is of *finite rank* if $|V_n| \leq L < \infty$ for all n. By telescoping we may assume that $|V_n| = K$ for all $n = 1, 2, \ldots$. We then say that (V, E) is of *rank* K, and write $\operatorname{rank}(V, E) = K$.

DEFINITION 2.3. We say that the Bratteli diagram (V, E) is *simple* if there exists a telescoping of (V, E) such that the resulting Bratteli diagram (V', E') has full connection between all consecutive levels, i.e. the entries of all the incidence matrices are non-zero.

Given a Bratteli diagram (V, E) we define the infinite path space associated to (V, E) by

$$X_{(V,E)} = \{(e_1, e_2, \ldots) \mid e_i \in E_i, r(e_i) = s(e_{i+1}), \forall i \geq 1\}.$$

Clearly $X_{(V,E)} \subseteq \prod_{n=1}^{\infty} E_n$, and we give $X_{(V,E)}$ the relative topology, where $\prod_{n=1}^{\infty} E_n$ has the product topology. Loosely speaking this means that two paths in $X_{(V,E)}$ are close if the initial parts of the two paths agree on a long initial stretch. Also, $X_{(V,E)}$ is a closed subset of $\prod_{n=1}^{\infty} E_n$, and so is compact.

Let $p = (e_1, e_2, \ldots, e_n) \in E_1 \circ \cdots \circ E_n$ be a finite path starting at $v_0 \in V_0$. We define the *cylinder set* $U(p) = \{(f_1, f_2, \ldots) \in X_{(V,E)} \mid f_i = e_i, i = 1, 2, \ldots, n\}$. The collection of cylinder sets is a basis for the topology on $X_{(V,E)}$. The cylinder sets are clopen sets, and so $X_{(V,E)}$ is a compact, totally disconnected metric space. An admissable metric d yielding the topology is $d(x, x') = 1/n$ if $x = (e_1, e_2, \ldots, e_{n-1}, e_n, \ldots)$, $y = (e_1, e_2, \ldots, e_{n-1}, e'_n, \ldots)$, where $e_n \neq e'_n$. If (V, E) is simple then $X_{(V,E)}$ has no isolated points, and so $X_{(V,E)}$ is a Cantor set. (We will in the sequel disregard the trivial case where $|X_{(V,E)}|$ is finite.)

Let $P_n = E_1 \circ \cdots \circ E_n$ be the set of finite paths of length n (starting at the top vertex). We define the truncation map $\tau_n \colon X_{(V,E)} \to P_n$ by $\tau_n((e_1, e_2, \ldots)) = (e_1, e_2, \ldots, e_n)$. If $m \geq n$ we have the obvious truncation map $\tau_{m,n} \colon P_m \to P_n$.

There is an obvious notion of isomorphism between Bratteli diagrams (V, E) and (V', E'); namely, there exists a pair of bijections between V and V' preserving the gradings and intertwining the respective source and range maps. Let $\sim$ denote the equivalence relation on Bratteli diagrams generated by isomorphism and telescoping. One can show that $(V, E) \sim (V', E')$ if and only if there exists a Bratteli diagram (W, F) such that telescoping (W, F) to odd levels $0 < 1 < 3 < \cdots$ yields a diagram isomorphic to some telescoping of (V, E), and telescoping (W, F) to even levels $0 < 2 < 4 < \cdots$ yields a diagram isomorphic to some telescoping of (V', E').

An *ordered Bratteli diagram* $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ in E so that edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. In other words, we have a linear order on each set $r^{-1}(v)$, $v \in V \setminus V_0$. Assume $|r^{-1}(v)| = m$ and the edge $f \in r^{-1}(v)$ has order k, where $1 \leq k \leq m$. Then we will say that f has *ordinal* k, and we will write $\mathrm{ordinal}(f) = k$. (In Figure 5 this is illustrated; the edge f shown there has ordinal 5, so $\mathrm{ordinal}(f) = 5$.) We let $E_{\min}$ and $E_{\max}$, respectively, denote the minimal and maximal edges of the partially ordered set E.

Note that if $(V, E, \geq)$ is an ordered Bratteli diagram and $k < \ell$ in $\mathbb{Z}^+$, then the set $E_{k+1} \circ E_{k+2} \circ \cdots \circ E_\ell$ of paths from V_k to V_ℓ with the same range can be given an induced (lexicographic) order as follows:

$$(e_{k+1} \circ e_{k+2} \circ \cdots \circ e_\ell) > (f_{k+1} \circ f_{k+2} \circ \cdots \circ f_\ell)$$

if for some i with $k+1 \leq i \leq \ell$, $e_j = f_j$ for $i < j \leq \ell$ and $e_i > f_i$. If (V', E') is a telescoping of (V, E) then, with this induced order from $(V, E, \geq)$, we get again an ordered Bratteli diagram $(V', E', \geq)$.

DEFINITION 2.4. We say that the ordered Bratteli diagram $(V, E, \geq)$, where (V, E) is a simple Bratteli diagram, is *properly ordered* if there exists a unique min path $x_{\min} = (e_1, e_2, \ldots)$ and a unique max path $x_{\max} = (f_1, f_2, \ldots)$ in $X_{(V,E)}$. (That is, $e_i \in E_{\min}$ and $f_i \in E_{\max}$ for all $i = 1, 2, \ldots$.)

Let (V, E) be a properly ordered Bratteli diagram, and let $X_{(V,E)}$ be the path space associated to (V, E). Then $X_{(V,E)}$ is a Cantor set. Let $T_{(V,E)}$ be the *lexicographic map* on $X_{(V,E)}$, i.e. if $x = (e_1, e_2, \ldots) \in X_{(V,E)}$ and $x \neq x_{\max}$ then $T_{(V,E)}x$ is the successor of x in the lexicographic ordering. Specifically, let k be the smallest natural number so that $e_k \notin E_{\max}$. Let f_k be the successor of e_k (and so $r(e_k) = r(f_k)$). Let $(f_1, f_2, \ldots, f_{k-1})$ be the unique least element in $E_1 \circ E_2 \circ \cdots \circ E_{k-1}$ from $s(f_k) \in V_{k-1}$ to the top vertex $v_0 \in V_0$. Then $T_{(V,E)}((e_1, e_2, \ldots)) = (f_1, f_2, \ldots, f_k, e_{k+1}, e_{k+2}, \ldots)$. We define $T_{(V,E)}x_{\max} = x_{\min}$. Then it is easy to check that $T_{(V,E)}$ is a minimal homeomorphism on $X_{(V,E)}$. We note that if $x \neq x_{\max}$ then x and $T_{(V,E)}x$ are *cofinal*, i.e. the edges making up x and $T_{(V,E)}x$, respectively, agree from a certain level on. We will call the Cantor minimal system $(X_{(V,E)}, T_{(V,E)})$ a *Bratteli-Vershik system*. There is an obvious way to telescope a properly ordered Bratteli diagram, getting another properly ordered Bratteli diagram, such that the associated Bratteli-Vershik systems are conjugate (cf. Definition 2.10) – the map implementing the conjugacy is the obvious one. By telescoping we may assume without loss of generality that the properly ordered Bratteli diagram has the property that at each level all the minimal edges (respectively the maximal edges) have the same source, cf. [3, Proposition 2.8].

We use the term *dynamical system* to mean a compact metric space X together with a homeomorphism $T\colon X \to X$, and we will denote this by (X, T). We say (X, T) is *minimal* if all T-orbits are dense. (Equivalently $T(A) = A$ for some closed $A \subseteq X$ implies that $A = X$ or $A = \varnothing$.) If X is a Cantor set and T is minimal, then we say that (X, T) is a Cantor minimal system.

THEOREM 2.5 ([3]). *Let (X, T) be a Cantor minimal system. Then there exists a properly ordered Bratteli diagram $(V, E, \geq)$ such that the associated Bratteli-Vershik system $(X_{(V,E)}, T_{(V,E)})$ is conjugate to (X, T).*

REMARK 2.6. The simplest Bratteli-Vershik model $(V, E, \geq)$ for the odometer (see below) (G_α, T) associated to $\alpha = (a_i)_{i\in\mathbb{N}}$ is obtained by letting $V_n = 1$ for all n, and the number of edges between V_{n-1} and V_n be a_n.

Let $(G_\alpha, \rho_{\widehat{\mathbb{1}}})$ denote the *odometer* (also called *adding machine*) associated to the α-adic group

$$G_\alpha = \prod_{i=1}^{\infty}\left\{0, 1, \ldots, \frac{p_i}{p_{i-1}} - 1\right\},$$

where $\alpha = \{p_i/p_{i-1}\}_{i\in\mathbb{N}}$ (we set $p_0 = 1$) and where $\rho_{\widehat{\mathbb{1}}}(x) = x + \widehat{1}$, where $\widehat{1} = (1, 0, 0, \ldots)$. We note that G_α is naturally isomorphic to the inverse limit group

$$\mathbb{Z}/p_1\mathbb{Z} \xleftarrow{\phi_1} \mathbb{Z}/p_2\mathbb{Z} \xleftarrow{\phi_2} \mathbb{Z}/p_3\mathbb{Z} \xleftarrow{\phi_3} \cdots,$$

where $\phi_i(n)$ is the residue of n modulo p_i. It is a fact that the family consisting of compact groups G that are both monothetic (i.e. contains a dense copy of $\mathbb{Z}$, which of course implies that G is abelian) and Cantor (as a topological space), coincides with the family of α-adic groups. It is also noteworthy that all minimal rotations (in particular rotations by $\widehat{1}$) on such groups are conjugate. This is a consequence of the fact that the dual group of an α-adic group is a torsion group. If $\alpha = \{p\}_{i\in\mathbb{N}}$, where p is a prime, then G_α is the p-adic integers. (We refer to [4, Vol. 1] for background information on α-adic groups.)

REMARK 2.7. It is well known, and easy to prove, that the Cantor minimal system (X, T) is conjugate (cf. Definition 2.10) to an odometer if and only if it is the inverse limit of a sequence of periodic systems.

DEFINITION 2.8. (X, T) is *expansive* if there exists $\delta > 0$ such that if $x \neq y$ then $\sup_{n\in\mathbb{Z}} \mathrm{d}(T^n x, T^n y) > \delta$, where d is a metric that gives the topology of X.

(Expansiveness is independent of the metric as long as the metric gives the topology of X.)

Let $\Lambda = \{a_1, a_2, \dots, a_n\}$, $n \geq 2$, be a finite alphabet and let $Z = \Lambda^{\mathbb{Z}}$ be the set of all bi-infinite sequences of symbols from Λ with Z given the product topology – thus Z is a Cantor set. Let $S\colon Z \to Z$ denote the shift map, $S\colon (x_n) \to (x_{n+1})$. If X is a closed subset of Z such that $S(X) = X$, we say that (X, S) is a *subshift*, where we denote the restriction of S to X again by S. Subshifts are easily seen to be expansive. We state the following well-known fact as a proposition. (Cf. [6, Theorem 5.24].)

PROPOSITION 2.9. *Let (X, T) be a Cantor minimal system. Then (X, T) is conjugate to a minimal subshift on a finite alphabet if and only if (X, T) is expansive.*

DEFINITION 2.10. We say that a dynamical system (Y, S) is a *factor* of (X, T) and that (X, T) is an *extension* of (Y, S) if there exists a continuous surjection $\pi\colon X \to Y$ which satisfies $S(\pi(x)) = \pi(Tx)$, $\forall x \in X$. We call π a *factor map*. If π is a bijection then we say that (X, T) and (Y, S) are *conjugate*, and we write $(X, T) \cong (Y, S)$.

Let $(V, E, \geq)$ be a properly ordered Bratteli diagram, and let $(X_{(V,E)}, T_{(V,E)})$ be the associated Bratteli-Vershik system. For each $k \in \mathbb{N}$, let P_k as above denote the paths from V_0 to V_k, i.e. the paths from $v_0 \in V_0$ to some $v \in V_k$. For $x \in X_{(V,E)}$ we associate the bi-infinite sequence $\pi_k(x) = \big(\tau_k(T^n_{(V,E)}x)\big)_{n=-\infty}^{\infty} \in P_k^{\mathbb{Z}}$ over the finite alphabet P_k, where $\tau_k\colon X_{(V,E)} \to P_k$ is the truncation map. Let S_k denote the shift map on $P_k^{\mathbb{Z}}$. Then the following diagram commutes

$$\begin{tikzcd} X_{(V,E)} \arrow[r, "T_{(V,E)}"] \arrow[d, "\pi_k"'] & X_{(V,E)} \arrow[d, "\pi_k"] \\ X_k \arrow[r, "S_k"] & X_k \end{tikzcd}$$

where $X_k = \pi_k(X_{(V,E)})$. One observes that π_k is a continuous map, and so X_k is a compact shift-invariant subset of $P_k^{\mathbb{Z}}$. So (X_k, S_k) is a factor of $(X_{(V,E)}, T_{(V,E)})$, and hence (X_k, S_k) is minimal. For $k > \ell$ there is an obvious factor map $\pi_{k,\ell}\colon X_k \to X_\ell$, and one can show that $(X_{(V,E)}, T_{(V,E)})$ is the inverse limit of the system $\{(X_k, S_k)\}_{k\in\mathbb{N}}$. We write $(X_{(V,E)}, T_{(V,E)}) = \varprojlim (X_k, S_k)$. All the systems (X_k, S_k) are clearly expansive. One has the following result which will be important for us. (See the remarks prededing Theorem 1 of [1].)

PROPOSITION 2.11. *Assume $(X_{(V,E)}, T_{(V,E)})$ is expansive. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $(X_{(V,E)}, T_{(V,E)})$ is conjugate to (X_k, S_k) by the map $\pi_k\colon X_{(V,E)} \to X_k$.*

PROOF. Since the π_k's are factor maps, all we need to show is that there exists k_0 such that π_k is injective for all $k \geq k_0$. Recall that $(X_{(V,E)}, T_{(V,E)})$ being expansive means that there exists $\delta > 0$ such that given $x \neq y$ there exists $n_0 \in \mathbb{Z}$ such that $d(T^{n_0}_{(V,E)}x, T^{n_0}_{(V,E)}y) > \delta$, where d is some metric on $X_{(V,E)}$ compatible with the topology. Choose k_0 such that $d(x, y) < \delta$ if x and y agree (at least) on the k_0 first edges. Now assume that $\pi_k(x) = \pi_k(y)$ for some $k \geq k_0$. By the definition of π_k this means that, for all $n \in \mathbb{Z}$, $\tau_k(T^n_{(V,E)}x) = \tau_k(T^n_{(V,E)}y)$, and so $d(T^n_{(V,E)}x, T^n_{(V,E)}y) < \delta$ for all $n \in \mathbb{Z}$ because of our choice of k_0. This contradicts that $d(T^{n_0}_{(V,E)}x, T^{n_0}_{(V,E)}y) > \delta$. Hence π_k is injective for all $k \geq k_0$, proving the proposition.

We draw the following conclusions from the above: let $(X_{(V,E)}, T_{(V,E)})$ be the Bratteli-Vershik system associated to the properly ordered Bratteli diagram $(V, E, \geq)$. Then $(X_{(V,E)}, T_{(V,E)})$ is not expansive if and only if $\pi_k\colon X_{(V,E)} \to X_k$ $(= \pi_k(X_{(V,E)}))$ is not injective for $k = 1, 2, 3, \ldots$.

3. Key definitions and basic properties

Set $X = X_{(V,E)}$ and $T = T_{(V,E)}$, where $(X_{(V,E)}, T_{(V,E)})$ is the Bratteli-Vershik system associated to the properly ordered Bratteli diagram $(V, E, \geq)$. (We will use the notation introduced in Section 2 as well as the one in [1], and we adopt the terminology of [1].)

Consider a pair (x, x') of distinct points in X such that $\pi_i(x) = \pi_i(x')$ for some $i \geq 1$. We call such a pair *i-compatible*. Observe that (x, x') is then k-compatible if $k \leq i$. Since $x \neq x'$, there exists some $j > i$ such that $\pi_j(x) \neq \pi_j(x')$. We say that the pair is *j-separated*. The largest index i_0 for which the pair (x, x') is i_0-compatible (and hence it is $(i_0 + 1)$-separated) will be called the *depth of compatibility* (*depth* for short) of this pair. In particular, equal elements have depth ∞. Let (x, x') be i-compatible and j-separated for some $j > i$. By telescoping between levels i and j we obtain that (x, x') is of depth i, which is easily seen.

We make some observations:

(i) If (x, x') is i-compatible and j-separated, then $(T^m x, T^m x')$ is i-compatible and j-separated for all $m \in \mathbb{Z}$. (This follows since $\pi_k(T^m y) = S_k^m \pi_k(y)$ for all $y \in X$, $k = 1, 2, 3, \ldots$.)

(ii) If (x, x') is of depth i, then $(T^m x, T^m x')$ is of depth i for all $m \in \mathbb{Z}$. (This is an immediate consequence of (i).)

(iii) If (x, x') is a pair of depth i and (x, x'') is a pair of depth $j > i$, then (x', x'') is a pair of depth i (and hence not equal). (Clearly the pair (x', x'') is i-compatible. There exists $m \in \mathbb{Z}$ such that $\tau_{i+1}(T^m x) \neq \tau_{i+1}(T^m x')$. Since $\tau_{i+1}(T^m x) = \tau_{i+1}(T^m x'')$, the assertion follows.)

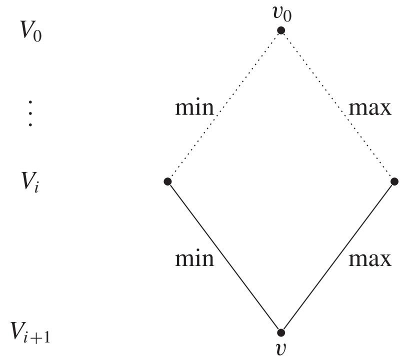

FIGURE 2.

An i-compatible and j-separated ($j > i$) pair (x, x') is said to have a *common j-cut* if for some $m \in \mathbb{Z}$, $\tau_j(T^m x)$ and $\tau_j(T^m x')$ are minimal paths, i.e. consisting of only minimal edges, between level j and level 0 (i.e. the top vertex). Note that if a pair has a common j-cut it also has a common j'-cut for every $i < j' \leq j$. It is obvious from the definitions that if (x, x') has a common j-cut, then $(T^\ell x, T^\ell x')$ also has a common j-cut for any $\ell \in \mathbb{Z}$. Observe also that if the pair (x, x') has no common j-cut the pair must be j-separated.

We make one important observation: let (x, x') be of depth i, and assume (x, x') has a common $(i + 1)$-cut. Then for some $m \in \mathbb{Z}$, the pair $(T^m x, T^m x')$ is of depth i such that $\tau_{i+1}(T^m x)$ and $\tau_{i+1}(T^m x')$ are minimal paths, *and* $r(\tau_{i+1}(T^m x)) \neq r(\tau_{i+1}(T^m x'))$. (In fact, by assumption there exists $k \in \mathbb{Z}$ such that $\tau_{i+1}(T^k x)$ and $\tau_{i+1}(T^k x')$ are minimal paths. If $v = r(\tau_{i+1}(T^k x)) = r(\tau_{i+1}(T^k x'))$ then ℓ iterates of T, say, applied to $T^k x$ and $T^k x'$ respectively, will "sweep over" all the paths between $v_0 \in V_0$ and $v \in V_{i+1}$, eventually reaching the max path, see Figure 2. Applying T one more time to $T^{k+\ell} x$ and $T^{k+\ell} x'$, respectively, will result in $\tau_{i+1}(T^p x)$ and $\tau_{i+1}(T^p x')$ are minimal paths. (Here $p = k + \ell + 1$.) If $r(\tau_{i+1}(T^p x)) \neq r(\tau_{i+1}(T^p x'))$ we are done, setting $m = p$. If $r(\tau_{i+1}(T^p x)) = r(\tau_{i+1}(T^p x'))$, we do the same procedure as above. If we get to a stage where the ranges are distinct we are done. If this does not happen, we play the same game on $T^k x$ and $T^k x'$, but now with iterates of T^{-1} instead of T. This must lead to a situation where the ranges are distinct, otherwise $\pi_{i+1}(x) = \pi_{i+1}(x')$, contradicting that (x, x') is $(i + 1)$-separated.)

4. Proof of Theorem 1.1

We assume that $(X_{(V,E)}, T_{(V,E)})$ it not expansive and so for all $i \geq 1$, $\pi_i \colon X \to X_i$ is not injective. This is easily seen to have as a consequence that for infinitely many levels i there exist pairs of points (x_i, x_i') of depth i. If we telescope

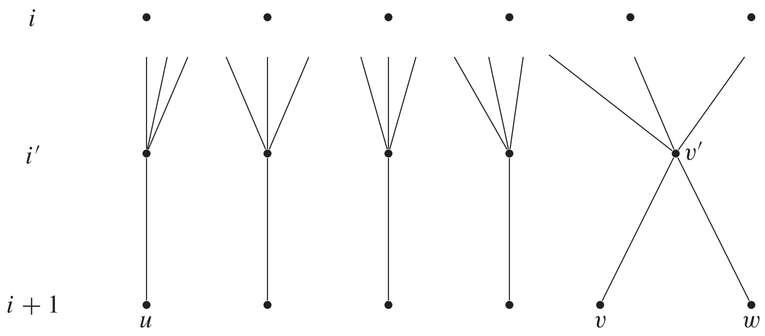

FIGURE 3. Rank $K = 6$.

between these levels we may assume that for every $i \geq 1$ there exists a pair (x_i, x_i') of depth i. We will show that $(X_{(V,E)}, T_{(V,E)})$ is an odometer, which will complete the proof. First we set the stage in the sense that we may assume that $(V, E, \geq)$ has the following properties:

(i) We may assume that $\text{rank}(V, E) = K$ (cf. Definition 2.2) is the smallest possible such that the Bratteli-Vershik system associated to $(V, E, \geq)$ is (conjugate to) the given one. (If $K = 1$ we have an odometer, so there is nothing more to prove.)

(ii) By telescoping we may assume that between consecutive levels there is full connection (cf. Definition 2.3) and, furthermore, that at each level all the minimal edges (respectively the maximal edges) have the same source, cf. [3, Proposition 2.8]. (This is not an essential assumption, but it makes it easier to visualize the Vershik map.)

Note that the property (i) is not affected by the operations performed in (ii).

As before we let $X = X_{(V,E)}$, $T = T_{(V,E)}$. There are two scenarios, mutually exclusive, cf. [1].

(1) There exists i_0 such that for all $i \geq i_0$ and every $j > i$ there exists a pair (x, x') of depth i with a common j-cut.

(2) For infinitely many i, any pair (x, x') of depth i has no common j-cuts for sufficiently large $j > i$. (Note that j depends upon (x, x')!)

The proof is different for case (1) and case (2).

We consider case (1). The idea is to find another properly ordered Bratteli diagram $(V', E', \geq)$ with $\text{rank}(V', E') < K$ (assuming $K > 1$), such that $(X_{(V',E')}, T_{(V',E')}) \cong (X, T)$. This contradiction will finish the proof in this case. Now choose any $i \geq i_0$. By the observation we made at the end of Section 3 we may assume that there exists a pair (x, x') of depth i such that

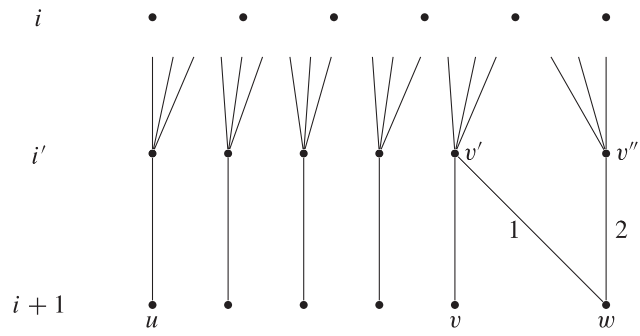

FIGURE 4. Rank $K = 6$.

$\tau_{i+1}(x)$ and $\tau_{i+1}(x')$ consist of minimal edges, and that $v = r(\tau_{i+1}(x)) \neq r(\tau_{i+1}(x')) = w$. If $|r^{-1}(v)| = |r^{-1}(w)|$ we may insert a new level (we name it i') between levels i and $i+1$ with ordering of the edges as shown in Figure 3. (The ordering at the vertex v' is the same as the ordering at v and w, the two latter being the same since (x, x') is of depth i.) The order of the edges ranging at vertices $u \in V_{i+1} - \{v, w\}$ is replicated at level i'. We notice that if we telescope between levels i and $i+1$ we get the original ordering. So the insertion of level i' does not change the Bratteli-Vershik map. Now we have obtained a level i' with $K-1$ number of vertices. If $|r^{-1}(v)| < |r^{-1}(w)|$, say, we insert a new level i' between levels i and $i+1$ as shown in Figure 4. The $|r^{-1}(v)|$ first edges ranging at v and w are ordered at v' as they are at v and w while the $|r^{-1}(w)| - |r^{-1}(v)|$ remaining edges ranging at w are ordered at v'' as they are at w. As before vertices $u \in V_{i+1} - \{v, w\}$ are just replicated at level i'. We observe that the number of vertices at level i' is the same as at level $i+1$, namely K. Now we claim that (x, x') separates at level i', and so (x, x') has depth i in the new diagram as well. In fact, by applying $L+1$ iterates of T to x and x', respectively, we see that they separate at level i'. Here L is the number of paths from the top vertex ranging at v. We observe that the number of edges between levels i and i' is strictly smaller than the number of edges between levels i and $i+1$. Now we repeat the same construction between levels i and i'. Since we decrease the number of edges each time, we must eventually arrive at the first case, where the number of edges ranging at v and w are the same. Doing the construction we did in the first case will then yield a level which has $K-1$ vertices.

After we have done this, we do the same construction between levels $(i+1)$ and $(i+2)$, etc. If we now telescope to the new levels with $K-1$ vertices we wind up with a properly ordered Bratteli diagram of rank $K-1$ which yields a Bratteli-Vershik system conjugate to the original. From this we conclude that K can not be larger than one and the proof is completed for case (1).

We now look at case (2). By telescoping to appropriate levels we may assume that we have the following scenario:

For each $i \geq 1$ there exists a pair of depth i that has no common $(i+1)$-cuts, and hence no common j-cuts for any $j > i$.

Now fix any $i_0 \geq 1$. We shall prove that (X_{i_0}, S_{i_0}) is periodic (and hence finite). This will imply that (X, T) is an odometer since $(X, T) = \varprojlim(X_k, S_k)$, hence finishing the proof.

Under the assumption that the above scenario holds we can prove the following lemma.

SUBLEMMA 4.1. *For any positive integer L there exist L distinct elements $y_1, y_2, \ldots, y_L$ in X which are pairwise i_0-compatible and pairwise have no common j-cuts for some $j \geq i_0$. In particular, they are pairwise j-separated. (Observe that for all $k \in \mathbb{Z}$ the elements $T^k y_1, T^k y_2, \ldots, T^k y_L$ have the same properties as $y_1, y_2, \ldots, y_L$.)*

PROOF OF SUBLEMMA 4.1. For each $i \in [i_0, i_0+L-1]$ let (x_i, x_i') be a pair of depth i which do not have a common $(i+1)$-cut. Let $\{n_k\}_k$ be a subsequence of natural numbers such that $T^{n_k}x_i \to y_0$ as $k \to \infty$, where y_0 is the unique minimal path $x_{\min}$ in $X = X_{(V,E)}$. (Because of minimality of (X, T) such a subsequence exists.) By compactness of X there exists a subsequence of $\{n_k\}_k$, which we again will denote by $\{n_k\}_k$, such that $T^{n_k}x_i' \to y_i$ for some $y_i \in X$. By continuity we get that $\pi_i(y_0) = \pi_i(y_i)$ since $\pi_i(T^{n_k}x_i) = \pi_i(T^{n_k}x_i')$ for all k. So (y_0, y_1) is i-compatible. We claim that (y_0, y_i) is $(i+1)$-separated, and hence (y_0, y_i) is of depth i. In fact, there exists k_0 such that for all $k \geq k_0$, $\tau_{i+1}(T^{n_k}x_i) = \tau_{i+1}(y_0)$ and $\tau_{i+1}(T^{n_k}x_i') = \tau_{i+1}(y_i)$. Since (x_i, x_i') do not have a common $(i+1)$-cut we conclude that $\tau_{i+1}(T^{n_k}x_i) \neq \tau_{i+1}(T^{n_k}x_i')$. Hence $\tau_{i+1}(y_0) \neq \tau_{i+1}(y_i)$, and so (y_0, y_i) is $(i+1)$-separated, hence of depth i. By (iii) in Section 3 we get that if $i < i'$, then $(y_i, y_{i'})$ is a pair of depth i' and hence, in particular, i_0-compatible. (In particular, the points $y_{i_0}, y_{i_0+1}, \ldots, y_{i_0+L-1}$ are distinct.) By assumption (2) there exists $j(i, i') > i'$ such that $(y_i, y_{i'})$ have no common $j(i, i')$-cut. Let $j = \max\{j(i, i') \mid i \neq i'\}$. Then $y_1, y_2, \ldots, y_L$ are distinct points in X which are pairwise i_0-compatible and pairwise have no common j-cut. (Here we rename the indices by letting $i_0 \to 1, i_0+1 \to 2, \ldots, i_0+L-1 \to L$.) This finishes the proof of Sublemma 4.1.

By telescoping between level i_0 and level j we may assume that the elements $y_1, y_2, \ldots, y_L$ in Sublemma 4.1 are pairwise of depth i_0 and have no common (i_0+1)-cuts. Choose L in Sublemma 4.1 to be

$$L = (K-1)2^{K-1} + 2.$$

(Note that $L \geq K + 1$.)

Let us in the sequel denote τ_{i_0} by τ_1 and τ_{i_0+1} by τ_2. Let $v \in V_{i_0+1}$ and let ℓ_v be the smallest (positive) difference of the ordinal numbers of any pair $(\tau_2(T^p y_i), \tau_2(T^p y_j))$ with common range v, i.e. $r(\tau_2(T^p y_i)) = r(\tau_2(T^p y_j)) = v$. Here $i, j \in \{1, 2, \ldots, L\}$, $i \neq j$, and p can be any integer such that the range condition is satisfied. (In Figure 5 we have illustrated this by assuming that ℓ_v is obtained at v by $(T^0 y_1 =)\, y_1 = (a, e, \ldots)$, $(T^0 y_2 =)\, y_2 = (a, f, \ldots)$. We see that $\ell_v = 5 - 1 = 4$. Actually, Figure 5 illustrates another point (setting aside that $K = 2$): in the general case, if we telescope between level 0 and level i_0, then we wind up with a scenario like the one in Figure 5 except that there are multiple edges instead of the single edge a (respectively b).)

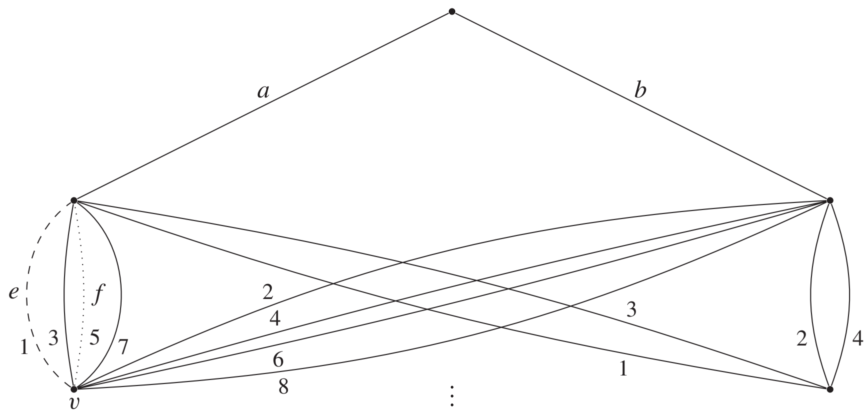

FIGURE 5. A Bratteli diagram where $i_0 = 1$, $|v| = 8$, $\ell_v = 4$, $y_1 = (a, e, \ldots)$, $y_2 = (a, f, \ldots)$ and $\pi_1(y_1) = \pi_1(y_2)$.

Assume ℓ_v is obtained at v with the pair $(\tau_2(T^p y_i), \tau_2(T^p y_j))$. Since $\pi_{i_0}(T^p y_i) = \pi_{i_0}(T^p y_j)$ this has the following consequence: $\widehat{v}(k) = \widehat{v}(k + \ell_v)$ for $k \in [1, |v| - \ell_v]$. Here $|v|$ denotes the number of paths in P_{i_0+1} ranging at v, and $\widehat{v}(k)$ is the element in P_{i_0} obtained by "cutting off" (or truncating) the path in P_{i_0+1} ranging at v with ordinal number k. (In Figure 5 we have $\ell_v = 4$, and we get that $\widehat{v}(1) = \widehat{v}(1+4) = a$, $\widehat{v}(2) = \widehat{v}(2+4) = b$, $\widehat{v}(3) = \widehat{v}(3+4) = a$, $\widehat{v}(4) = \widehat{v}(4+4) = b$.)

Now let $\widehat{y}$ denote the common image of $y_1, y_2, \ldots, y_L$ under π_{i_0}, i.e. $\pi_{i_0}(y_1) = \pi_{i_0}(y_2) = \cdots = \pi_{i_0}(y_L) = \widehat{y} \subseteq P_{i_0}^{\mathbb{Z}}$. Observe that by the definition of $\widehat{y}$ we have that $\widehat{y}(\ell) = \tau_1(T^\ell y_i)$ for $\ell \in \mathbb{Z}$ and any $i = 1, 2, \ldots, L$. In particular, $\widehat{y}(0) = \tau_1(y_i)$. We will say that the ℓ_v-periodicity law holds at the coordinate $n \in \mathbb{Z}$ of $\widehat{y}$ if $\widehat{y}(n) = \widehat{y}(n + \ell_v)$. We make one important observation: if, say, $r(\tau_2(T^n y_i)) = r(\tau_2(T^n y_j)) = v$ for some $y_i \neq y_j$ and $\tau_2(T^n y_i) < \tau_2(T^n y_j)$ $(n \in \mathbb{Z})$, then the ℓ_v periodicity law holds at the coordinate n of $\widehat{y}$. In fact, if the ordinal number of $\tau_2(T^n y_i)$ is k, then $k + \ell_v \leq$ (ordinal number of $\tau_2(T^n y_j)) \leq |v|$, and so $\widehat{v}(k) = \widehat{v}(k + \ell_v)$. By definition of $\widehat{y}$ it follows that $\widehat{v}(k) = \widehat{y}(n)$. Now $\tau_2(T^{n+\ell_v} y_i) \leq \tau_2(T^n y_j)$, and so $\widehat{y}(n+\ell_v) = \widehat{v}(k+\ell_v)$, and hence the ℓ_v-periodicity law holds at the coordinate n of $\widehat{y}$.

Let us order the vertices at level $i_0 + 1$ by $v_1, v_2, \ldots, v_K$ such that $\ell_{v_1} \leq \ell_{v_2} \leq \cdots \leq \ell_{v_K}$. (If there exists some vertex v at level $i_0 + 1$ such that no two $T^p y_i, T^p y_j$ ($p \in \mathbb{Z}, i \neq j$) range at v, then we just ignore that v. This will not cause any problem for the subsequent argument, so we may just as well assume that there exists no such v.) Retaining the set-up and the notation and terminology introduced above, we can prove the following lemma.

SUBLEMMA 4.2. *Assume that there exists some $k \in \mathbb{Z}$ such that*

$$|\{T^k y_i \mid r(\tau_2(T^k y_i)) = v_1,\ i = 1, 2, \ldots, L\}| \geq K + 1,$$

where $y_1, \ldots, y_L$ are distinct elements in X which are pairwise of depth i_0 and have no common $(i_0 + 1)$-cuts. Then (X_{i_0}, S_{i_0}) is periodic with periodicity ℓ_{v_1}.

PROOF OF SUBLEMMA 4.2. By renaming $T^k y_i$ as y_i, $i = 1, 2, \ldots, L$ (cf. Sublemma 4.1), we may assume

$$|\{y_i \mid r(\tau_2(y_i)) = v_1,\ i = 1, 2, \ldots, L\}| \geq K + 1.$$

Let I be the largest interval of integers (obviously containing 0) such that the ℓ_{v_1}-periodicity law holds. Specifically, if $i \in I$, then $\widehat{y}(i) = \widehat{y}(i + \ell_{v_1})$. If I is infinite at the right end, then $\widehat{y}(i) = \widehat{y}(i + \ell_{v_1})$ for all $i \geq 0$. Shifting $\widehat{y}$ to the left and using minimality of (X_{i_0}, S_{i_0}), we get that $\widehat{y}$ is periodic and so (X_{i_0}, S_{i_0}) is periodic, thus finishing the proof. If I has a right end, let $m \in \mathbb{Z}_+$ be the first integer to the right of I. At least two of the elements in $\{T^m y_i \mid r(\tau_2(y_i)) = v_1, i = 1, 2, \ldots, L\}$, say, $T^m y_i$ and $T^m y_j$ ($i \neq j$) are such that $r(\tau_2(T^m y_i)) = r(\tau_2(T^m y_j)) = v_k$ for some $k \geq 1$. We have $\widehat{y}(m) = \tau_1(T^m y_j) = \tau_1(T^m y_i)$. If $v_k = v_1$ then the ℓ_{v_1}-periodicity law holds at m by the observation we made above, contradicting our assumption. So $k > 1$. Let the ordinal numbers of $\tau_2(T^m y_i)$ and $\tau_2(T^m y_j)$ be s and t, respectively, and assume $s < t$. Now $\ell_{v_1} \leq \ell_{v_k} \leq t - s$, and so the ordinal number $s + \ell_{v_1}$ exists for paths in P_{i_0+1} ranging at v_k.

Applying $T^{-(t-s)}$ to $T^m y_j$ results in the following:

$$\tau_2(T^m y_i) = \tau_2(T^{-(t-s)}(T^m y_j)) = \tau_2(T^{m-(t-s)} y_j).$$

In particular, the ordinal numbers of $\tau_2(T^{m-(t-s)} y_j)$ and $\tau_2(T^m y_i)$ are the same, both equal to s. We also get

$$\widehat{y}(m) = \tau_1(T^m y_i) = \tau_1(T^{m-(t-s)} y_j) = \widehat{y}(m - (t - s)).$$

Assume we can prove that $m-(t-s)>0$. Then $m-(t-s)\in I$ and so the ℓ_{v_1} periodicity law holds at $m-(t-s)$ i.e. $\widehat{y}(m-(t-s)+\ell_{v_1})=\widehat{y}(m-(t-s))$. If we apply $T^{\ell_{v_1}}$ to both $T^m y_i$ and $T^{m-(t-s)}y_j$, respectively, we get

$$\tau_2(T^{m+\ell_{v_1}}y_i)=\tau_2(T^{m-(t-s)+\ell_{v_1}}y_j)$$

(both having ordinal number $s+\ell_{v_1}$), and so

$$\widehat{y}(m+\ell_{v_1})=\widehat{y}(m-(t-s)+\ell_{v_1})=\widehat{y}(m-(t-s))=\widehat{y}(m).$$

So the ℓ_{v_1}-periodicity law holds at m which contradicts our assumption that I has a (finite) right end, thus finishing the proof. It remains to prove that $m-(t-s)>0$. Assume by contradiction that $m-(t-s)\le 0$, and so $m\le t$. Thus $r(\tau_2(T^{-m}(T^m y_j)))=v_k$, which is impossible since $r(\tau_2(y_j))=v_1$. This finishes the proof of Sublemma 4.2.

We can now finish the proof of Theorem 1.1. By Sublemma 4.2 we have that if there exists some $k\in\mathbb{Z}$ such that

$$|\{T^k y_i \mid r(\tau_2(T^k y_i))=v_1,\ i=1,2,\ldots,L\}|\ge K+1, \qquad (*)$$

then we can prove that (X_{i_0},S_{i_0}) is periodic. So assume that this is not the case. In other words, for all $k\in\mathbb{Z}$ we have

$$|\{T^k y_i \mid r(\tau_2(T^k y_i))=v_1,\ i=1,2,\ldots,L\}|\le K.$$

Assume now that there exists some $k\in\mathbb{Z}$ such that

$$|\{T^k y_i \mid r(\tau_2(T^k y_i))=v_2,\ i=1,2,\ldots,L\}|\ge 2K. \qquad (**)$$

Now we argue exactly as above letting I and m be as above. There will then exist at least two elements in $\{T^m y_i \mid r(\tau_2(y_i))=v_2,\ i=1,2,\ldots,L\}$, say $T^m y_i$ and $T^m y_j$ ($i\neq j$) such that $r(\tau_2(T^m y_i))=r(\tau_2(T^m y_j))=v_k$, where $k\ge 2$. By exactly the same argument as above, we get that (X_{i_0},S_{i_0}) is periodic. If both $(*)$ and $(**)$ do not occur, we assume there exists $k\in\mathbb{Z}$ such that

$$|\{T^k y_i \mid r(\tau_2(T^k y_i))=v_3,\ i=1,2,\ldots,L\}|\ge 4K-2. \qquad (***)$$

We repeat the same argument as above, again getting that (X_{i_0},S_{i_0}) is periodic. We continue this process, and it must eventually stop. The "worst" case scenario is that for all $j,k,\ell,\ldots,p,\ldots,t\in\mathbb{Z}$ the following simultaneously holds:

$$\left.\begin{array}{r}
|\{T^j y_i \mid r(\tau_2(T^j y_i)) = v_1,\ i = 1, 2, \ldots, L\}| \le K \\
|\{T^k y_i \mid r(\tau_2(T^k y_i)) = v_2,\ i = 1, 2, \ldots, L\}| \le 2K - 1 \\
|\{T^\ell y_i \mid r(\tau_2(T^\ell y_i)) = v_3,\ i = 1, 2, \ldots, L\}| \le 4K - 3 \\
\vdots \\
|\{T^p y_i \mid r(\tau_2(T^p y_i)) = v_q,\ i = 1, 2, \ldots, L\}| \le 2^{q-1}K - (2^{q-1} - 1) \\
\vdots \\
|\{T^t y_i \mid r(\tau_2(T^t y_i)) = v_{K-1},\ i = 1, 2, \ldots, L\}| \le 2^{K-2}K - (2^{K-2} - 1)
\end{array}\right\} \quad (****)$$

Adding up the right hand side of $(****)$ we get $(K-1)2^{K-1}$. Now $L = (K-1)2^{K-1} + 2$, and so for every $n \in \mathbb{Z}$ (in particular, for $n = 0$) we have

$$|\{T^n y_i \mid r(\tau_2(y_i)) = v_K,\ i = 1, 2, \ldots, K\}| \ge 2.$$

Recall the observation we made just before stating Sublemma 4.2 – adapted to our setting: if $r(\tau_2(T^n y_i)) = r(\tau_2(T^n y_j) = v_K$ for some $i \neq j$, then the ℓ_{v_K}-periodicity law holds at the coordinate n, i.e. $\widehat{y}(n) = \widehat{y}(n + \ell_{v_K})$. This immediately implies that (X_{i_0}, S_{i_0}) is periodic, and so the proof of Theorem 1.1 is complete.

The following corollary gives a positive answer to Question 1 raised in [1] about finding a smaller L than the one given in the so-called "Infection Lemma" in [1], namely $L = K^{K+1}+1$. In fact, the way the proof of Theorem 1.1 (Case 2) is structured, at each stage seeking the minimal number of compatible paths in order to ensure periodicity, makes it plausible to conjecture that the L we have found is optimal.

COROLLARY 4.3. *Let $K > 1$. If there exists at least*

$$L = (K-1)2^{K-1} + 2$$

points y_k, $k \in [1, L]$, that are i-compatible and have no common j-cut for some $j > i$ (and hence are j-separated), then (X_i, S_i) is periodic.

REMARK 4.4. We find some of the assertions at the beginning of "Proof in case (2)" of Theorem 1 (the same as our Theorem 1.1) in [1, pp. 744–745] needing some explanations. For example, it is stated that under assumption (2) the following holds: for each $i \ge 1$ every (sic) pair of depth i has no common $(i+1)$-cuts. We do not see why this should be true. However, it does follow from assumption (2) that there exists (sic) a pair with the desired properties, and that is sufficient for the proof to work. We also find the subsequent argument for the existence of appropriately many i_0 compatible and j-separated elements with no common j-cuts needing some further explanation.

That said, we have only high praise for the [1] paper. In fact, Downarowicz and Maass had the insight to realize that such a remarkable result as Theorem 1.1 holds, and also the ingenuity of finding a proof, the basic idea of which we use in our new proof, though stated in more Bratteli diagram terms.

REFERENCES

1. Downarowicz, T., and Maass, A., *Finite-rank Bratteli-Vershik diagrams are expansive*, Ergodic Theory Dynam. Systems 28 (2008), no. 3, 739–747.
2. Giordano, T., Putnam, I. F., and Skau, C. F., *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. 469 (1995), 51–111.
3. Herman, R. H., Putnam, I. F., and Skau, C. F., *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. 3 (1992), no. 6, 827–864.
4. Hewitt, E., and Ross, K. A., *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations*, Die Grundlehren der mathematischen Wissenschaften, Bd. 115, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
5. Høynes, S.-M., *Toeplitz flows and their ordered K-theory*, Ergodic Theory Dynam. Systems 36 (2016), no. 6, 1892–1921.
6. Walters, P., *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982.

DEPARTMENT OF MATHEMATICAL SCIENCES
NTNU, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
7491 TRONDHEIM
NORWAY
E-mail: siri.m.hoynes@ntnu.no