MATH. SCAND. 120 (2017), 161–180

A FUNCTION ON THE SET OF ISOMORPHISM CLASSES IN THE STABLE CATEGORY OF MAXIMAL COHEN-MACAULAY MODULES OVER A GORENSTEIN RING: WITH APPLICATIONS TO LIAISON THEORY

TONY J. PUTHENPURAKAL

Abstract

Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 1$. Let $\underline{\mathrm{CM}}(A)$ be the stable category of maximal Cohen-Macaulay A-modules and let $\underline{\mathrm{ICM}}(A)$ denote the set of isomorphism classes in $\underline{\mathrm{CM}}(A)$. We define a function $\xi\colon \underline{\mathrm{ICM}}(A) \to \mathbb{Z}$ which behaves well with respect to exact triangles in $\underline{\mathrm{CM}}(A)$. We then apply this to (Gorenstein) liaison theory. We prove that if $\dim A \geq 2$ and A is not regular then the even liaison classes of $\{\mathfrak{m}^n \mid n \geq 1\}$ is an infinite set. We also prove that if A is Henselian with finite representation type with $A/\mathfrak{m}$ uncountable then for each $m \geq 1$ the set $\mathscr{C}_m = \{I \mid I \text{ is a codim 2 CM-ideal with } e_0(A/I) \leq m\}$ is contained in finitely many even liaison classes $L_1, \dots, L_r$ (here r may depend on m).

1. Introduction

Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 1$ and residue field k. We say an ideal $\mathfrak{q}$ is a Gorenstein ideal if it is perfect and $A/\mathfrak{q}$ is a Gorenstein ring. We should remark that some authors *do not* require in the definition of Gorenstein ideals that $\mathfrak{q}$ be perfect. *However we will require it to be so.*

We begin by recalling the definition of (Gorenstein) linkage.

Definition 1.1. Ideals I and J of A are (algebraically) *linked* by a Gorenstein ideal $\mathfrak{q}$ if

(a) $\mathfrak{q} \subseteq I \cap J$, and

(b) $I = (\mathfrak{q}\colon J)$ and $J = (\mathfrak{q}\colon I)$.

We write it as $I \sim_{\mathfrak{q}} J$.

If $\mathfrak{q}$ is a complete intersection ideal then we say that I is CI-linked to J. We say ideals I and J are in the *same linkage class* if there is a sequence of ideals $I_0, \dots, I_n$ in A and Gorenstein ideals $\mathfrak{q}_0, \dots, \mathfrak{q}_{n-1}$ such that

Received 28 January 2015, in final form 5 November 2015.
DOI: https://doi.org/10.7146/math.scand.a-25728

(i) $I_j \sim_{\mathfrak{q}_j} I_{j+1}$, for $j = 0, \dots, n-1$.

(ii) $I_0 = I$ and $I_n = J$.

If n is even then we say that I and J are *evenly linked*. We can analogously define CI-linkage classes and even CI-linkage classes.

The notion of linkage has been extended to modules [8]. See section 4 for the definition. Note that ideals I and J are linked as ideals if and only if the cyclic modules A/I and A/J are linked as modules; see [8, Proposition 1]. In this paper we prove three results in liaison theory of modules.

Result 1: In [10], Polini and Ulrich investigated when an ideal is the unique maximal element of its CI-linkage class, in the sense that it contains every ideal of the class. They showed that if $(A, \mathfrak{m})$ is a Gorenstein local ring of dimension $d \geq 2$, with $d \geq 3$ if A is regular, then every ideal in the linkage class of $\mathfrak{m}^t$ is contained in $\mathfrak{m}^t$ provided that the associated graded ring $G(A) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is Cohen-Macaulay, or A is a complete intersection, or ecodim $A \leq 3$, or $t \leq 3$ (here ecodim stands for embedding codimension of A). They conjectured that this holds in general. This was proved by Wang, see [14, Theorem 1.1]. We note that this result has a non-trivial application in constructing equimultiple ideals of reduction number one, see [14, Theorem 1.2].

In particular if A is regular and $\dim A \geq 3$ then the CI-liaison classes of $\mathfrak{m}^n$ for $n \geq 1$ are all distinct. This fails spectacularly for Gorenstein liaison. If $A = K[[X_1, \dots, X_n]]$ then $\mathfrak{m}^n$ is evenly linked to $\mathfrak{m}^{n-1}$ for all $n \geq 2$ (this follows from [6, Theorem 3.6]).

If $(A, \mathfrak{m})$ is a one-dimensional Gorenstein local ring then one can prove that there exists $s \geq 1$ such that $\mathfrak{m}^{sn+r}$ is evenly linked to $\mathfrak{m}^{s(n-1)+r}$ for all $n \gg 0$ and $r = 0, 1, \dots, s-1$; see Proposition 5.1 (here we can choose $s = 1$ if the residue field of A is infinite). A natural question is when is the set of ideals $\{\mathfrak{m}^n \mid n \geq 1\}$ is contained in finitely many even liaison classes. Our first result implies that the above two cases are essentially the only ones when the above condition holds. We prove the following more general result:

THEOREM 1.2. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring. Let M be a finitely generated A-module of dimension $r \geq 2$. If there exists finitely many even liaison classes of modules $L_1, L_2, \dots, L_m$ such that*

$$M/\mathfrak{m}^n M \in \bigcup_{i=1}^{m} L_i, \quad \textit{for all } n \geq 1,$$

then A is regular.

Result 2: Assume $(A, \mathfrak{m})$ is a complete equi-characteristic Gorenstein local ring. Let I be an ideal in A generated by a regular sequence. Using results in [6, Theorem 3.6] it can be proved that I^n is evenly linked to I^{n-1} for all $n \geq 2$, see Proposition 6.1. Thus the modules A/I^n is evenly linked to A/I^{n-1} for all $n \geq 2$. It follows that if F is a finitely generated free A-module then F/I^nF is evenly linked to $F/I^{n-1}F$ for all $n \geq 2$. A natural question is whether the set of modules $\{M/I^nM \mid n \geq 1\}$ is contained in finitely many even liaison classes when M is a maximal Cohen-Macaulay A-module. We prove the following surprising result:

THEOREM 1.3. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 2$. Let M be a maximal Cohen-Macaulay A-module. Let $x_1, \dots, x_r$ be an A-regular sequence with $r \geq 2$ and let $I = (x_1, \dots, x_r)$. If there exists finitely many even liaison classes of modules $L_1, L_2, \dots, L_m$ such that*

$$M/I^nM \in \bigcup_{i=1}^{m} L_i, \quad \textit{for all } n \geq 1,$$

then M is free.

Note that in the above result we *do not* assume that A is complete or contains a field. We do not know whether the result holds if $r = 1$. Our result implies that for $r \geq 2$ a regular sequence of length r can determine whether a maximal Cohen-Macaulay module is free.

Result 3: Let I be a perfect ideal of codimension 2. It is well-known that I is licci (i.e., it is CI-linked to a complete intersection). However an arbitrary codimension two Cohen-Macaulay ideal need not be licci. For instance if $(A, \mathfrak{m})$ is non-regular Gorenstein ring of dimension 2 then $\mathfrak{m}$ is not a licci-ideal (this is so because if I is licci then projdim A/I is finite.) So a natural question is whether codimension two Cohen-Macaulay ideals are contained in finitely many even liaison classes. Again this is not possible. Let $(A, \mathfrak{m})$ be a non-regular Gorenstein ring of dimension 2. Then by Theorem 1.2 the set of ideals $\{\mathfrak{m}^n \mid n \geq 1\}$ is not contained in finitely many even liaison classes of ideals in A. Note that $\ell(A/\mathfrak{m}^n) \to \infty$ as $n \to \infty$. So we reformulate the question. Let $\mathscr{C}_m = \{I \mid I \text{ is a codim 2 CM-ideal with } e_0(A/I) \leq m\}$. Here $e_0(A/I)$ is the multiplicity of the ring A/I with respect to its maximal ideal. Our question is whether $\mathscr{C}_m$ contained in finitely many even liaison classes of ideals. Regular rings trivially have this property. Our next result shows that most rings of finite representation type have this property. Recall a Henselian Cohen-Macaulay local ring B is said to be of finite representation type if it has only finitely many indecomposable maximal Cohen-Macaulay modules.

If A is Gorenstein and it is of finite representation type then it is an abstract hypersurface ring [4, 1.2]. We prove:

THEOREM 1.4. *Let* $(A, \mathfrak{m})$ *be a Henselian Gorenstein ring of finite representation type and dimension* $d \geq 2$. *Assume* $k = A/\mathfrak{m}$ *is uncountable. For* $m \geq 1$ *let*

$$\mathscr{C}_m = \{I \mid I \text{ is a codim } 2 \text{ CM-ideal with } e_0(A/I) \leq m\}.$$

Then for every $m \geq 1$ *there exists finitely many even liaison classes* $L_1, \dots, L_r$ (*depending on* m) *such that*

$$\mathscr{C}_m \subseteq \bigcup_{i=1}^{r} L_i.$$

For examples of hypersurfaces with finite representation type see [7]. The assumption k is uncountable is a bit irritating, however it is essential in our proof. We conjecture that the converse of this theorem is also true.

The technique to prove the above three results is new and involves a construction of "triangle functions" on the stable category of A. Let $\mathrm{CM}(A)$ denote the full subcategory of maximal Cohen-Macaulay A-modules and let $\underline{\mathrm{CM}}(A)$ denote the stable category of maximal Cohen-Macaulay A-modules. Recall that objects in $\underline{\mathrm{CM}}(A)$ are the same as objects in $\mathrm{CM}(A)$. However the set of morphisms $\underline{\mathrm{Hom}_A}(M, N)$ between M and N is equal to $\mathrm{Hom}_A(M, N)/P(M, N)$, where $P(M, N)$ is the set of A-linear maps from M to N which factor through a finitely generated free module. It is well-known that $\underline{\mathrm{CM}}(A)$ is a triangulated category with translation functor Ω^{-1}. Here $\Omega(M)$ denotes the syzygy module of M and $\Omega^{-1}(M)$ denotes the cosyzygy module of M. Also recall that an object M is zero in $\underline{\mathrm{CM}}(A)$ if and only if it is free considered as an A-module. Furthermore $M \cong N$ in $\underline{\mathrm{CM}}(A)$ if and only if there exists finitely generated free modules F, G with $M \oplus F \cong N \oplus G$ as A-modules. Let $\underline{\mathrm{ICM}}(A)$ denote the set of isomorphism classes in $\underline{\mathrm{CM}}(A)$ and for an object $M \in \underline{\mathrm{CM}}(A)$ denote its isomorphism class by $[M]$.

We say a function $\xi\colon \underline{\mathrm{ICM}}(A) \to \mathbb{Z}$ is a *triangle* function if it satisfies the following properties:

(1) $\xi([M]) \geq 0$, for all $M \in \underline{\mathrm{CM}}(A)$;

(2) $\xi([M]) = 0$ if and only if $M = 0$ in $\underline{\mathrm{CM}}(A)$;

(3) $\xi([M_1 \oplus M_2]) = \xi([M_1]) + \xi([M_2])$, for all $M_1, M_2 \in \underline{\mathrm{CM}}(A)$;

(4) (*sub-additivity*) if $M \to N \to L \to \Omega^{-1}(M)$ is an exact triangle in $\underline{\mathrm{CM}}(A)$ then
   (a) $\xi([N]) \leq \xi([M]) + \xi([L])$,
   (b) $\xi([L]) \leq \xi([N]) + \xi([\Omega^{-1}(M)])$ and
   (c) $\xi([\Omega^{-1}(M)]) \leq \xi([L]) + \xi([\Omega^{-1}(N)])$.

REMARK 1.5.
(i) Since rotations of exact triangles are exact it follows that if ξ satisfies (4)(b) for all exact triangles then it will also satisfy 4(a),(c).
(ii) Axiom (3) implies that $\xi([M]) = 0$ if $M = 0$ in $\underline{\mathrm{CM}}(A)$. However note that axiom (2) also implies that if $\xi([M]) = 0$ then $M = 0$ in $\underline{\mathrm{CM}}(A)$.

We have the following result on existence of triangle functions. Let $\ell(N)$ denote the length of an A-module N.

THEOREM 1.6. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 1$. Then the function*

$$e_A^T([M]) = \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \ell\left(\mathrm{Tor}_1^A\left(M, \frac{A}{\mathfrak{m}^{n+1}}\right)\right), \quad \textit{where } [M] \in \underline{\mathrm{ICM}}(A),$$

is a triangle function on $\underline{\mathrm{ICM}}(A)$.

Unlike the multiplicity function which can be defined uniquely through a set of axioms, triangle functions are highly non-unique. In §3.5 we will construct infinitely many triangle functions. However e_A^T is the simplest triangle function that we have constructed. It also behaves well with generic hyperplane sections, see Proposition 2.9 for details.

The existence of triangle functions has non-trivial implications in liaison theory. In fact we prove results 1 and 2 by using any triangle function. However for the third result we need some additional properties of e_A^T.

We now briefly describe the contents of the paper. In section 2 we introduce the function $e_A^T(-)$ and prove some of its basic properties. In section 3 we prove Theorem 1.6. In section 4 we discuss some results on liaison theory of modules and discuss the notion of maximal Cohen-Macaulay approximations. In section 5, 6 and 7 we prove Theorems 1.2, 1.3 and 1.4 respectively.

2. Pre-triangles in CM(A)

In this paper all rings are commutative Noetherian local and all modules are assumed to be finitely generated. In this section $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension $d \geq 1$ and residue field k. Let $\mathrm{ICM}(A)$ denote the set of isomorphism classes of maximal Cohen-Macaulay A-modules and for an object $M \in \mathrm{CM}(A)$, we denote its isomorphism class by $[M]$. In this section we study the function

$$e_A^T([M]) = \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \ell\left(\operatorname{Tor}_1^A\left(M, \frac{A}{\mathfrak{m}^{n+1}}\right)\right), \quad \text{where } [M] \in \mathrm{ICM}(A).$$

We also abstract some of its properties and call the notion a *pre-triangle* function.

2.1. Let M be an A-module. We denote it's first syzygy-module by $\Omega(M)$. If we have to specify the ring, then we write it as $\Omega_A(M)$.

Set $\Omega^1(M) = \Omega(M)$. For $i \geq 2$, define $\Omega^i(M) = \Omega(\Omega^{i-1}(M))$. It can be easily proved that $\Omega^i(M)$ are invariants of M.

2.2. The function $e_A^T(-)$ arose in the author's study of certain aspects of the theory of Hilbert functions [11], [12]. Let N be an A-module of dimension r. It is well-known that there exists a polynomial $P_N(z) \in \mathbb{Q}[z]$ of degree r such that $P_N(n) = \ell(N/\mathfrak{m}^{n+1}N)$ for all $n \gg 0$. We write

$$P_N(z) = \sum_{i=0}^{r} (-1)^i e_i(N) \binom{z+r-i}{r-i}.$$

Then $e_0(N), \ldots, e_r(N)$ are integers and are called the *Hilbert coefficients* of N. The number $e_0(N)$ is called the *multiplicity* of N. It is positive if N is non-zero. The number $e_1(N)$ is *non-negative* if N is Cohen-Macaulay; see [11, Proposition 12]. Also note that

$$\sum_{n\geq 0} \ell(N/\mathfrak{m}^{n+1}N) z^n = \frac{h_N(z)}{(1-z)^{r+1}},$$

where $h_N(z) \in \mathbb{Z}[z]$ with $e_i(N) = h_N^{(i)}(1)/i!$, for $i = 0, \ldots, r$.

2.3. Let $M \in \mathrm{CM}(A)$. In [11, Proposition 17], we proved that the function

$$n \mapsto \ell\left(\operatorname{Tor}_1^A\left(M, \frac{A}{\mathfrak{m}^{n+1}}\right)\right)$$

is of polynomial type, i.e., it coincides with a polynomial $t_M(z)$ for all $n \gg 0$. In [11, Theorem 18], we also proved that:

(1) M is free if and only if $\deg t_M(z) < d-1$;
(2) if M is not free then $\deg t_M(z) = d-1$ and the normalized leading coefficient of $t_M(z)$ is $\mu(M)e_1(A) - e_1(M) - e_1(\Omega(M))$, here $\mu(M)$ denotes the minimal number of generators of M;

(3) for any $M \in \mathrm{CM}(A)$,

$$e_A^T(M) = \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \ell\left(\mathrm{Tor}_1^A\left(M, \frac{A}{\mathfrak{m}^{n+1}}\right)\right)$$
$$= \mu(M)e_1(A) - e_1(M) - e_1(\Omega(M)).$$

By (1) note that $e_A^T(M) = 0$ if and only if M is free. Otherwise $e_A^T(M) > 0$. In fact $e_A^T(M) \geq e_0(\Omega(M))$, see [11, Lemma 19].

Our first result shows that we need not confine to a minimal presentation to compute $e_A^T(M)$.

LEMMA 2.4. *Let $M \in \mathrm{CM}(A)$ and let $0 \to N \to F \to M \to 0$ be an exact sequence in* $\mathrm{CM}(A)$ *with F free. Then*

$$e_A^T(M) = e_1(F) - e_1(M) - e_1(N).$$

PROOF. By Schanuel's Lemma [9, Lemma 3, section 19] we have $A^{\mu(M)} \oplus N \cong F \oplus \Omega(M)$. So

$$\mu(M)e_1(A) + e_1(N) = e_1(F) + e_1(\Omega(M)).$$

The result follows.

Our next result shows that $e_1(-)$ is sub-additive over short-exact sequences in $\mathrm{CM}(A)$.

PROPOSITION 2.5. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short-exact sequence in* $\mathrm{CM}(A)$. *Then*

$$e_1(M_2) \geq e_1(M_1) + e_1(M_3).$$

PROOF. Note $e_0(M_2) = e_0(M_1) + e_0(M_3)$. For $n \geq 0$ we define modules K_n by the exact sequence

$$0 \to K_n \to \frac{M_1}{\mathfrak{m}^{n+1}M_1} \to \frac{M_2}{\mathfrak{m}^{n+1}M_2} \to \frac{M_3}{\mathfrak{m}^{n+1}M_3} \to 0.$$

It follows that

$$\sum_{n\geq 0} \ell(K_n)z^n = \frac{h_{M_1}(z) - h_{M_2}(z) + h_{M_3}(z)}{(1-z)^{d+1}}.$$

Since $e_0(M_2) = e_0(M_1) + e_0(M_3)$, we have that $h_{M_1}(z) - h_{M_2}(z) + h_{M_3}(z) = (1 - z)\ell_K(z)$ for some $\ell_K(z) \in \mathbb{Z}[z]$. So we have

$$\sum_{n\geq 0} \ell(K_n)z^n = \frac{\ell_K(z)}{(1-z)^d}.$$

Notice $\ell_K(1) = e_1(M_2) - e_1(M_1) - e_1(M_3)$. It follows that for all $n \gg 0$

$$\ell(K_n) = (e_1(M_2) - e_1(M_1) - e_1(M_3)) \frac{n^{d-1}}{(d-1)!} + \text{lower order terms in } n.$$

So $e_1(M_2) \geq e_1(M_1) + e_1(M_3)$.

We now prove that $e_A^T(-)$ is sub-additive over short-exact sequences in $\mathrm{CM}(A)$.

THEOREM 2.6. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short-exact sequence in* $\mathrm{CM}(A)$. *Then*

$$e_A^T(M_2) \leq e_A^T(M_1) + e_A^T(M_3).$$

PROOF. By a standard result in homological algebra we have the following diagram with exact rows and columns, with F_i free A-modules for $i = 1, 2, 3$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & N_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & F_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_3 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note $F_2 \cong F_1 \oplus F_3$. So $e_1(F_2) = e_1(F_1) + e_1(F_3)$. However $e_1(M_2) \geq e_1(M_1) + e_1(M_3)$ and $e_1(N_2) \geq e_1(N_1) + e_1(N_3)$; see Proposition2.5.

By Lemma 2.4, we have $e_A^T(M_i) = e_1(F_i) - e_1(M_i) - e_1(N_i)$, for $i = 1, 2, 3$. The result follows.

2.7. Let us recall the definition of superficial elements. Let N be an A-module. An element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ is said to be N-*superficial* if there exists $c > 0$

such that $(\mathfrak{m}^{n+1}N : x) \cap \mathfrak{m}^c N = \mathfrak{m}^n N$, for all $n \gg 0$. It is well-known that superficial elements exist if k is infinite. If depth $N > 0$, then one can prove that an N-superficial element x is N-regular. Furthermore $(\mathfrak{m}^{n+1}N : x) = \mathfrak{m}^n N$, for all $n \gg 0$.

2.8. *Behavior of Hilbert coefficients with respect to superficial elements*: assume N is an A-module with depth $N > 0$ and dimension $r \geq 1$. Let x be N-superficial. Then by [11, Corollary 10] we have

$$e_i(N/xN) = e_i(N), \quad \text{for } i = 0, \ldots, r-1.$$

Our next result shows that $e_A^T(-)$ behaves well modulo superficial elements.

Proposition 2.9. *Suppose* $\dim A \geq 2$ *and let* $M \in \mathrm{CM}(A)$. *Assume the residue field k is infinite. Let x be $A \oplus M \oplus \Omega_A(M)$-superficial. Set $B = A/(x)$ and $N = M/xM$. Then*

$$e_B^T(N) = e_A^T(M).$$

Proof. Note

$$\begin{aligned} e_A^T(M) &= e_1(A)\mu(M) - e_1(M) - e_1(\Omega_A(M)), \\ &= e_1(B)\mu(N) - e_1(N) - e_1(\Omega_A(M)/x\Omega_A(M)). \end{aligned}$$

The result follows from observing that $\Omega_A(M)/x\Omega_A(M) \cong \Omega_B(M/xM)$.

2.10. We now abstract some of the essential properties of $e_A^T(-)$.

We say a function $\xi : \mathrm{ICM}(A) \to \mathbb{Z}$ is a *pre-triangle* function if it satisfies the following properties:

(1) $\xi([M]) \geq 0$ for all $M \in \mathrm{CM}(A)$;

(2) $\xi([M]) = 0$ if and only if M is free;

(3) $\xi([M_1 \oplus M_2]) = \xi([M_1]) + \xi([M_2])$ for all $M_1, M_2 \in \mathrm{CM}(A)$;

(4) (*sub-additivity*) if $0 \to M \to N \to L \to 0$ is an exact sequence in $\mathrm{CM}(A)$, then
$$\xi([N]) \leq \xi([M]) + \xi([L]).$$

We state our basic existence result for pre-triangle functions.

Theorem 2.11. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$. Then the function*

$$e_A^T([M]) = \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \ell\left(\mathrm{Tor}_1^A\left(M, \frac{A}{\mathfrak{m}^{n+1}}\right)\right), \quad \textit{where } [M] \in \underline{\mathrm{ICM}}(A),$$

is a pre-triangle function on $\mathrm{ICM}(A)$.

PROOF. Properties (1), (2) are satisfied by §2.3. Property (3) is trivially satisfied. Property (4) is satisfied by Theorem 2.6.

2.12. If ξ is a pre-triangle function then trivially $k\xi$ is a pre-triangle function for any $k \geq 1$. Perhaps less-obvious is the following:

PROPOSITION 2.13. *Let ξ be a pre-triangle function. Then the function $\xi^{(i)}\colon \mathrm{ICM}(A) \to \mathbb{Z}$ defined by*

$$\xi^{(i)}([M]) = \xi([\Omega^i(M)])$$

is a pre-triangle function for all $i \geq 0$.

PROOF. Note $\xi^{(0)} = \xi$. Also note that for $i \geq 2$, we have

$$\xi^{(i)} = \big(\xi^{(i-1)}\big)^{(1)}.$$

So it suffices to prove that $\nu = \xi^{(1)}$ is a pre-triangle function.

It is very easy to prove that ν satisfies properties (1), (2) and (3) and is left to the reader. We prove that ν satisfies property (4). Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathrm{CM}(A)$. Note that we have a short exact sequence

$$0 \to \Omega(M_1) \to \Omega(M_2) \oplus F \to \Omega(M_3) \to 0,$$

where F is a finitely generated free A-module (possibly zero). Since ξ is a pre-triangle function we have

$$\xi([\Omega(M_2)]) = \xi([\Omega(M_2) \oplus F]) \leq \xi([\Omega(M_1)]) + \xi([\Omega(M_3)])$$

The result follows.

REMARK 2.14. In general $\xi^{(i)}$ will be different from ξ. For instance if $\xi = e_A^T(-)$ and if the Betti-numbers of M are unbounded, then since $e_A^T(M) \geq e_0(\Omega(M)) \geq \mu(\Omega(M))$, see [11, Lemma 19], we get that for $i \gg 0$ that $e^T(\Omega^i(M)) > e^T(M)$. So in this case $\xi^{(i)}(M) \neq \xi(M)$.

The following easy proposition (proof left to the reader) combined with 2.12 and Proposition 2.13 yields yet another abundant number of pre-triangle functions.

PROPOSITION 2.15. *Let ξ_1, ξ_2 be two pre-triangle functions. Then $\xi = \xi_1 + \xi_2$ is a pre-triangle function.*

3. Triangle functions on $\underline{\mathrm{CM}}(A)$

For the rest of the paper $(A, \mathfrak{m})$ denotes a Gorenstein local ring of dimension $d \geq 1$ with residue field k. Let $\mathrm{CM}(A)$ denote the full subcategory of maximal Cohen-Macaulay A-modules and let $\underline{\mathrm{CM}}(A)$ denote the stable category of maximal Cohen-Macaulay A-modules. Let $\underline{\mathrm{ICM}}(A)$ denote the set of isomorphism classes in $\underline{\mathrm{CM}}(A)$ and for an object $M \in \underline{\mathrm{CM}}(A)$ we denote its isomorphism class by $[M]$. In this section we prove Theorem 1.6. We also construct a large class of triangle functions on $\underline{\mathrm{ICM}}(A)$.

3.1. Let $M \in \mathrm{CM}(A)$. By M^* we mean the dual of M, i.e., $M^* = \mathrm{Hom}_A(M, A)$. Note $M \cong M^{**}$. By $\Omega^{-1}(M)$ we mean the *co-syzygy* of M. Recall this is constructed as follows. Let $F \to G \xrightarrow{\epsilon} M^* \to 0$ be a minimal presentation of M^*. Dualizing we get an exact sequence $0 \to M \xrightarrow{\epsilon^*} G^* \to F^*$. Then $\Omega^{-1}(M) = \operatorname{coker} \epsilon^*$. It can be easily shown that if $F' \to G' \xrightarrow{\eta} M^* \to 0$ is another minimal presentation of M^* then $\operatorname{coker} \epsilon^* \cong \operatorname{coker} \eta^*$.

3.2. *The triangulated category structure on* $\underline{\mathrm{CM}}(A)$. The reference for this topic is [3, §4.7]. We first describe a basic exact triangle. Let $f: M \to N$ be a morphism in $\mathrm{CM}(A)$. Note that we have an exact sequence $0 \to M \xrightarrow{i} Q \to \Omega^{-1}(M) \to 0$, with Q free. Let $C(f)$ be the pushout of f and i. Thus we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "f"] & Q \arrow[r, "p"] \arrow[d] & \Omega^{-1}(M) \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & N \arrow[r, "i'"] & C(f) \arrow[r, "p'"] & \Omega^{-1}(M) \arrow[r] & 0
\end{tikzcd}$$

Here j is the identity map on $\Omega^{-1}(M)$. As $N, \Omega^{-1}(M) \in \mathrm{CM}(A)$ it follows that $C(f) \in \mathrm{CM}(A)$. Then the projection of the sequence

$$M \xrightarrow{f} N \xrightarrow{i'} C(f) \xrightarrow{-p'} \Omega^{-1}(M)$$

in $\underline{\mathrm{CM}}(A)$ is a basic exact triangle. Exact triangles in $\underline{\mathrm{CM}}(A)$ are triangles isomorphic to a basic exact triangle.

REMARK 3.3. If $0 \to M \xrightarrow{f} N \to L \to 0$ is an exact sequence in $\mathrm{CM}(A)$ then we have an exact triangle $M \to N \to L \to \Omega^{-1}(M)$ in $\underline{\mathrm{CM}}(A)$. To see this we do the basic construction with the map f. We have the following exact sequence:

$$0 \to Q \to C(f) \to L \to 0.$$

As A is Gorenstein and Q is free we get $C(f) \cong Q \oplus L$. It follows that $C(f) \cong L$ in $\underline{\mathrm{CM}}(A)$. The result follows.

The main result of this section is

THEOREM 3.4. *Let* $\xi\colon \mathrm{ICM}(A) \to \mathbb{Z}$ *be a pre-triangle function. Then* ξ *induces a triangle function* $\xi'\colon \underline{\mathrm{ICM}}(A) \to \mathbb{Z}$ *defined as*

$$\xi'([M]) = \xi(\langle M\rangle).$$

(Here by $\langle M\rangle$ *we mean the isomorphism class of* M *in* $\mathrm{CM}(A)$*).*

PROOF. We first show that ξ' is a well-defined function. Let $[M] = [N]$. Then there exists free modules F and G such that $M \oplus F \cong N \oplus G$. So $\langle M \oplus F\rangle = \langle N \oplus G\rangle$ in $\mathrm{ICM}(A)$. Thus $\xi(\langle M \oplus F\rangle) = \xi(\langle N \oplus G\rangle)$. But ξ is a pre-triangle function. So

$$\xi(\langle M \oplus F\rangle) = \xi(\langle M\rangle) + \xi(\langle F\rangle) = \xi(\langle M\rangle).$$

Similarly $\xi(\langle N \oplus G\rangle) = \xi(\langle N\rangle)$. It follows that ξ' is a well-defined function.

Properties (1), (2) and (3) are trivial to show and are left to the reader. We prove property (4). Let $M \to N \to L \to \Omega^{-1}(M)$ be an exact triangle in $\underline{\mathrm{CM}}(A)$. Then it is isomorphic to a basic triangle $M' \xrightarrow{f} N' \to C(f) \to \Omega^{-1}(M)$. We have an exact sequence $0 \to N' \to C(f) \to \Omega^{-1}(M') \to 0$. As ξ is a pre-triangle we have

$$\xi(\langle C(f)\rangle) \le \xi(\langle N'\rangle) + \xi(\langle \Omega^{-1}(M')\rangle).$$

Note $C(f) \cong L$, $\Omega^{-1}M \cong \Omega^{-1}(M')$ and $N \cong N'$ in $\underline{\mathrm{CM}}(A)$. So we have

$$\xi'([L]) \le \xi'([N]) + \xi'([\Omega^{-1}(M)]).$$

Thus we have shown property 4(b) for all exact triangles. By 1.5 it follows that properties 4(a) and (c) are also satisfied for all exact triangles.

We now give

PROOF OF THEOREM 1.6. This follows from Theorem 2.11 and Theorem 3.4.

3.5. We now give a construction of infinitely many triangle functions on $\underline{\mathrm{CM}}(A)$. Since we have one pre-triangle function on $\mathrm{ICM}(A)$, we constructed in §2.12, Proposition 2.13 and Proposition 2.15 infinitely many pre-triangle functions. Each of these will yield a triangle function on $\underline{\mathrm{CM}}(A)$.

4. Some preliminaries on liaison of modules and maximal Cohen-Macaulay approximation

In this section we recall the definition of linkage of modules as given in [8]. We also recall the notion of maximal Cohen-Macaulay approximations and then breifly explain its connection with liaison theory.

4.1. Let us recall the definition of the transpose of a module. Let $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ be a minimal presentation of M. Let $(-)^* = \operatorname{Hom}(-, A)$. The *transpose* $\operatorname{Tr}(M)$ is defined by the exact sequence

$$0 \to M^* \to F_0^* \xrightarrow{\phi^*} F_1^* \to \operatorname{Tr}(M) \to 0.$$

DEFINITION 4.2. Two A-modules M and N are said to be *horizontally linked* if $M \cong \Omega(\operatorname{Tr}(N))$ and $N \cong \Omega(\operatorname{Tr}(M))$.

Next we define linkage in general.

DEFINITION 4.3. Two A-modules M and N are said to be *linked* via a Gorenstein ideal $\mathfrak{q}$ if

(1) $\mathfrak{q} \subseteq \operatorname{ann} M \cap \operatorname{ann} N$, and

(2) M and N are horizontally linked as $A/\mathfrak{q}$-modules.

We write it as $M \sim_{\mathfrak{q}} N$.

REMARK 4.4. It can be shown that ideals I and J are linked by a Gorenstein ideal $\mathfrak{q}$ (Definition 1.1 in the introduction) if and only if the module A/I is linked to A/J by $\mathfrak{q}$, see [8, Proposition 1].

4.5. We say M, N are in the *same linkage class* of modules if there is a sequence of A-modules $M_0, \ldots, M_n$ and Gorenstein ideals $\mathfrak{q}_0 \ldots, \mathfrak{q}_{n-1}$ such that

(i) $M_j \sim_{\mathfrak{q}_j} M_{j+1}$, for $j = 0, \ldots, n-1$,

(ii) $M_0 = M$ and $M_n = N$.

If n is even then we say that M and N are *evenly linked*.

4.6. (*MCM-approximations*) An MCM approximation of a A-module M is a short exact sequence $0 \to Y \to X \to M \to 0$, where X is maximal Cohen-Macaulay and $\operatorname{projdim} Y < \infty$. If $0 \to Y' \to X' \to M \to 0$ is another MCM approximation of M then X and X' are stably isomorphic, i.e., there exists free modules F and G with $X \oplus F \cong X' \oplus G$. Thus we have a well-defined object X_M in $\underline{\mathrm{CM}}(A)$.

The relation between liaison theory and MCM approximation is the following result by Martsinkovsky and Strooker [8, Theorem 13].

THEOREM 4.7. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring and let M and N be two A-modules. If M is evenly linked to N then $X_M \cong X_N$ in $\underline{\mathrm{CM}}(A)$.*

4.8. If M is Cohen-Macaulay then maximal Cohen-Macaulay approximation of M are very easy to construct, see [1, p. 7]. Set $\operatorname{codim} M = \dim A - \dim M$. The following result is well-known.

PROPOSITION 4.9. *Let M, N and L be Cohen-Macaulay A-modules with* codim $= n$. *Suppose we have an exact sequence* $0 \to M \to N \to L \to 0$. *Then we have an exact triangle*

$$X_M \to X_N \to X_L \to \Omega^{-1}(X_M)$$

in $\underline{\mathrm{CM}}(A)$.

5. Proof of Theorem 1.2

First we prove that for one-dimensional rings the set of even liaison classes of $\{\mathfrak{m}^n \mid n \geq 1\}$ is a finite set.

PROPOSITION 5.1. *Let* $(A, \mathfrak{m})$ *be a one-dimensional Gorenstein ring. Then there exists* $s \geq 1$ *such that* $\mathfrak{m}^{sn+r}$ *is evenly linked to* $\mathfrak{m}^{s(n-1)+r}$ *for all* $n \gg 0$ *and* $r = 0, 1, \ldots, s-1$.

PROOF. Let $a \in \mathfrak{m}^s \setminus \mathfrak{m}^{s+1}$ be such that image of a in $\mathfrak{m}^s/\mathfrak{m}^{s+1}$ is a parameter for the associated graded ring $G = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Then it can be shown that a is a non-zero divisor of A and $(\mathfrak{m}^{n+s} \colon a) = \mathfrak{m}^n$ for all $n \gg 0$. We also have that $\mathfrak{m}^{n+s} = a\mathfrak{m}^n$ for all $n \gg 0$.

It is easily verified that for all $n \gg 0$ we have $(a^n \colon \mathfrak{m}^{sn-r}) = (a^{n-1} \colon \mathfrak{m}^{s(n-1)-r})$ for $r = 0, 1, \ldots, s-1$. Therefore $\mathfrak{m}^{sn-r}$ is evenly linked to $\mathfrak{m}^{s(n-1)-r}$ for $r = 0, 1, \ldots, s-1$ and for all $n \gg 0$.

REMARK 5.2. If k is infinite, then note we can choose $s = 1$ in the above Proposition, see [2, 1.5.12]. So we get $\mathfrak{m}^n$ is evenly linked to $\mathfrak{m}^{n-1}$ for all $n \gg 0$.

5.3. By [6, Theorem 3.6], it follows that if K is a field and $R = K[[X_1, \ldots, X_n]]$ then $\mathfrak{n}^i$ is evenly linked to $\mathfrak{n}^{i-1}$ for all $i \geq 2$; here $\mathfrak{n}$ is the maximal ideal of R. We do not know whether in general for a regular local ring $(R, \mathfrak{n})$ with $\dim R \geq 3$ we have $\mathfrak{n}^i$ is evenly linked to $\mathfrak{n}^{i-1}$. We also do not know whether the set of even liaison classes of $\{\mathfrak{n}^i \mid i \geq 1\}$ is a finite set.

5.4. Let M be an A-module of dimension r. The function

$$H(M, n) = \ell(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M), \quad n \geq 0,$$

is called the *Hilbert function* of M. It is well-known that it is of polynomial type of degree $r - 1$. In particular, if $r \geq 2$ then $H(M, n) \to \infty$ as $n \to \infty$.

We now give:

PROOF OF THEOREM 1.2. For $n \geq 0$ we have an exact sequence of finite length A-modules

$$0 \to \frac{\mathfrak{m}^n M}{\mathfrak{m}^{n+1} M} \to \frac{M}{\mathfrak{m}^{n+1} M} \to \frac{M}{\mathfrak{m}^n M} \to 0.$$

For $n \geq 0$, let X_n, Y_n denote the maximal Cohen-Macaulay approximations of $\mathfrak{m}^n M/\mathfrak{m}^{n+1}M$ and $M/\mathfrak{m}^{n+1}M$ respectively. Note $X_n \cong X_k^{H(M,n)}$ in $\underline{\mathrm{CM}}(A)$. By Proposition 4.9, for all $n \geq 1$ we have an exact triangle in $\underline{\mathrm{CM}}(A)$

$$X_n \to Y_n \to Y_{n-1} \to \Omega^{-1}(X_n). \tag{5.4.1}$$

Suppose if possible that $M/\mathfrak{m}^n M \in \bigcup_{i=1}^m L_i$ for some finitely many even liaison classes $L_1, \ldots, L_m$ and for all $n \geq 1$. Choose $V_i \in L_i$ for $i = 1, \ldots, m$. Then for all $n \geq 0$ we have $Y_n \cong X_{V_i}$ in $\underline{\mathrm{CM}}(A)$ for some i (depending on n). Notice we also have $\Omega^{-1}(Y_n) \cong \Omega^{-1}(X_{V_i})$ in $\underline{\mathrm{CM}}(A)$.

Let ξ be any triangle function on $\underline{\mathrm{ICM}}(A)$. Then by Proposition 5.4.1 we have

$$\xi([\Omega^{-1}(X_n)]) \leq \xi([Y_{n-1}]) + \xi([\Omega^{-1}(Y_n)]). \tag{5.4.2}$$

Let

$$\alpha = \max\{\xi([X_{V_i}]) \mid i = 1, \ldots, m\},$$
$$\beta = \max\{\xi([\Omega^{-1}(X_{V_i})]) \mid i = 1, \ldots, m\}.$$

Also note that

$$\Omega^{-1}(X_n) = (\Omega^{-1}X_k)^{H(M,n)}, \quad \text{in } \underline{\mathrm{CM}}(A).$$

By Proposition 5.4.2 we have

$$H(M,n)\xi([\Omega^{-1}X_k]) \leq \alpha + \beta.$$

Since $\dim M \geq 2$ we have that $H(M,n) \to \infty$ as $n \to \infty$. It follows that $\xi([\Omega^{-1}X_k]) = 0$. Therefore $\Omega^{-1}(X_k)$ is free. It follows that X_k is free. Therefore $\operatorname{projdim} k < \infty$. This implies that A is regular.

6. Proof of Theorem 1.3

The following result follows easily from [6, Theorem 3.6]. However we sketch a proof as we do not have a reference. It also explains the significance of Theorem 1.3.

PROPOSITION 6.1. *Let $(A, \mathfrak{m})$ be a complete equi-characteristic Gorenstein local ring. Let I be an ideal generated by a regular sequence. The I^n is evenly linked to I^{n-1} for all $n \geq 2$.*

To prove this result we need the following general result which is easy to prove.

LEMMA 6.2. *Let $\phi\colon (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a faithfully flat homomorphism of Gorenstein local rings. Let I, J be ideals in A and let $\mathfrak{q}$ be a Gorenstein ideal in A such that $I \sim_{\mathfrak{q}} J$. Then*

(1) $\mathfrak{q}B$ *is a Gorenstein ideal in* B.

(2) $IB \sim_{\mathfrak{q}B} JB$.

As an easy consequence we have

COROLLARY 6.3. *Let* K *be a field. Let* $R = K[[X_1, \dots, X_n]]$. *Fix* $r \geq 1$. *Set* $I = (X_1, \dots, X_r)$. *Then* I^n *is evenly linked to* I^{n-1} *for* $n \geq 2$.

PROOF. Let $T = K[[X_1, \dots, X_r]]$ and let $\mathfrak{m} = (X_1, \dots, X_r)$. The inclusion $T \to R$ is flat. By [6, Theorem 3.6], $\mathfrak{m}^n$ is evenly linked to $\mathfrak{m}^{n-1}$ for $n \geq 2$. By Lemma 6.2 we have that I^n is evenly linked to I^{n-1} for $n \geq 2$.

We now give

PROOF OF PROPOSITION 6.1. Let $I = (x_1, \dots, x_r)$. Extend this regular sequence to a system of parameters $x_1, \dots, x_d$ of A. Assume $A = K[[Y_1, \dots, Y_m]]/I$. Consider the subring $B = K[[x_1, \dots, x_d]]$ of A. Then note that

(1) A is finitely generated as a B-module.

(2) $B \cong K[[X_1, \dots, X_d]]$ the power series ring over K in d-variables.

(3) As A is Cohen-Macaulay we have that A is free as a B-module. Thus the inclusion $i \colon B \to A$ is flat.

By Corollary 6.3, we have that the B-ideal $J = (x_1, \dots, x_r)$ has the property that J^n is evenly linked to J^{n-1} for all $n \geq 2$. By Lemma 6.2 it follows that I^n is evenly linked to I^{n-1} for all $n \geq 2$.

REMARK 6.4 (With the hypotheses of Proposition 6.1). Note that as modules, A/I^n is evenly linked to A/I^{n-1} for all $n \geq 2$. It follows that if F is a finitely generated free A-module then F/I^nF is evenly linked to $F/I^{n-1}F$ for all $n \geq 2$.

We now give

PROOF OF THEOREM 1.3. As M is a maximal Cohen-Macaulay A-module it follows that $x_1, \dots, x_r$ is an M-regular sequence. Note that $I^nM/I^{n+1}M \cong (M/IM)^{\gamma_n}$ where $\gamma_n = \binom{n+r-1}{r-1}$, see [2, Theorem 1.1.8]. For all $n \geq 0$ we also have an exact sequence

$$0 \to \frac{I^nM}{I^{n+1}M} \to \frac{M}{I^{n+1}M} \to \frac{M}{I^nM} \to 0. \tag{6.4.1}$$

Inductively one can prove that M/I^nM is a Cohen-Macaulay A-module of codimension r for all $n \geq 1$. Thus (6.4.1) is an exact sequence of codimension r Cohen-Macaulay A-modules. For $n \geq 0$ let X_n and Y_n denote maximal Cohen-Macaulay approximations of $I^nM/I^{n+1}M$ and $M/I^{n+1}M$ respectively.

Therefore by Proposition 4.9, for all $n \geq 1$, we have the following exact triangle in $\underline{\mathrm{CM}}(A)$

$$X_n \to Y_n \to Y_{n-1} \to \Omega^{-1}(X_n).$$

Suppose if possible that $M/I^nM \in \bigcup_{i=1}^m L_i$ for some finitely many even liaison classes $L_1, \ldots, L_m$ and for all $n \geq 1$. Choose $V_i \in L_i$ for $i = 1, \ldots, m$. Then for all $n \geq 0$ we have $Y_n \cong X_{V_i}$ in $\underline{\mathrm{CM}}(A)$ for some i (depending on n). Notice we also have $\Omega^{-1}(Y_n) \cong \Omega^{-1}(X_{V_i})$ in $\underline{\mathrm{CM}}(A)$.

Let ξ be any triangle function on $\underline{\mathrm{ICM}}(A)$. Then by (5.4.1) we have

$$\xi([\Omega^{-1}(X_n)]) \leq \xi([Y_{n-1}]) + \xi([\Omega^{-1}(Y_n)]). \tag{6.4.2}$$

Let

$$\alpha = \max\{\xi([X_{V_i}]) \mid i = 1, \ldots, m\},$$
$$\beta = \max\{\xi([\Omega^{-1}(X_{V_i})]) \mid i = 1, \ldots, m\}.$$

Also note that

$$\Omega^{-1}(X_n) = (\Omega^{-1}X_{M/IM})^{\gamma_n}, \quad \text{in } \underline{\mathrm{CM}}(A).$$

By (6.4.2) we have

$$\gamma_n \xi([\Omega^{-1}X_{M/IM}]) \leq \alpha + \beta.$$

Since $r \geq 2$ we have that $\gamma_n \to \infty$ as $n \to \infty$. It follows that $\xi([\Omega^{-1}X_{M/IM}]) = 0$. Therefore $\Omega^{-1}(X_{M/IM})$ is free. It follows that $X_{M/IM}$ is free. Therefore $\operatorname{projdim}_A M/IM < \infty$. As $x_1, \ldots, x_r$ is an M-regular sequence it follows that $\operatorname{projdim}_A M$ is finite. So M is free.

7. Proof of Theorem 1.4

Let $r \geq 1$. Let $\mathrm{CM}^r(A)$ denote the full sub-category of Cohen-Macaulay A-modules of codimension r. In this section we define an invariant of modules in $\mathrm{CM}^r(A)$ and then use it to prove Theorem 1.4.

DEFINITION 7.1. Let $N \in \mathrm{CM}^r(A)$. Let X_N be a maximal Cohen-Macaulay approximation of N. Set $\theta_A(N) = e_A^T([X_N])$.

As $e_A^T(-)$ is a triangle function on $\underline{\mathrm{CM}}(A)$ it follows that $\theta_A(N)$ is a well-defined invariant of M.

The number $\theta_A(-)$ behaves well modulo superficial sequences. Let us recall the notion of a superficial sequence. Let N be an A-module of dimension r. We say $\mathbf{x} = x_1, \ldots, x_s$ (with $s \leq r$) is an N-superficial sequence if x_1 is N-superficial, x_i is $N/(x_1, \ldots, x_{i-1})N$ superficial for $2 \leq i \leq s$.

PROPOSITION 7.2. *Let $N \in \mathrm{CM}^r(A)$ with $r \geq 1$ and let $\dim A = d$. Let $0 \to Y \to X \to N \to 0$ be a maximal Cohen-Macaulay approximation of*

M. Let $\mathbf{x} = x_1, \dots, x_{d-r}$ be an $A \oplus X \oplus \Omega(X) \oplus N$ superficial sequence. Set $B = A/(\mathbf{x})$. Then

$$\theta_A(N) \leq e_0(N)\theta_B(k).$$

PROOF. Note $\mathbf{x}$ is a $X \oplus Y \oplus N$ regular sequence. So $0 \to Y/\mathbf{x}Y \to X/\mathbf{x}X \to N/\mathbf{x}N \to 0$ is a maximal Cohen-Macaulay approximation of the B-module $N/\mathbf{x}N$. Note as $r \geq 1$ we have that $d-r \leq d-1$. Using Proposition 2.9 we get $e^T_B(X/\mathbf{x}X) = e^T_A(X)$. So we have $\theta_A(N) = \theta_B(N/\mathbf{x}N)$. It suffices to prove $\theta_B(N/\mathbf{x}N) \leq e_0(N)\theta_B(k)$. By §2.8 we get that $N/\mathbf{x}N$ is a B-module of length $e_0(N)$.

Let L be a finite length B-module. We prove by induction on $\ell(L)$ that $\theta_B(L) \leq \ell(L)\theta_B(k)$. We have nothing to prove if $\ell(L) = 1$. So assume $\ell(L) = m \geq 2$ and the result is proved for all B-modules of length $\leq m - 1$.

We have an exact sequence $0 \to V \to L \to k \to 0$, where $\ell(V) = m - 1$. In $\underline{\mathrm{CM}}(B)$ we have an exact triangle

$$X_V \to X_L \to X_k \to \Omega^{-1}(X_V).$$

It follows that $e^T_B(X_L) \leq e^T_B(X_V) + e^T_B(X_k) \leq m e^T_B(k)$. Thus $\theta_B(L) \leq m\theta_B(k)$.

Our proof of Theorem 1.4 uses the following result by Herzog and Kühl, [5, 2.1].

THEOREM 7.3. *Let R be a local Gorenstein domain with infinite residue field k. Let $0 \to F_1 \to M_1 \to I_1 \to 0$ and $0 \to F_2 \to M_2 \to I_2 \to 0$ be any two Bourbaki sequences (i.e., F_1, F_2 are free, M_1, M_2 are maximal Cohen-Macaulay modules and I_1, I_2 are Cohen-Macaulay ideals of codimension 2). Then the following two statements are equivalent:*

(1) *M_1 and M_2 are stably isomorphic.*
(2) *I_1 and I_2 are evenly linked by a complete intersection.*

We should remark that a Bourbaki sequence is simply a maximal Cohen-Macaulay approximation of I, where I is a codimension 2 Cohen-Macaulay ideal. We now give

PROOF OF THEOREM 1.4. Suppose if possible that for some $m \geq 1$ the set $\mathscr{C}_m$ is not contained in any collection of finitely many even liaison classes. For $j \geq 1$ let I_j be ideals with $e_0(A/I_j) \leq m$ such that the liaison classes L_j of I_j are all distinct.

For $j \geq 1$ let $0 \to Y_j \to X_j \to A/I_j \to 0$ be maximal Cohen-Macaulay approximation of A/I_j.

As the residue field of A is uncountable it can be easily shown that there exists $\mathbf{x} = x_1, \dots, x_{d-2}$ such that $\mathbf{x}$ is a $A \oplus X_j \oplus \Omega(X_j) \oplus A/I_j$-superficial for all $j \geq 1$.

Set $B = A/(\mathbf{x})$. By Proposition 7.2 we get that

$$e_A^T(X_j) = \theta_A(A/I_j) \leq m\theta_B(k).$$

Set $c = m\theta_B(k)$. Let $M_1, M_2, \dots, M_m$ be all the indecomposable non-free maximal Cohen-Macaulay A-modules. Write

$$X_j = M_1^{a_{1,j}} \oplus \dots \oplus M_m^{a_{m,j}} \oplus A^{l_j}.$$

Here $a_{i,j}, l_j \geq 0$. Note that

$$\sum_{j=1}^{m} a_{i,j} \leq e_A^T(X_j) \leq c, \quad \text{for all } \ j \geq 1.$$

By the pigeon-hole principle it follows that there exists r and s with $r < s$ such that X_r is stably isomorphic to X_s. Note $\Omega(X_r)$ (and a free summand) will give a maximal Cohen-Macaulay approximation of I_r. By the result of Herzog and Kühl we have that I_r is evenly linked to I_s. So $L_r = L_s$ a contradiction. Thus $\mathscr{C}_m$ is contained in finitely many even liaison classes of A.

REMARK 7.4. If $e_0(A/I) = m$ and X is a MCM approximation of A/I it does not follow that X decomposes as a direct sum of at most m indecomposable MCM A-modules. For a counterexample let k be an algebraically closed field of characteristic zero and let $A = k[[X, Y, Z]]/(XY - Z^2)$ be the A_2 singularity. Then $\mathrm{Syz}_A^2(k)$ decomposes as a direct sum of two indecomposable non-free MCM A-modules, see [13, Theorem B]. It follows that if X is a MCM-approximation of k then X is at least the direct sum of two non-free indecomposable MCM A-modules.

REFERENCES

1. Auslander, M., and Buchweitz, R.-O., *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.), Colloque en l'honneur de Pierre Samuel (Orsay, 1987) (1989), no. 38, 5–37.
2. Bruns, W., and Herzog, J., *Cohen-Macaulay rings*, revised ed., Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1998.
3. Buchweitz, R.-O., *Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings*, preprint Univ. Hannover, http://hdl.handle.net/1807/16682, 1986.
4. Herzog, J., *Ringe mit nur endlich vielen Isomorphieklassen von maximalen, unzerlegbaren Cohen-Macaulay-Moduln*, Math. Ann. 233 (1978), no. 1, 21–34.
5. Herzog, J., and Kühl, M., *Maximal Cohen-Macaulay modules over Gorenstein rings and Bourbaki-sequences*, Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 65–92.
6. Kleppe, J. O., Migliore, J. C., Miró-Roig, R., Nagel, U., and Peterson, C., *Gorenstein liaison, complete intersection liaison invariants and unobstructedness*, Mem. Amer. Math. Soc. 154 (2001), no. 732.

7. Knörrer, H., *Cohen-Macaulay modules on hypersurface singularities. I*, Invent. Math. 88 (1987), no. 1, 153–164.
8. Martsinkovsky, A., and Strooker, J. R., *Linkage of modules*, J. Algebra 271 (2004), no. 2, 587–626.
9. Matsumura, H., *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989.
10. Polini, C., and Ulrich, B., *Linkage and reduction numbers*, Math. Ann. 310 (1998), no. 4, 631–651.
11. Puthenpurakal, T. J., *Hilbert-coefficients of a Cohen-Macaulay module*, J. Algebra 264 (2003), no. 1, 82–97.
12. Puthenpurakal, T. J., *The Hilbert function of a maximal Cohen-Macaulay module*, Math. Z. 251 (2005), no. 3, 551–573.
13. Takahashi, R., *Direct summands of syzygy modules of the residue class field*, Nagoya Math. J. 189 (2008), 1–25.
14. Wang, H.-J., *Links of symbolic powers of prime ideals*, Math. Z. 256 (2007), no. 4, 749–756.

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
POWAI
MUMBAI 400 076
INDIA
E-mail: tputhen@math.iitb.ac.in