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A SIMPLE PROOF OF THE EXISTENCE OF HAAR MEASURE ON AMENABLE GROUPS

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(Dedicated to Joseph Diestel)

Abstract
A simple proof of the existence of Haar measure on amenable groups is given.

It is very well known that every locally compact group has a Haar measure and that the Haar measure is unique up to a positive multiplicative constant. Several different proofs have been given, all of them somewhat difficult. (See [2] for two proofs as well as references to others.) There are two special cases in which simple proofs of the existence of Haar measure are known. In the case of compact groups, a simple proof was given by von Neumann [3] (or see [4]), and in the case of locally compact *abelian* groups, a simple proof was given by the author [1]. The purpose of this short note is to give a simple proof for the case of *amenable* groups, which includes both compact groups and abelian groups.

There are many different equivalent definitions of amenable group. We recall the definition we will use here. Given a locally compact group G, let $C_b(G)$ denote the space of all bounded continuous real-valued functions on G. A *mean* on G is (for our purpose) a positive linear functional L on $C_b(G)$ such that $L(1) = 1$. Given a function f on G and $a \in G$, define the left translate f_a of f by a by the formula $f_a(x) = f(ax)$. A mean L on G is said to be (*left*) *invariant* if $L(f_a) = L(f)$ for all $f \in C_b(G)$ and all $a \in G$. A locally compact group G is said to be *amenable* if there exists an invariant mean on G.

The key to the proof below is the following lemma which is proved in [1] as an easy application of Zorn's lemma.

LEMMA 1.1. *Suppose G is a topological group and N is a neighborhood of the identity in G that is symmetric (i.e., $N^{-1} = N$). Then there exists a subset S of G such that for each g in G the set $gN \cdot N$ contains at* least *one element of S and the set gN contains at* most *one element of S.*

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PROOF OF THE EXISTENCE OF HAAR MEASURE ON AMENABLE GROUPS. Let G be an amenable group. Let $C_c(G)$ denote the space of compactly supported continuous real-valued functions on G. Fix a symmetric neighborhood N of the identity in G with compact closure, and let S be as in the above lemma. Define $T : C_c(G) \to C_b(G)$ by

$$(Tf)(g) = \sum_{s \in S} f(gs) \qquad (f \in C_c(G), g \in G). \tag{1}$$

Given $f \in C_c(G)$, the support of f can be covered by finitely many, n say, left translates of N. It is then easily shown that for each value of $g \in G$, the sum in (1) contains at most n nonzero terms, and hence the sum defines a bounded function. Furthermore, the sum is easily shown to be locally finite and hence yields a continuous function on G. Thus (1) does indeed define a map from $C_c(G)$ to $C_b(G)$. Note that T is a positive linear operator, in the sense that T sends nonnegative functions to nonnegative functions. Note also that T commutes with left translation (i.e., $T(f_a) = (Tf)_a$).

Since G is amenable, there exists an invariant mean L on G. Let $\Lambda = L \circ T$. Obviously Λ is a positive linear functional on $C_c(G)$, and it is easily verified that Λ is (left) invariant (i.e., $\Lambda(f_a) = \Lambda(f)$). Finally, choose a nonnegative function $f \in C_c(G)$ such that f is equal to 1 on $N \cdot N$, and note that then $(Tf)(g) \geq 1$ for all $g \in G$, and hence, $\Lambda(f) = L(Tf) \geq L(1) = 1$. Thus Λ is nonzero, and the proof is complete.

It had long been a mystery to the author why there are simple proofs of the existence of Haar measure on compact groups and on abelian groups, but seemingly, no simple proof in the case of arbitrary locally compact groups. The above argument provides an explanation: there are simple proofs of the existence of an invariant mean in both these special cases, and the existence of a Haar measure follows easily from the existence of an invariant mean.

A different proof of the existence of Haar measure on amenable groups that also uses ideas from [1], was found independently by Khadime Salame [5].

The referee pointed out that it is an unsolved problem whether every hypergroup has a Haar measure. Inspired by the present author's paper [1], Benjamin Willson established the existence of Haar measure on certain amenable hypergroups [6]. This leads to the question of whether the argument presented above can be applied to hypergroups.

It is a pleasure to dedicate this paper to Joseph Diestel whose beautiful talk about Haar measure at the Seventh Conference on Function Spaces in Edwardsville, Illinois on May 23, 2014 inspired me to think one more time about the existence of Haar measure.

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