MATH. SCAND. 120 (2017), 249–271

ON A CLASS OF OPERATORS IN THE HYPERFINITE II_1 FACTOR

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Abstract

Let R be the hyperfinite II_1 factor and let u, v be two generators of R such that $u^*u = v^*v = 1$ and $vu = e^{2\pi i\theta}uv$ for an irrational number θ. In this paper we study the class of operators $uf(v)$, where f is a bounded Lebesgue measurable function on the unit circle S^1. We calculate the spectrum and Brown spectrum of operators $uf(v)$, and study the invariant subspace problem of such operators relative to R. We show that under general assumptions the von Neumann algebra generated by $uf(v)$ is an irreducible subfactor of R with index n for some natural number n, and the C^*-algebra generated by $uf(v)$ and the identity operator is a generalized universal irrational rotation C^*-algebra.

1. Introduction

Let M be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. A closed subspace $\mathscr{K}$ of $\mathscr{H}$ is said to be affiliated with M if the projection of $\mathscr{H}$ onto $\mathscr{K}$ belongs to M. For $T \in M$, a subspace K is said to be T-invariant if $T\mathscr{K} \subseteq \mathscr{K}$ or equivalently $P_{\mathscr{K}}TP_{\mathscr{K}} = TP_{\mathscr{K}}$. The invariant subspace problem relative to a von Neumann algebra M asks whether every operator $T \in M$ has a non-trivial, closed, invariant subspace $\mathscr{K}$ affiliated with M, and the hyperinvariant subspace problem asks whether one can always choose such a $\mathscr{K}$ to be hyperinvariant for T, i.e., it is S-invariant for every $S \in \mathscr{B}(\mathscr{H})$ that commutes with T. If the subspace $\mathscr{K}$ is T-hyperinvariant, then $P_{\mathscr{K}} \in W^*(T) = \{T, T^*\}''$.

Let M be a finite von Neumann algebra with a faithful normal tracial state τ. The *Fuglede-Kadison determinant*, $\Delta\colon M \to [0, \infty)$, is given by

$$\Delta(T) = \exp\{\tau(\ln |T|)\}, \quad T \in M,$$

* The second author is partially supported by National Natural Science Foundation of China (Grant No. 11222104 and 11431011), Program for NCET in University and the Fundamental Research Funds for the Central Universities. The third author is partially supported by NSFC (Grant No. 11401071) and the Fundamental Research Funds for the Central Universities (Grant No. DUT16RC(4)57).

Received 9 December 2014.

DOI: https://doi.org/10.7146/math.scand.a-25625

with $\exp\{-\infty\} := 0$ [9]. For an arbitrary element T in M the function $\lambda \to \ln \Delta(T - \lambda 1)$ is subharmonic on $\mathbb{C}$, and its Laplacian

$$d\mu_T(\lambda) := \frac{1}{2\pi} \nabla^2 \ln \Delta(T - \lambda 1),$$

in the distribution sense, defines a probability measure μ_T on $\mathbb{C}$, called *Brown's spectral distribution* or the *Brown measure* of T [3]. From the definition, the Brown measure μ_T only depends on the joint distribution of T and T^*, i.e., the (non-commutative) mixed moments of T and T^*. If T is normal, then μ_T is the trace τ composed with the spectral projections of T. If $M = M_n(\mathbb{C})$, then μ_T is the normalized counting measure $(\delta_{\lambda_1} + \delta_{\lambda_2} + \cdots + \delta_{\lambda_n})/n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of T repeated according to root multiplicity. Recently, Uffe Haagerup and Hanne Schultz made a huge advance on the invariant subspace problem relative to a type II_1 factor [11]. They proved that if the Brown measure of an operator T in a type II_1 factor is not concentrated in one point, then the operator T has a non-trivial, closed, invariant subspace $\mathscr{K}$ affiliated with M and moreover, this subspace is hyperinvariant. However, the calculation of Brown measures of non-normal operators is complicated in general (see [10], [1], [7]). Note that the support of the Brown measure of an operator is contained in the spectrum of the operator.

As regards the invariant subspace problem relative to the von Neumann algebra, the following question remains open: if T is an operator in a type II_1 factor M and if the Brown measure μ_T is a Dirac measure, for example if T is quasinipotent, does T have a non-trivial, closed, invariant subspace affiliated with M? In [4], Dykema and Haagerup introduced the family of DT-operators and they studied many of their properties. In [5] they showed that every quasinilpotent DT-operator T has a one-parameter family of non-trivial hyperinvariant subspaces. In particular, they proved that for $t \in [0, 1]$,

$$\mathscr{H}_t = \left\{ \xi \in \mathscr{H} : \limsup_n \left(\frac{k}{e} \|T^k \xi\| \right)^{2/k} \leq t \right\}$$

is a closed, hyperinvariant subspace of T. In [16], Tucci introduced a class of quasinilpotent operators in the hyperfinite II_1 factor R. He showed that the quasinilpotent operator generates R and it has non-trivial, closed invariant subspaces affiliated to R. However the existence of non-trivial hyperinvariant subspaces of such class of operators remains open.

Let R be the hyperfinite II_1 factor and let $\theta \in (0, 1)$ be an irrational number. Then there are two unitary operators u, v in R such that $R = \{u, v\}''$ and $vu = e^{2\pi i\theta} uv$. In this paper we study the class of operators $uf(v)$ in R, where f is a bounded Lebesgue measurable function on the unit circle S^1. A natural example of the class of operators is $u+\lambda v$, where $\lambda \in \mathbb{C}$. Indeed, if let $w = u^*v$, then $R = \{u, w\}''$ and $wu = e^{2\pi i\theta}uw$. Note that $u + \lambda v = u(1 + \lambda w)$. The operator $u + \lambda v$ is closely related to the so-called almost Mathieu operators, which can be viewed as the operator $(u + \lambda e^{2\pi i\beta}v) + (u + \lambda e^{2\pi i\beta}v)^*$ in R (see [13] for a recent historical account and for the physics background of almost Mathieu operators).

The above class of operators are analogues of R-diagonal operators. Recall that if u and v are free Haar unitary operators in a finite von Neumann algebra M and f is a bounded measurable function on the unit circle S^1 then $uf(v)$ is an R-diagonal operator [14]. In [10], Haagerup and Larson calculated the spectrum and Brown spectrum of R-diagonal operators. In [15], Sniady and Speicher proved that every R-diagonal operator has a continuous family of invariant subspaces affiliated with M.

In sections 2 and 3 of this paper, we calculate the spectrum of $uf(v)$ in R, where f is a continuous function on S^1. The main result is that the spectrum of $uf(v)$ is given by

$$\sigma(uf(v)) = \begin{cases} \Delta(f(v))S^1, & f(v) \text{ is invertible,} \\ \overline{\mathbb{B}(0, \Delta(f(v)))}, & f(v) \text{ is not invertible,} \end{cases}$$

where $\Delta(f(v)) = \exp\left(\int_0^1 \ln|f(e^{2\pi ix})|\,dx\right)$ is the Fuglede-Kadison determinant of $f(v)$. In section 2 we show that the spectral radius of $uf(v)$ is $\Delta(f(v))$. A key idea in the calculation is to use Birkhoff's Ergodic theorem and the unique ergodicity of the irrational rotation. Then in section 3 we prove the main result. The main difficulty is to show that $\sigma(uf(v))$ is connected. This is done by using an averaging technique. We also point out that the above formula for the spectrum of $uf(v)$ does not hold for some $f \in L^\infty(S^1, m)$.

In section 4, we study the von Neumann algebra generated by $uf(v)$. We show that if the zero set of $f(z) \in L^\infty(S^1, m)$ has Lebesgue measure zero, then $W^*(uf(v))$ is an irreducible subfactor of R with index n, for some positive integer n.

In section 5, we consider the invariant subspace problem for $uf(v)$ relative to R. Firstly we calculate the Brown measure of $uf(v)$. We will show that the Brown measure of $uf(v)$ (in R) is the Haar measure on $\Delta(f(v))S^1$ for all $f \in L^\infty(S^1, m)$. As a corollary of Haagerup and Schultz's result, if $\Delta(f(v)) > 0$, for example f is a polynomial, then $uf(v)$ has a continuous family of invariant subspaces affiliated with M. On the other hand, if $\Delta(f(v)) = 0$, we show that the known methods are unable to determine whether or not the operator $uf(v)$ has a non-trivial, closed, invariant subspace affiliated with R. Thus such class of operators are interesting candidates for the question of the invariant subspace problem relative to R.

Recall that a generalized universal irrational rotation C^*-algebra $A_{\theta,\gamma}$ is the universal C^*-algebra generated by x and w satisfying the following properties [8]:

$$w^*w = ww^* = 1, \tag{1.1}$$

$$x^*x = \gamma(w), \tag{1.2}$$

$$xx^* = \gamma(e^{-2\pi i\theta}w), \tag{1.3}$$

$$xw = e^{-2\pi i\theta}wx, \tag{1.4}$$

where $\gamma(z) \in C(S^1)$ is a positive function. If $\gamma(z) \equiv 1$ (or $\gamma(z)$ does not have any zeros), then $A_{\theta,\gamma}$ is the universal irrational rotation C^*-algebra. In [8], many properties of generalized universal irrational rotation C^*-algebras are studied, including tracial state spaces, simplicity, K-groups, and classification of simple generalized universal irrational rotation C^*-algebras. For instance, the following results are Theorem 5.7 and Theorem 6.6 of [8] respectively.

THEOREM 1.1. *Let Y be the set of zeros of γ. If $\emptyset \neq Y \neq S^1$, then*

$$K_1(A_{\theta,\gamma}) \cong \mathbb{Z}$$

and there exists a split short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_0(A_{\theta,\gamma}) \longrightarrow C(Y, \mathbb{Z}) \longrightarrow 0.$$

In particular, if Y has n points, then

$$K_0(A_{\theta,\gamma}) \cong \mathbb{Z}^{n+1}.$$

THEOREM 1.2. *Let θ_1 and θ_2 be two irrational numbers, γ_1 and $\gamma_2 \in C(S^1)$ be non-negative functions and let Y_i be the set of zeros of γ_i, $i = 1, 2$. Suppose that A_{θ_i,γ_i} is simple. Then $A_{\theta_1,\gamma_1} \cong A_{\theta_2,\gamma_2}$ if and only if the following hold:*

$$\theta_1 = \pm\theta_2 \ (\mathrm{mod}\ \mathbb{Z}) \qquad \text{and} \qquad C(Y_1, \mathbb{Z})/\mathbb{Z} \cong C(Y_2, \mathbb{Z})/\mathbb{Z}.$$

In particular, when γ_1 has only finitely many zeros, then $A_{\theta_1,\gamma_1} \cong A_{\theta_2,\gamma_2}$ if and only if $\theta_1 = \pm\theta_2$ $(\mathrm{mod}\ \mathbb{Z})$ and γ_2 has the same number of zeros.

In section 6, we show that the C^*-algebra generated by $uf(v)$ and the identity operator is closely related to the generalized universal irrational rotation algebra. Precisely, we will prove the following result. Let Y be the zero set of $f(z)$. If Y satisfies $\phi^n(Y) \cap Y = \emptyset$ for every integer $n \neq 0$, where $\phi(z) = e^{2\pi i\theta}z$, then $C^*(uf(v), 1)$ is a generalized universal irrational rotation C^*-algebra. Furthermore, if $|f|(z)$ is not a periodic function, then

$C^*(uf(v), 1) \cong A_{\theta,|f|^2}$. As a corollary, we will show that if $A_{\theta,\gamma}$ is a simple C^*-algebra, then $A_{\theta,\gamma}$ is generated by an element $uf(v)$ and the identity operator for some $f(z) \in C(S^1)$.

ACKNOWLEDGEMENTS. The authors thank Chunlan Jiang and Liu Zhengwei for valuable discussions on the paper. The authors thank the referee for valuable comments and suggestions on the paper.

2. The spectral radius of $uf(v)$

Let $\alpha = e^{2\pi i\theta}$. Since $vu = \alpha uv$, we get $f(v)u = uf(\alpha v)$ for all $f \in L^\infty(S^1, m)$. So

$$(uf(v))^2 = uf(v)uf(v) = u^2 f(\alpha v)f(v),$$
$$(uf(v))^3 = uf(v)uf(v)uf(v) = u^3 f(\alpha^2 v)f(\alpha v)f(v).$$

By induction, we have

$$(uf(v))^n = u^n f(\alpha^{n-1}v)f(\alpha^{n-2}v)\cdots f(v).$$

Let $r(uf(v))$ be the spectral radius of $uf(v)$. Then

$$\begin{aligned} r(uf(v)) &= \lim_{n\to+\infty} \|(uf(v))^n\|^{1/n} \\ &= \lim_{n\to+\infty} \|f(\alpha^{n-1}v)f(\alpha^{n-2}v)\cdots f(v)\|^{1/n}. \end{aligned}$$

Since v is a Haar unitary operator, we may identify v with the multiplication operator M_z on $L^2(S^1, m)$, where m is the Haar measure on S^1. Hence,

$$\begin{aligned} \|(uf(v))^n\|^{1/n} &= \|f(\alpha^{n-1}v)f(\alpha^{n-2}v)\cdots f(v)\|^{1/n} \\ &= \|f(\alpha^{n-1}z)f(\alpha^{n-2}z)\cdots f(z)\|_\infty^{1/n} \\ &= \big(\operatorname*{ess\,sup}_{z\in S^1} |f(\alpha^{n-1}z)f(\alpha^{n-2}z)\cdots f(z)|\big)^{1/n}. \end{aligned}$$

LEMMA 2.1. *If $f(z) \in L^\infty(S^1, m)$ and $\int_0^1 \big|\big(\ln|f(e^{2\pi ix})|\big)\big|\,dx < \infty$, then $r(uf(v)) \geq \Delta(f(v))$.*

PROOF. Let $T\colon x \to x+\theta \pmod 1$. Then T is a measure preserving ergodic transformation of $[0, 1]$. By Birkhoff's Ergodic Theorem and the assumption $\int_0^1 \big|\big(\ln|f(e^{2\pi ix})|\big)\big|\,dx < \infty$, for almost all $x \in [0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|f(\alpha^k e^{2\pi ix})| = \int_0^1 \ln|f(e^{2\pi ix})|\,dx.$$

Let $\epsilon > 0$. Then there is a measurable subset E of $[0, 1]$ with $m(E) > 0$ and $N \in \mathbb{N}$ such that for all $x \in E$ and $n \geq N$

$$\frac{1}{n}\sum_{k=0}^{n-1} \ln|f(\alpha^k e^{2\pi i x})| \geq \int_0^1 \ln|f(e^{2\pi i x})|\,dx - \epsilon,$$

i.e.,

$$\left|f(\alpha^{n-1}e^{2\pi i x})f(\alpha^{n-2}e^{2\pi i x})\cdots f(e^{2\pi i x})\right|^{1/n} \geq \exp\left(\int_0^1 \ln|f(e^{2\pi i x})|\,dx - \epsilon\right).$$

This implies that

$$\begin{aligned} r(uf(v)) &\geq \limsup_{n\to\infty}\ \operatorname*{ess\,sup}_{x\in E}\left|f(\alpha^{n-1}e^{2\pi i x})f(\alpha^{n-2}e^{2\pi i x})\cdots f(e^{2\pi i x})\right|^{1/n} \\ &\geq \exp\left(\int_0^1 \ln|f(e^{2\pi i x})|\,dx - \epsilon\right). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $r(uf(v)) \geq \exp\bigl(\int_0^1 \ln|f(e^{2\pi i x})|\,dx\bigr) = \Delta(f(v))$.

Recall that a continuous transformation $T\colon X \to X$ of a compact metrisable space X is called *uniquely ergodic* if there is only one T invariant Borel probability measure μ on X. If T is uniquely ergodic, then $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$ converges uniformly to $\int_X f(x)\,d\mu(x)$ for every $f \in C(X)$ (see Theorem 6.19 of [17]). It is well-known that the irrational rotation of the unit circle is uniquely ergodic. If we apply the above fact to $\ln|f(z)|$, then we easily see the following lemma.

LEMMA 2.2. *If both $f(z), f(z)^{-1} \in C(S^1)$ and $\epsilon > 0$, then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in [0, 1]$,*

$$\begin{aligned} &\left(\left|f(\alpha^{n-1}e^{2\pi i x})f(\alpha^{n-2}e^{2\pi i x})\cdots f(e^{2\pi i x})\right|\right)^{1/n} \\ &\qquad\qquad \leq \exp\left(\int_0^1 \ln|f(e^{2\pi i x})|\,dx + \epsilon\right). \end{aligned}$$

THEOREM 2.3. *If $f(z) \in C(S^1)$, then $r(uf(v)) = \Delta(f(v))$.*

PROOF. We may assume that $|f(z)| \leq 1$ for all $z \in S^1$. For $k \in \mathbb{N}$, define $f_k(z) = \max\{|f(z)|, 1/k\}$. Then $f_k(z), f_k(z)^{-1} \in C(S^1)$. Let $\epsilon > 0$. By

Lemma 2.2, there exists an $N \in \mathbb{N}$ such that for $n \geq N$ and all $x \in [0, 1]$,

$$\begin{aligned}
&\left| f(\alpha^{n-1}e^{2\pi ix}) f(\alpha^{n-2}e^{2\pi ix}) \cdots f(e^{2\pi ix}) \right|^{1/n} \\
&\quad \leq \left| f_k(\alpha^{n-1}e^{2\pi ix}) f_k(\alpha^{n-2}e^{2\pi ix}) \cdots f_k(e^{2\pi ix}) \right|^{1/n} \\
&\quad \leq \exp\left(\int_0^1 \ln f_k(e^{2\pi ix})\, dx + \epsilon \right).
\end{aligned}$$

Let $n \to \infty$, then

$$r(uf(v)) \leq \exp\left(\int_0^1 \ln f_k(e^{2\pi ix})\, dx + \epsilon \right).$$

Since $\epsilon > 0$ is arbitrary,

$$r(uf(v)) \leq \exp\left(\int_0^1 \ln f_k(e^{2\pi ix})\, dx \right), \qquad \forall k \in \mathbb{N}.$$

Note that

$$1 = \frac{1}{f_1(e^{2\pi ix})} \leq \frac{1}{f_2(e^{2\pi ix})} \leq \cdots \leq \frac{1}{f_n(e^{2\pi ix})} \leq \cdots,$$

and

$$\lim_{k\to\infty} \frac{1}{f_k(e^{2\pi ix})} = \frac{1}{|f(e^{2\pi ix})|}, \qquad \forall x \in [0, 1].$$

So

$$0 \leq -\ln f_1(e^{2\pi ix}) \leq -\ln f_2(e^{2\pi ix}) \leq \cdots \leq -\ln f_n(e^{2\pi ix}) \leq \cdots,$$

and

$$\lim_{k\to\infty} -\ln f_k(e^{2\pi ix}) = -\ln |f(e^{2\pi ix})|, \qquad \forall x \in [0, 1].$$

The monotone convergence theorem implies that

$$\lim_{k\to\infty} \int_0^1 -\ln f_k(e^{2\pi ix})\, dx = -\int_0^1 \ln |f(e^{2\pi ix})|\, dx$$

and therefore,

$$r(uf(v)) \leq \lim_{k\to\infty} \exp\left(\int_0^1 \ln f_k(e^{2\pi ix})\, dx \right) = \exp\left(\int_0^1 \ln |f(e^{2\pi ix})|\, dx \right).$$

Now if $\int_0^1 \ln|f(e^{2\pi ix})|\,dx = -\infty$, then

$$r(uf(v)) \le \exp\left(\int_0^1 \ln|f(e^{2\pi ix})|\,dx\right) = 0$$

and hence $r(uf(v)) = \exp\big(\int_0^1 \ln|f(e^{2\pi ix})|\,dx\big) = 0$. If $\int_0^1 \ln|f(e^{2\pi ix})|\,dx > -\infty$, then $\int_0^1 \big|\big(\ln|f(e^{2\pi ix})|\big)\big|\,dx < \infty$. By Lemma 2.1,

$$r(uf(v)) \ge \exp\left(\int_0^1 \ln|f(e^{2\pi ix})|\,dx\right)$$

and hence $r(uf(v)) = \exp\big(\int_0^1 \ln|f(e^{2\pi ix})|\,dx\big) = \Delta(f(v))$.

3. The spectrum of $uf(v)$

LEMMA 3.1. *Let $f_n(z) \in L^2(S^1, m)$ for $n \in \mathbb{Z}$ and assume $T = \sum_{n=-\infty}^{\infty} u^n f_n(v) \in R$. Then for each $n \in \mathbb{Z}$,*

$$\|f_n(v)\| \le \|T\|. \tag{3.1}$$

PROOF. Let N be the von Neumann algebra generated by v and let E_N be the faithful normal conditional expectation of R onto N preserving the unique trace on R. Then for any $n \in \mathbb{Z}$, we have

$$\|f_n(v)\| = \|E_N(u^{-n}T)\| \le \|T\|.$$

LEMMA 3.2. *Let $f(z) \in L^\infty(S^1, m)$ and let $x = uf(v)$. Then the spectrum $\sigma(x)$ of x is connected.*

PROOF. Suppose the spectrum $\sigma(x)$ of x is not connected. Then the Riesz spectral decomposition theorem gives a non-trivial idempotent p in the Banach algebra generated by x. Let τ be the faithful trace on R. We may assume that $0 < \tau(p) \le 1/2$. Let $\epsilon > 0$ be sufficiently small and let $a = \lambda + \sum_{n=1}^N \lambda_n x^n$ be such that $\|p - a\| < \epsilon/(2\|p\|^2 + 2)$. Since $|\tau(a) - \tau(p)| \le \|p - a\| < \epsilon/(2\|p\|^2 + 2)$, we may assume that $0 \le \lambda < 3/4$. Then

$$\begin{aligned}\|a^2 - a\| \le \|a^2 - pa\| + \|pa - p^2\| + \|p - a\| \\ \le \big((\|p\| + \epsilon) + \|p\| + 1\big)\|p - a\| < \epsilon.\end{aligned}$$

This implies that

$$\left\|\left(\sum_{n=1}^N \lambda_n x^n + \lambda\right)\left(\sum_{n=1}^N \lambda_n x^n + \lambda\right) - \left(\sum_{n=1}^N \lambda_n x^n + \lambda\right)\right\| < \epsilon.$$

By Lemma 3.1, we have $|\lambda^2 - \lambda| < \epsilon$. Since $0 < \lambda < 3/4$, we get $\lambda < \lambda^2 + \epsilon < \frac{3}{4}\lambda + \epsilon$. Therefore $0 \le \lambda < 4\epsilon$. Thus $\tau(p) \le \tau(a) + \epsilon/(2\|p\|^2 + 2) < 5\epsilon$. Since $\epsilon > 0$ is arbitrary, $\tau(p) = 0$. So $p = 0$. This is a contradiction.

THEOREM 3.3. *Let $f(z) \in C(S^1)$ and let $x = uf(v)$. Then the spectrum $\sigma(x)$ of x is given as follows:*

(1) *if $f(v)$ is invertible, then $\sigma(uf(v)) = \Delta(f(v))S^1$.*

(2) *if $f(v)$ is not invertible, then $\sigma(uf(v)) = \overline{\mathbb{B}(0, \Delta(f(v)))}$.*

Here $\Delta(f(v))$ is the Fuglede-Kadison determinant of $f(v)$.

PROOF. Suppose $f(v)$ is not invertible, then $0 \in \sigma(uf(v))$. By Theorem 2.3, we have $r(uf(v)) = \Delta(f(v))$. Clearly $\sigma(uf(v))$ is rotationally symmetric. By Lemma 3.2, $\sigma(uf(v)) = \overline{\mathbb{B}(0, \Delta(f(v)))}$.

Suppose $f(v)$ is invertible. By Theorem 2.3, $r(uf(v)) = \Delta(f(v))$. Note that $(uf(v))^{-1} = f(v)^{-1}u^* = u^* f(e^{-2\pi i\theta}v)^{-1}$. So

$$r((uf(v))^{-1}) = \Delta(f(e^{-2\pi i\theta}v)^{-1}) = \exp\Big(\int_0^1 \ln|f(e^{-2\pi i\theta}e^{2\pi ix})|^{-1}\, dx\Big)$$

$$= \exp\Big(\int_0^1 \ln|f(e^{2\pi ix})|^{-1} dx\Big) = \Delta(f(v)^{-1}) = (\Delta(f(v)))^{-1}.$$

So $\sigma(uf(v))$ is contained in $\Delta(f(v))S^1$. Since $\sigma(uf(v))$ is rotationally symmetric, $\sigma(uf(v)) = \Delta(f(v))S^1$.

REMARK 3.4. A natural question is that if the above theorem can be generalized to $f \in L^\infty(S^1)$. It can be shown that the above theorem can be generalized to a larger class of functions which are the essentially upper semi-continuous functions. However the formula in the above theorem does not hold for some $f \in L^\infty(S^1, m)$. Indeed one can construct a proper open subset E of S^1 such that $r(u\chi_E(v)) = 1$ but $\Delta(\chi_E) = 0$.

4. Von Neumann algebras generated by $uf(v)$

LEMMA 4.1. *Let $f \in L^\infty(S^1)$. If $f(v)$ is not a scalar operator, then the von Neumann subalgebra generated by u and $f(v)$ is an irreducible subfactor of R.*

PROOF. Let N be the von Neumann algebra generated by u and $f(v)$. Suppose $x \in R$ commutes with N. Write $x = \sum_{m,n\in\mathbb{Z}} \alpha_{mn} v^m u^n$, where $\sum_{m,n\in\mathbb{Z}} |\alpha_{mn}|^2 < \infty$. Then $xu = ux$ and $vu = e^{2\pi i\theta}uv$ imply that

$$\sum_{m,n\in\mathbb{Z}} \alpha_{mn} v^m u^{n+1} = \sum_{m,n\in\mathbb{Z}} \alpha_{mn} e^{-2\pi i m\theta} v^m u^{n+1}.$$

Since $\{v^m u^n\}$ is an orthonormal basis of $L^2(R, \tau)$, it follows that $x = \sum_{n\in\mathbb{Z}} \alpha_n u^n$, where $\sum_{n\in\mathbb{Z}} |\alpha_n|^2 < \infty$. Now $xf(v) = f(v)x$ implies that

$$\sum_{n\in\mathbb{Z}} \alpha_n u^n f(v) = \sum_{n\in\mathbb{Z}} \alpha_n u^n f(e^{2\pi i n\theta} v).$$

So $\alpha_n u^n f(v) = \alpha_n u^n f(e^{2\pi i n\theta} v)$, for all $n \in \mathbb{Z}$. If for some $n \neq 0$ we have $\alpha_n \neq 0$, then $f(v) = f(e^{2\pi i n\theta} v)$. This implies that $f(v)$ is a scalar operator, which contradicts the assumption. Therefore, x is a scalar operator and N is an irreducible subfactor of R.

THEOREM 4.2. *Let $f \in L^\infty(S^1)$ such that $f(v)$ is not a scalar operator. Let N be the irreducible subfactor of R generated by u and $f(v)$. Then the Jones index [12] $[R : N]$ is a finite integer. Furthermore, the following conditions are equivalent:*

(1) *$f(z)$ is a periodic function with minimal period $e^{2\pi i/n}$;*

(2) *if $f(z) = \sum_{k\in\mathbb{Z}} \alpha_k z^k$ is the Fourier series of $f(z)$, then $n = \gcd\{k : \alpha_k \neq 0\}$;*

(3) *$N = W^*(u, v^n)$.*

PROOF. (1) $\Leftrightarrow$ (2). Suppose $f(z)$ is a periodic function with minimal period $e^{2\pi i/n}$ and $m = \gcd\{k : \alpha_k \neq 0\}$. Then $f(z) = f(e^{2\pi i/m} z)$, for almost all $z \in S^1$. So $f(z)$ is a periodic function period $e^{2\pi i/m}$. Since $e^{2\pi i/n}$ is a minimal period of $f(z)$, we have $n = mj$, for some positive integer j. Suppose $j \geq 2$. Then there exists k_0 such that $\alpha_{k_0} \neq 0$ and n is not a factor of k_0. Since $f(z) = f(e^{2\pi i/n} z)$, by comparing the coefficients of both sides, we have

$$\alpha_{k_0} z^{k_0} = \alpha_{k_0} e^{2\pi i k_0/n} z^{k_0}.$$

This is a contradiction. Thus $n = m$.

(2) $\Rightarrow$ (3). Note that

$$(u^*)^k f(v) u^k = f(e^{2\pi i k\theta} v).$$

So $f(e^{2\pi i k\theta} v) \in N$, for all $k \in \mathbb{Z}$. Since $\{e^{2\pi i k\theta} : k \in \mathbb{Z}\}$ is dense in the unit circle S^1, we have $f(e^{2\pi i t} v) \in N$, for all $t \in [0, 1]$. For $k \in \mathbb{Z}$ and $z \in S^1$,

$$g_k(z) = \int_0^1 e^{2\pi i k t} f(z e^{-2\pi i t})\, dt = \alpha_k z^k.$$

Thus if $\alpha_k \neq 0$, then $\alpha_k^{-1} g_k(v) = v^k \in N$. Since $n = \gcd\{k : \alpha_k \neq 0\}$, we get $v^n \in N$. This proves that $W^*(u, v^n) \subseteq N$. Clearly, $N \subseteq W^*(u, v^n)$. So $N = W^*(u, v^n)$.

(3) $\Rightarrow$ (2). Suppose $N = W^*(u, v^n)$ and $m = \gcd\{k : \alpha_k \neq 0\}$. By (2) $\Rightarrow$ (3), $W^*(u, v^n) = N = W^*(u, v^m)$. Hence $m = n$.

By the above proof, $N = W^*(u, v^n)$ for some positive integer n. Therefore, $[R : N]$ is n.

COROLLARY 4.3. *Suppose the set of zeros of $f(z) \in L^\infty(S^1)$ has Lebesgue measure zero and $|f|(v)$ is not a scalar operator. Then $W^*(uf(v))$ is an irreducible subfactor of R with index n, for some positive integer n.*

PROOF. Let N be the von Neumann subalgebra generated by $uf(v)$ and let $f(v) = w|f|(v)$ be the polar decomposition of $f(v)$. Then $R = \{uw, v\}''$ and $vuw = e^{2\pi i\theta}uwv$. Therefore, there is an automorphism θ of R such that $\theta(u) = uw$ and $\theta(v) = v$. As a consequence, $\{uw, |f|(v)\}$ has the same $*$-distribution as $\{u, |f|(v)\}$. Note that $|f|^2(v) = (uf(v))^*(uf(v)) \in N$. So $|f|(v) \in N$ and $|f|^{-1}(v)$ is an (unbounded) operator affiliated with N. Thus $uw = uw|f|(v)|f|^{-1}(v)$ is a bounded operator affiliated with N. Therefore, $uw \in N$. By Theorem 4.2, $N = W^*(uw, |f|(v))$ is an irreducible subfactor of R with index n, for some positive integer n.

The above corollary shows that under a very general assumption $W^*(uf(v))$ is an irreducible subfactor of R. Recall that an operator $x \in R$ is called a *strongly irreducible operator relative to R* if there does not exist a non-trivial idempotent p in $\{x\}' \cap R$ [8]. So an operator $x \in R$ is strongly irreducible relative to R if and only if for every invertible operator $z \in R$, $W^*(zxz^{-1})$ is an irreducible subfactor of R.

THEOREM 4.4. *Suppose $f(z) \in C(S^1)$ such that the set of zeros of $f(z)$ has Lebesgue measure zero and is non-empty. Then $uf(v)$ is strongly irreducible relative to R.*

PROOF. Let $x = uf(v)$ and $y = \sum_{n=-\infty}^{\infty} u^n f_n(v) \in \{x\}' \cap R$. We have $f_0(e^{2\pi i\theta}v)f(v) = f(v)f_0(v)$ by comparing the coefficients of u. By the assumption of the theorem, $f_0(e^{2\pi i\theta}v) = f_0(v)$. By the ergodicity of irrational rotation, $f_0(v) = \lambda_0$, for some complex number λ_0. We have

$$f(e^{2\pi in\theta}v)f_n(v) = f_n(e^{2\pi i\theta}v)f(v)$$

by comparing the coefficients of u^{n+1} for $n \geq 1$. Thus,

$$\frac{f_n(e^{2\pi i\theta}v)}{f(e^{2\pi in\theta}v)f(e^{2\pi i(n-1)\theta}v)\cdots f(e^{2\pi i\theta}v)} = \frac{f_n(v)}{f(e^{2\pi i(n-1)\theta}v)f(e^{2\pi i(n-2)\theta}v)\cdots f(v)}.$$

By the ergodicity of irrational rotation,

$$f_n(v) = \lambda_n f(e^{2\pi i(n-1)\theta}v) f(e^{2\pi i(n-2)\theta}v) \cdots f(v)$$

for a complex number λ_n. Note that

$$x^n = uf(v)uf(v)\cdots uf(v) = u^n f(e^{2\pi i(n-1)\theta}v) f(e^{2\pi i(n-2)\theta}v) \cdots f(v).$$

So $u^n f_n(v) = \lambda_n x^n$. We have $f_{-n}(e^{2\pi i\theta}v) f(v) = f(e^{-2\pi in\theta}v) f_{-n}(v)$ by comparing the coefficients of $u^{-(n-1)}$ for $n \geq 1$. Similarly,

$$f_{-n}(v) = \lambda_{-n}\big(f(e^{-2\pi in\theta}v)\cdots f(e^{-2\pi i\theta}v)\big)^{-1}.$$

By the assumption of the theorem, $\lambda_{-n} = 0$, since

$$\big(f(e^{-2\pi in\theta}v)\cdots f(e^{-2\pi i\theta}v)\big)^{-1}$$

is an unbounded operator. Thus $y = \sum_{n=0}^{\infty} \lambda_n x^n$. If $y^2 = y$, then clearly $y = 0$ or $y = 1$. So x is strongly irreducible relative to R.

Now we have the following corollary, which was first proved in [8] by a different method.

COROLLARY 4.5. *$u + v$ is strongly irreducible relative to R.*

5. Invariant subspaces of $uf(v)$ relative to R

The following theorem is Theorem 2.2 of [11].

THEOREM 5.1. *Let $T \in M$, and for $n \in \mathbb{N}$, let $\mu_n \in$ Prob$([0,\infty))$ denote the distribution of $(T^n)^*T^n$ with respect to τ. Let ν_n denote the push-forward measure of μ_n under the map $t \to t^{\frac{1}{n}}$. Moreover, let ν denote the push-forward measure of μ_T under the map $z \to |z|^2$, i.e., ν is determined by*

$$\nu([0,t^2]) = \mu_T(\overline{\mathbb{B}(0,t)}), \qquad t > 0.$$

Then $\nu_n \to \nu$ weakly in Prob$([0,\infty))$.

LEMMA 5.2. *If $f(z) \in L^\infty(S^1)$, then for almost all $z \in S^1$,*

$$\lim_{n\to\infty}\left|\prod_{k=0}^{n-1}|f|(\alpha^k z)\right|^{1/n} = \Delta(f(v)).$$

PROOF. If $\int_0^1 \ln|f(e^{2\pi ix})|\,dx > -\infty$, then $\int_0^1 \big|\big(\ln|f(e^{2\pi ix})|\big)\big|\,dx < \infty$. By Birkhoff's Ergodic Theorem, the lemma holds. If $\int_0^1 \ln|f(e^{2\pi ix})|\,dx =$

$-\infty$, then $\Delta(f(v)) = 0$. We may assume that $|f(z)| \leq 1$, for all $z \in S^1$. For $m \in \mathbb{N}$, define $f_m(z) = \max\{|f(z)|, 1/m\}$. For each m and every z,

$$\limsup_{n\to\infty} \left| \prod_{k=0}^{n-1} |f|(\alpha^k z) \right|^{1/n} \leq \lim_{n\to\infty} \left| \prod_{k=0}^{n-1} f_m(\alpha^k z) \right|^{1/n}.$$

Hence, for almost all $z \in S^1$,

$$\limsup_{n\to\infty} \left| \prod_{k=0}^{n-1} |f|(\alpha^k z) \right|^{1/n} \leq \Delta(f_m(v)).$$

By the proof of Theorem 2.3, $\lim_{m\to\infty} \Delta(f_m(v)) = \Delta(f(v)) = 0$. So for almost all $z \in S^1$,

$$\limsup_{n\to\infty} \left| \prod_{k=0}^{n-1} |f|(\alpha^k z) \right|^{1/n} \leq 0.$$

This implies the lemma.

The next result follows from Theorem 5.4 of [6]. (The authors thank the referee for pointing this out.) In this paper we give a more direct proof.

THEOREM 5.3. *If $f(z) \in L^\infty(S^1)$, then the Brown measure of $uf(v)$ is the Haar measure on the circle $\Delta(f(v))S^1$.*

PROOF. Let $T = uf(v)$, and let ν and ν_n be the measures defined as in Theorem 5.1. Then ν_n converges weakly to ν. On the other hand, $((T^n)^*T^n)^{1/n} = |f(v)\cdots f(\alpha^{n-1}v)|^{2/n}$, where $\alpha = e^{2\pi i\theta}$. So we can view $((T^n)^*T^n)^{1/n}$ as the multiplication operator on $L^2[0,1]$ corresponding to the function

$$\left| \prod_{k=0}^{n-1} \big(|f|^2(\alpha^k z)\big) \right|^{1/n}.$$

By Lemma 5.2, for almost all $z \in S^1$,

$$\lim_{n\to\infty} \left| \prod_{k=0}^{n-1} \big(|f|^2(\alpha^k z)\big) \right|^{1/n} = \Delta(f(v))^2.$$

Thus ν_n converges weakly to the Dirac measure $\delta_{\Delta(f(v))^2}$ in $\mathrm{Prob}([0,\infty))$. Therefore, ν is the Dirac measure $\delta_{\Delta(f(v))^2}$ and the support of μ_T is contained in $\Delta(f(v))S^1$. Since μ_T is rotationally invariant, μ_T is the Haar measure on $\Delta(f(v))S^1$.

In [2], the spectrum and the Brown measure of $u + \lambda v$ are calculated. As an application of Theorem 3.3 and Theorem 5.3, we give another method to calculate the spectrum and Brown spectrum of $u + \lambda v$. Note that our method is very different from the method used in [2], which uses analytical function theory to calculate the spectrum and Brown spectrum of $u + \lambda v$.

COROLLARY 5.4. *The spectrum of $u + \lambda v$ is*

$$\sigma(u+\lambda v) = \begin{cases} S^1 & |\lambda < 1|; \\ \overline{\mathbb{B}(0,1)} & |\lambda| = 1; \\ \lambda S^1 & |\lambda| > 1, \end{cases}$$

and the Brown spectrum of $u + \lambda v$ is the Haar measure on λS^1.

PROOF. Let $w = u^* v$. Then $R = W^*(u, v) = W^*(u, w)$, $vu = e^{2\pi i\theta} uv$ and $wu = e^{2\pi i\theta} uw$. Therefore, there is an automorphism θ of R such that $\theta(u) = u$ and $\theta(v) = w$. Now $\theta(u(1 + \lambda v)) = u(1 + \lambda w) = u + \lambda v$, so the spectrum and Brown spectrum of $u + \lambda v$ and $u(1 + \lambda v)$ are the same. Now the corollary follows from Theorem 3.3 and Theorem 5.3.

Combining with the main result of [11] and Theorem 5.3, we have the following.

COROLLARY 5.5. *If $\Delta(f(v)) > 0$, then $uf(v)$ has a continuous family of hyperinvariant subspaces affiliated with R. In particular, if f is a polynomial then $\Delta(f(v)) > 0$.*

PROOF. Suppose $f(z)$ is a polynomial. Then $f(z) = \alpha(z - z_1)\ldots(z - z_n)$. So $\Delta(f(v)) = |\alpha|\Delta(v - z_1)\cdots\Delta(v - z_n)$. If $|z_i| \neq 1$, then $v - z_i$ is an invertible operator. Therefore, $\Delta(v - z_i) > 0$. If $|z_i| = 1$, then $\Delta(v - z_i) = \Delta(v - 1) = 1$. Thus $\Delta(f(v)) > 0$.

On the other hand, there are $f \in C(S^1)$ such that $\Delta(f(v)) = 0$. For example, for $p \geq 1$, let

$$g(x) = \begin{cases} 0, & x = 0, \\ \exp(-1/x^p), & 0 < x \leq 1/2, \\ \exp(-1/(1-x)^p), & 1/2 \leq x < 1, \\ 0, & x = 1. \end{cases}$$

Then $g(x)$ is a continuous function on $[0, 1]$ and $g(x) = g(1-x)$ for $x \in [0, 1]$. Therefore, there exists a continuous function $f(z)$ on S^1 with a single zero such that $f(e^{2\pi ix}) = g(x)$. Now

$$\Delta(f(v)) = \exp\left(\int_0^1 \ln f(e^{2\pi ix})\,dx\right) = 0.$$

In this case the Brown measure of $uf(v)$ is the Dirac measure. So the main result of [11] does not apply to this case. In the following we will show that indeed the established methods can not determine whether $uf(v)$ has a non-trivial invariant subspace affiliated with R in this case.

Recall that if M is a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and $T \in M$, Haagerup's invariant subspace of T is defined by [5], [16]:

$$\mathscr{E}_r(T) := \left\{\xi \in \mathscr{H} : \limsup_n \gamma_n \|T^n(\xi)\|^{1/n} \le r\right\} \quad \text{and} \quad \mathscr{H}_r(T) = \overline{\mathscr{E}_r(T)}.$$

This subspace is closed, T-invariant, affiliated to M and moreover, hyperinvariant. However, we will prove that for any sequence $\{\gamma_n\}_n$ and for any $r > 0$ this subspace is trivial for $uf(v)$.

Proposition 5.6. *Let $r > 0$ and $\{\gamma_n\}_n$ be a sequence of positive numbers. The subspace $\mathscr{H}_r(uf(v))$ defined as above is either $\mathscr{H}$ or $\{0\}$ if the set of zeros of $f(z)$ has Lebesgue measure zero and $|f|(z)$ is not a scalar operator.*

Proof. First we show that $\mathscr{H}_r(uf(v))$ is also an invariant subspace of $(uf(v))^*$. Suppose

$$\limsup_n \gamma_n \|(uf(v))^n\xi\|^{1/n} \le r.$$

Let $f(v) = w|f|(v)$. Then $R = \{u, v\}'' = \{uw, v\}''$ and $v(uw) = e^{2\pi i\theta}(uw)v$. Replacing u by uw and $f(v)$ by $|f|(v)$, we may assume that $f(v)$ is a positive operator. So $(uf(v))^* = f(v)u^*$. Let $\alpha = e^{2\pi i\theta}$. Then

$$\begin{aligned}
\|(uf(v))^n(uf(v))^*\xi\| &= \|u^{n-1}f(\alpha^{n-2}v)\cdots f(v)f^2(\alpha^{-2}v)\xi\| \\
&= \|f(\alpha^{n-2}v)\cdots f(v)f^2(\alpha^{-2}v)\xi\| \\
&\le \|f^2(\alpha^{-2}v)\|\,\|f(\alpha^{n-2}v)\cdots f(v)\xi\|.
\end{aligned}$$

So

$$\|(uf(v))^n(uf(v))^*\xi\|^{1/n} \le \|f^2(\alpha^{-2}v)\|^{1/n}\|f(\alpha^{n-2}v)\cdots f(v)\xi\|^{1/n}.$$

Note that

$$\|(uf(v))^n\xi\| = \|u^nf(\alpha^{n-1}v)\cdots f(v)\xi\| = \|f(\alpha^{n-1}v)\cdots f(v)\xi\|.$$

Therefore,

$$\limsup_n \gamma_n \|(uf(v))^n (uf(v))^* \xi\|^{1/n} \le r.$$

This proves that $\mathscr{H}_r(uf(v))$ is also an invariant subspace of $(uf(v))^*$. So the projection $P_{\mathscr{H}_r}$ is in the commutant algebra of $W^*(uf(v))$. By Corollary 4.3, $\mathscr{H}_r(uf(v))$ is either $\mathscr{H}$ or $\{0\}$.

In [16], Tucci introduced a class of quasinilpotent operators in the hyperfinite type II_1 factor. He showed for such operators one can find a nilpotent operator S in the commutant algebra of the quasinilpotent operator. Thus the range projection of S is a non-trivial invariant subspace of such operator. The following result tells us this idea does not apply to our case.

PROPOSITION 5.7. *Suppose $f(z) \in C(S^1)$ such that the set of zeros of $f(z)$ has Lebesgue measure zero and is non-empty. If S is a nilpotent operator in the commutant algebra of $uf(v)$, then $S = 0$.*

PROOF. By the proof of Theorem 4.4, $S = \sum_{n=0}^{\infty} \alpha_n (uf(v))^n$. So if S is a nilpotent operator, then $S = 0$.

QUESTION. Suppose $\Delta(f(v)) = 0$, the set of zero points of $f(z)$ has Lebesgue measure zero and $|f|(z)$ is not a scalar operator. Does $uf(v)$ have a non-trivial invariant subspace affiliated with R?

6. C^*-algebras generated by $uf(v)$ and 1

In this section, $f(z) \in C(S^1)$. Let $C^*(uf(v), 1)$ be the C^*-algebra generated by $uf(v)$ and 1. Let $x = uf(v)$. Recall that $\alpha = e^{2\pi i\theta}$. Since $vu = \alpha uv$, $g(v)u = ug(\alpha v)$ and $g(v)u^* = u^* g(\alpha^{-1} v)$ for $g(z) \in C(S^1)$. Repeatedly using the above two equations, we obtain

$$\begin{aligned}(x^*)^n x^n &= \bar{f}(v)u^* \cdots \bar{f}(v)u^* \bar{f}(v)u^* \bar{f}(v)u^* uf(v)uf(v)uf(v) \cdots uf(v) \\ &= \bar{f}(v)u^* \cdots \bar{f}(v)u^* |f|^2(v)|f|^2(\alpha v)uf(v) \cdots uf(v) \\ &= |f|^2(v)|f|^2(\alpha v) \cdots |f|^2(\alpha^{n-1} v)\end{aligned}$$

and

$$\begin{aligned}x^n (x^*)^n &= uf(v) \cdots uf(v)uf(v)uf(v)\bar{f}(v)u^* \bar{f}(v)u^* \bar{f}(v)u^* \cdots \bar{f}(v)u^* \\ &= uf(v) \cdots uf(v)u|f|^2(v)|f|^2(\alpha^{-1} v)u^* \bar{f}(v)u^* \cdots \bar{f}(v)u^* \\ &= |f|^2(\alpha^{-1} v)|f|^2(\alpha^{-2} v) \cdots |f|^2(\alpha^{-n} v).\end{aligned}$$

Let A be the C^*-subalgebra of $C^*(uf(v), 1)$ generated by 1 and $\{(x^*)^n x^n, x^n(x^*)^n\}_{n=1}^{\infty}$. Then

$$A = C^*(1, |f|^2(v), |f|^2(\alpha^{-1}v), |f|^2(v)|f|^2(\alpha v),\\ |f|^2(\alpha^{-1}v)|f|^2(\alpha^{-2}v), |f|^2(v)|f|^2(\alpha v)|f|^2(\alpha^2 v), \ldots).$$

Note that for a positive operator h, $h \in A$ if and only if $\sqrt{h} \in A$. We have

$$A = C^*(1, |f|(v), |f|(\alpha^{-1}v), |f|(v)|f|(\alpha v),\\ |f|(\alpha^{-1}v)|f|(\alpha^{-2}v), |f|(v)|f|(\alpha v)|f|(\alpha^2 v), \ldots). \quad (6.1)$$

In the following we identify $C^*(v)$ with $C(S^1)$ by the Gelfand theorem and thus we view A as a unital subalgebra of $C(S^1)$. Note that

$$C^*(uf(v), 1) = C^*(uf(v), A).$$

THEOREM 6.1. *If $f(v)$ is an invertible operator, then $C^*(uf(v), 1) \cong C^*(u, v^n)$ for some $n = 0, 1, 2, \ldots$. Furthermore, if $|f|(z)$ is not a periodic function then $C^*(uf(v), 1) = C^*(u, v)$.*

PROOF. If $|f|(v)$ is a scalar operator, then $uf(v)$ is a Haar unitary operator. Therefore, $C^*(uf(v), 1) \cong C^*(u)$. Assume that $|f|(v)$ is a non-scalar invertible operator. Then $f(v) = u_1|f|(v)$, for some unitary operator $u_1 \in C^*(v)$. So $uu_1 = uf(v)|f|(v)^{-1} \in C^*(uf(v), 1)$. By (6.1), $A = C^*\{|f|(\alpha^k v) : k \in \mathbb{Z}\}$. If A separates points of S^1, then the Stone-Weierstrass theorem implies that $A = C^*(v)$. Thus $C^*(uf(v), 1) = C^*(uf(v), v) = C^*(u, v)$.

Now suppose A does not separate points of S^1. Then there exists $z_1 \neq z_2$, $z_1, z_2 \in S^1$, such that $|f|(\alpha^k z_1) = |f|(\alpha^k z_2)$, for all $k \in \mathbb{Z}$. Since $\{\alpha^k : k \in \mathbb{Z}\}$ is dense in S^1, we have $f(zz_1) = f(zz_2)$. Let $z_0 = z_2 z_1^{-1}$ and replace z by zz_1^{-1}. Then $|f|(z) = |f|(z_0 z)$, for all $z \in S^1$. Suppose $|f|(z) = \sum_{k\in\mathbb{Z}} \alpha_k z^k$ is the Fourier series of $|f|(z)$. Then $|f|(z_0 z) = \sum_{k\in\mathbb{Z}} \alpha_k z_0^k z^k = \sum_{k\in\mathbb{Z}} \alpha_k z^k$. If $\alpha_n \neq 0$, then $z_0^n = 1$. Let $n = \gcd\{k : \alpha_k \neq 0\}$. Then by the proof of Theorem 4.2, $|f|(z)$ is a periodic function with a minimal period $e^{2\pi i/n}$. Since $\{\alpha^k : k \in \mathbb{Z}\}$ is dense in the unit circle S^1, we get $|f|(e^{2\pi i t}v) \in A$, for all $t \in [0, 1]$. For $k \in \mathbb{Z}$ and $z \in S^1$,

$$g_k(z) = \int_0^1 e^{2\pi i k t}|f|(e^{-2\pi i t}z)\, dt = \alpha_k z^k.$$

Thus if $\alpha_k \neq 0$, then $\alpha_k^{-1} g_k(v) = v^k \in A$. Since $n = \gcd\{k : \alpha_k \neq 0\}$, we have $v^n \in A$. Conversely, since $|f|(\alpha^k z)$ has period $e^{2\pi i/n}$, $|f|(\alpha^k v) \in C^*(v^n)$.

Thus $A = C^*(v^n)$. Therefore,

$$\begin{aligned} C^*(uf(v), 1) &= C^*(uf(v), A) = C^*(uu_1|f|(v), v^n) \\ &= C^*(uu_1, v^n) \cong C^*(u, v^n). \end{aligned}$$

Let Y be the set of zeros of $f(z)$. Then Y is also the set of zeros of $|f|(z)$. In the following we assume that $Y \neq \emptyset$. Define $\phi(z) = \alpha z = e^{2\pi i\theta} z$. For $\xi \in S^1$ denote by

$$\mathrm{Orb}(\xi) = \{\phi^n(\xi) : n \in \mathbb{Z}\}$$

the orbit of ξ under the rotation ϕ. By Proposition 2.5 of [8], the following conditions are equivalent:

(1) $\phi^n(Y) \cap Y = \emptyset$ for every integer $n \neq 0$;

(2) for each $\xi \in S^1$, $\mathrm{Orb}(\xi) \cap Y$ contains at most one point;

(3) $Y_1 \cap Y_2 = \emptyset$, where $Y_1 = \cup_{n \geq 0} \phi^n(Y)$ and $Y_2 = \cup_{k \geq 1} \phi^{-k}(Y)$.

By the proof of Corollary 4.6 of [8], if F is a Lebesgue measurable subset of S^1 satisfying the above conditions, then $m(F) = 0$. Recall that

$$\begin{aligned} A = C^*(1, |f|(v), |f|(\alpha^{-1}v), |f|(v)|f|(\alpha v), \\ |f|(\alpha^{-1}v)|f|(\alpha^{-2}v), |f|(v)|f|(\alpha v)|f|(\alpha^2 v), \ldots). \end{aligned}$$

LEMMA 6.2. *Let Y be the zero set of $f(z)$. If Y satisfies one of the above conditions* (1)–(3)*, then $A = C^*(v^n)$ for some natural number $n \geq 1$. Furthermore, $A = C^*(v)$ if and only if $|f|(z)$ is not a periodic function.*

PROOF. By the Stone-Weierstrass theorem, if A separates points of S^1 then $A = C^*(v)$. Otherwise, there exists $z_1 \neq z_2$, $z_1, z_2 \in S^1$, such that $g(z_1) = g(z_2)$ for all $g \in A$. Suppose $|f|(\alpha^k z_1) = |f|(\alpha^k z_2) \neq 0$ for all $k = 0, 1, 2, \ldots$. Since $\{\alpha^k : k \in \mathbb{N}\}$ is dense in S^1, $|f|(zz_1) = |f|(zz_2)$ for all $z \in S^1$. Replacing z by zz_1^{-1}, we have $|f|(z) = |f|(zz_0)$, where $z_0 = z_2 z_1^{-1}$. Thus $|f|(z)$ is a periodic function with period z_0. Suppose $|f|(\alpha^k z_1) = |f|(\alpha^k z_2) = 0$ for some $k = 0, 1, 2, \ldots$. Then we claim $|f|(\alpha^{-k} z_1) = |f|(\alpha^{-k} z_2) \neq 0$ for all $k \in \mathbb{N}$. Otherwise $|f|(\alpha^{-k'} z_1) = |f|(\alpha^{-k'} z_2) = 0$ for some $k' \in \mathbb{N}$. Now both $\alpha^k z_1$ and $\alpha^{-k'} z_1$ belong to Y. This contradicts condition (2) before the lemma. Thus $|f|(\alpha^{-k} z_1) = |f|(\alpha^{-k} z_2) \neq 0$ for all $k \in \mathbb{N}$. A similar argument shows that $|f|(z)$ is a periodic function. Let $e^{2\pi i/n}$ be a minimal period of $f(z)$. We claim $A = C^*(v^n)$. Let X be the quotient space $\{e^{2\pi it} : t \in [0, 2\pi/n]\}/\{1, e^{2\pi i/n}\}$. Then A and v^n can be viewed as continuous functions on X. Note that v^n separates points of X. By the Stone-Weierstrass theorem, $A \subseteq C^*(v^n)$. We claim A also separates points of X. Otherwise, there exists $z_1 \neq z_2$, $z_1, z_2 \in X$, such that $g(z_1) = g(z_2)$ for all $g \in A$. By a similar argument we have $|f|(z) = |f|(zz_0)$, where $z_0 = z_2 z_1^{-1}$. So $e^{\frac{2\pi i}{n}}$ is not a minimal period of $f(z)$. This is a contradiction. Thus $C^*(v^n) \subseteq A$ and hence $A = C^*(v^n)$.

THEOREM 6.3. *Let Y be the zero set of $f(z)$. If Y satisfies one of the conditions* (1)–(3) *before Lemma 6.2, then $C^*(uf(v), 1)$ is a generalized universal irrational rotation C^*-algebra. Furthermore, if $|f|(z)$ is not a periodic function, then $C^*(uf(v), 1) \cong A_{\theta,|f|^2}$.*

PROOF. By Lemma 6.2, if $|f|(z)$ is not a periodic function then $A = C^*(v)$ and if $|f|(z)$ is a periodic function, then $A = C^*(v^n)$ for some $n \geq 2$. In the first case, let $f(v) = u_1|f|(v)$ be the polar decomposition of $f(v)$. Since Y satisfies one of the conditions above Lemma 6.2, $m(Y) = 0$. Thus u_1 is a unitary operator in the von Neumann algebra generated by v. So

$$C^*(uf(v), 1) = C^*(uf(v), A) = C^*(uu_1|f(v)|, v) \cong A_{\theta,|f|^2}.$$

In the second case, $f(v) = u_1|f|(v)$ and $|f| \in C^*(v^n)$. So there exists a positive continuous function $g(z)$ on the unit circle such that $f(v) = g(v^n)$. Therefore,

$$\begin{aligned} C^*(uf(v), 1) &= C^*(uf(v), A) = C^*(uu_1|f(v)|, v^n) = C^*(uu_1|g(v^n)|, v^n) \\ &= C^*(uu_1|g(w)|, w) \cong A_{n\theta,|g|^2}. \end{aligned}$$

PROPOSITION 6.4. *Suppose $|f|(z)$ is not a periodic function and Y is the zero points of $f(z)$. Then the following conditions are equivalent:*

(1) *$C^*(uf(v), 1)$ is a simple algebra;*

(2) *$\phi^n(Y) \cap Y = \emptyset$ for all integers $n \neq 0$;*

(3) *for each $\xi \in S^1$,* Orb(ξ) *$\cap\, Y$ contains at most one point;*

(4) *$Y_1 \cap Y_2 = \emptyset$, where $Y_1 = \cup_{n\geq 0}\phi^n(Y)$ and $Y_2 = \cup_{k\geq 1}\phi^{-k}(Y)$.*

PROOF. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follows from Proposition 2.5 of [8]. (4) $\Rightarrow$ (1) follows from Theorem 6.3. We need to prove (1) $\Rightarrow$ (4). Suppose $Y_1 \cap Y_2 \neq \emptyset$. Let $x = uf(v)$ and $\gamma(z) = |f|^2(z)$. Then there exists $\lambda \in S^1$, $m \geq 0$, $n \geq 1$ such that λ is a zero of $\gamma(e^{2\pi in\theta}z)$ and $\gamma(e^{-2\pi im\theta}z)$. Consider the subset

$$\begin{aligned} J = \{\varphi(v) : \varphi(v) \in A \text{ and } &\varphi(e^{2\pi in\theta}\lambda) \\ &= \cdots = \varphi(\lambda) = \cdots = \varphi(e^{-2\pi im\theta}\lambda) = 0\} \end{aligned}$$

of $C^*(v)$. By the definition (6.1) of A,

$$|f|(\alpha^{-m}v)\cdots|f|(\alpha^{-1}v)|f|(v)|f|(\alpha v)\cdots|f|(\alpha^n v) \in J.$$

So J is a non-empty ideal of A.

We claim that $I = C^*(x, 1)JC^*(x, 1)$ is a two-sided ideal of $C^*(x, 1)$. Otherwise, there exists $\varphi_i(v) \in J$,

$$a_i = \sum_{n=1}^{K}(x^*)^n g_{-n}^i(v) + g^i(v) + \sum_{n=1}^{K} g_n^i(v)x^n,$$

and

$$b_i = \sum_{n=1}^{K}(x^*)^n h_{-n}^i(v) + h^i(v) + \sum_{n=1}^{K} h_n^i(v)x^n,$$

with $g_n^i, g^i, h_n^i, h^i \in C(\mathbb{T})$ and $K \in \mathbb{N}$ sufficiently large such that

$$\left\| \sum_{i=1}^{N} a_i\varphi_i(v)b_i - 1 \right\| < 1,$$

By Lemma 3.1 and simple computations, we have

$$\begin{aligned}
\Bigg\| \sum_{i=1}^{N} & g_{-K}^i(e^{2\pi iK\theta}v)\varphi_i(e^{2\pi iK\theta}v)h_K^i(e^{2\pi iK\theta}v)\gamma(e^{2\pi i(K-1)\theta}v)\cdots\gamma(v) \\
&+ g_{-(K-1)}^i(e^{2\pi i(K-1)\theta}v)\varphi_i(e^{2\pi i(K-1)\theta}v)h_{K-1}^i(e^{2\pi i(K-1)\theta}v)\gamma(e^{2\pi i(K-2)\theta}v)\cdots\gamma(v) \\
&+ \cdots + g_{-1}^i(e^{2\pi i\theta}v)\varphi_i(e^{2\pi i\theta}v)h_1^i(e^{2\pi i\theta}v)\gamma(v) \\
&+ g^i(v)\varphi_i(v)h^i(v) + g_1^i(v)\varphi_i(e^{-2\pi i\theta}v)h_{-1}^i(v)\gamma(e^{-2\pi i\theta}v) \\
&+ \cdots + g_{K-1}^i(v)\varphi_i(e^{-2\pi i(K-1)\theta}v)h_{-(K-1)}^i(v)\gamma(e^{-2\pi(K-1)i\theta}v)\cdots\gamma(e^{-2\pi i\theta}v) \\
&+ g_K^i(v)\varphi_i(e^{-2\pi iK\theta}v)h_{-K}^i(v)\gamma(e^{-2\pi Ki\theta}v)\cdots\gamma(e^{-2\pi i\theta}v) - 1 \Bigg\| < 1.
\end{aligned}$$

Let

$$\begin{aligned}
\psi(z) = \sum_{i=1}^{N} & g_{-K}^i(e^{2\pi iK\theta}z)\varphi_i(e^{2\pi iK\theta}z)h_K^i(e^{2\pi iK\theta}z)\gamma(e^{2\pi i(K-1)\theta}z)\cdots\gamma(z) \\
&+ \cdots + g_{-1}^i(e^{2\pi i\theta}z)\varphi_i(e^{2\pi i\theta}z)h_1^i(e^{2\pi i\theta}z)\gamma(z) + g^i(z)\varphi_i(z)h^i(z) \\
&+ g_1^i(z)\varphi_i(e^{-2\pi i\theta}z)h_{-1}^i(z)\gamma(e^{-2\pi i\theta}z) \\
&+ \cdots + g_K^i(z)\varphi_i(e^{-2\pi iK\theta}z)h_{-K}^i(z)\gamma(e^{-2\pi Ki\theta}z)\cdots\gamma(e^{-2\pi i\theta}z).
\end{aligned}$$

Since $\varphi_i(z) \in J$, $\varphi_i(e^{2\pi in\theta}\lambda) = \cdots = \varphi_i(\lambda) = \cdots = \varphi_i(e^{-2\pi im\theta}\lambda) = 0$. Note that $\gamma(e^{2\pi in\theta}\lambda) = \gamma(e^{-2\pi im\theta}\lambda) = 0$. So $\psi(\lambda) = 0$. Hence $\|\psi(z) - 1\| \geq$

1 and $\|\psi(v) - 1\| \geq 1$. By Lemma 3.1,

$$\left\| \sum_{i=1}^{N} a_i \varphi_i(v) b_i - 1 \right\| \geq \|\psi(v) - 1\| \geq 1.$$

This is a contradiction.

COROLLARY 6.5. *If $A_{\theta,\gamma} = C^*(u\gamma^{1/2}(v), v)$ is a simple generalized universal irrational rotation C^*-algebra, then $A_{\theta,\gamma}$ is generated by an element $uf(v)$ and the identity operator for some $f(z) \in C(S^1)$.*

PROOF. Since $A_{\theta,\gamma} = C^*(u\gamma(v)^{1/2}, v)$ is simple, by Corollary 6.5 of [8] the zeros of $\gamma(z)^{1/2}$ satisfies conditions (1)–(3) above Lemma 6.2. If $\gamma(z)$ is not a periodic function, then $\gamma(z)^{1/2}$ is not a periodic function. By Theorem 6.3,

$$A_{\theta,\gamma} = C^*(u\gamma(v)^{1/2}, v) \cong C^*(u\gamma(v)^{1/2}, 1).$$

If $\gamma(z)$ is a periodic function, then either $\gamma(z)|2+z|$ or $\gamma(z)|3+z|$ is not a periodic function. Otherwise $\frac{|3+z|}{|2+z|}$ will be a periodic function. Assume that $\gamma(z)|2+z|$ is not a periodic function. Then

$$\begin{aligned} A_{\theta,\gamma} &= C^*(u\gamma(v)^{1/2}, v) = C^*(u\gamma(v)^{1/2}|2+v|^{1/2}, v) \\ &\cong C^*(u\gamma(v)^{1/2}|2+v|^{1/2}, 1). \end{aligned}$$

COROLLARY 6.6. *Suppose $f(z)$ has a single zero. Then $A = C^*(v)$ and $C^*(uf(v), 1) = C^*(uf(v), v) \cong A_{\theta,|f|^2}$ is a simple generalized universal irrational rotation C^*-algebra.*

LEMMA 6.7. *Suppose $f(z)$ has two zeros. Then $A = C^*(v)$ or $A = C^*(v^2)$. Furthermore, $A = C^*(v)$ if and only if $|f|(z)$ is not a periodic function.*

PROOF. By the Stone-Weierstrass theorem, if A separates points of S^1 then $A = C^*(v)$. Otherwise, there exists $z_1 \neq z_2$, $z_1, z_2 \in S^1$, such that $g(z_1) = g(z_2)$ for all $g \in A$. Since $f(z)$ has two zeros, $|f|(z)$ has two zeros. Suppose $|f|(\alpha^k z_1) = |f|(\alpha^k z_2) \neq 0$ for all $k = 0, 1, 2, \ldots$. Since $\{\alpha^k : k \in \mathbb{N}\}$ is dense in S^1, $|f|(zz_1) = |f|(zz_2)$ for all $z \in S^1$. Replacing z by zz_1^{-1}, we have $|f|(z) = |f|(zz_0)$, where $z_0 = z_2 z_1^{-1}$. Thus $|f|(z)$ is a periodic function with period z_0. Since $|f|(z)$ has exactly two zeros, $|f|(z)$ is periodic function with minimal period $e^{\pi i}$. Thus $A = C^*(v^2)$. Suppose $|f|(\alpha^k z_1) = |f|(\alpha^k z_2) = 0$ for some $k = 0, 1, 2, \ldots$. Then claim $|f|(\alpha^{-k} z_1) = |f|(\alpha^{-k} z_2) \neq 0$ for all $k \in \mathbb{N}$. Otherwise $|f|(\alpha^{-k'} z_1) = |f|(\alpha^{-k'} z_2) = 0$ for some $k' \in \mathbb{N}$. Since $|f|(z)$ has exactly two zeros, $\{\alpha^k z_1, \alpha^k z_2\} = \{\alpha^{-k'} z_1, \alpha^{-k'} z_2\}$. Since $\alpha = e^{2\pi i\theta}$ and θ is irrational, this is impossible. Thus $|f|(\alpha^{-k} z_1) = |f|(\alpha^{-k} z_2) \neq 0$ for all $k \in \mathbb{N}$. A similar argument shows that $|f|(z)$ is periodic function with minimal period $e^{\pi i}$. Thus $A = C^*(v^2)$.

Proposition 6.8. *Suppose $f(z)$ has two zeros. Then $C^*(uf(v), 1)$ is a generalized universal irrational rotation C^*-algebra. Furthermore, if $|f|(z)$ is not a periodic function, then $C^*(uf(v), 1) \cong A_{\theta,|f|^2}$.*

Proof. By Lemma 6.7, if $|f|(z)$ is not a periodic function then $A = C^*(v)$ and if $|f|(z)$ is a periodic function, then $A = C^*(v^2)$. In the first case, let $f(v) = u_1|f|(v)$ be the polar decomposition of $f(v)$. Then u_1 is a unitary operator in the von Neumann algebra generated by v. So

$$C^*(uf(v), 1) = C^*(uf(v), A) = C^*(uu_1|f(v)|, v) \cong A_{\theta,|f|^2}.$$

In the second case, $f(v) = u_1|f|(v)$ and $|f| \in C^*(v^2)$. So there exists a positive continuous function $g(z)$ on the unit circle such that $f(v) = g(v^2)$. Therefore,

$$\begin{aligned} C^*(uf(v), 1) = C^*(uf(v), A) = C^*(uu_1|f(v)|, v^2) = C^*(uu_1|g(v^2)|, v^2) \\ = C^*(uu_1|g(w)|, w) \cong A_{2\theta,|g|^2}. \end{aligned}$$

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