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ESSENTIAL NORM ESTIMATES FOR HANKEL OPERATORS ON CONVEX DOMAINS IN $\mathbb{C}^2$

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Abstract

Let $\Omega \subset \mathbb{C}^2$ be a bounded convex domain with C^1-smooth boundary and $\varphi \in C^1(\overline{\Omega})$ such that φ is harmonic on the non-trivial disks in the boundary. We estimate the essential norm of the Hankel operator H_φ in terms of the $\overline{\partial}$ derivatives of φ "along" the non-trivial disks in the boundary.

Let Ω be a domain in $\mathbb{C}^n$ for $n \geq 1$ and $b\Omega$ denote the boundary of Ω. Furthermore, let dV denote the volume measure on Ω and $A^2(\Omega)$ be the Bergman space on Ω, the space of square integrable holomorphic functions on Ω with respect to dV. The Bergman projection P is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. For $\varphi \in L^\infty(\Omega)$ we define the Hankel operator $H_\varphi\colon A^2(\Omega) \to L^2(\Omega)$ by

$$H_\varphi f = (I - P)(\varphi f),$$

where I denotes the identity operator on $L^2(\Omega)$.

In [4] we studied compactness of Hankel operators on smooth bounded pseudoconvex domains with the symbols smooth up to the boundary. Our most complete result is attained on smooth bounded convex domains in $\mathbb{C}^2$. On such domains we characterize compactness of H_φ in terms of the behavior of φ on the analytic disks in $b\Omega$. Throughout this paper $\mathbb{D}$ will denote the unit open disk in $\mathbb{C}$.

THEOREM ([4]). *Let Ω be a smooth bounded convex domain in $\mathbb{C}^2$ and $\varphi \in C^\infty(\overline{\Omega})$. Then H_φ is compact if and only if $\varphi \circ F$ is holomorphic for all holomorphic $F\colon \mathbb{D} \to b\Omega$.*

In this paper we continue our study of compactness of Hankel operators and obtain estimates on their essential norms. The essential norm $\|T\|_e$ of a bounded linear operator $T\colon X \to Y$, where X and Y are normed linear spaces, is defined as

$$\|T\|_e = \inf\big\{\|T - K\| : K\colon X \to Y \text{ is a compact linear operator}\big\}.$$

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That is, the essential norm of T is the distance from T to the subspace of compact operators.

The first estimates for the essential norms of Hankel operators were obtained by Lin and Rochberg [7] in 1993, for the case of the Bergman space on $\mathbb{D}$. They showed that the essential norm estimates of H_φ, acting on $A^2(\mathbb{D})$, are analogous to the estimates on the Hardy space which is a famous theorem of Adamjan, Arov and Kreĭn [1]. The Lin-Rochberg results were later generalized by Asserda [2] to higher dimensions when the domain is a strongly pseudoconvex.

As in [4] our approach uses the connection between Hankel operators and the $\overline{\partial}$-Neumann operator. Due to this connection, we are able to consider more general domains; however, our symbols are more restricted. As a result, our estimates are of a different type to Lin and Rochberg's estimates. In our case, the estimates depend on the behavior of the symbol on the analytic disks in the boundary of domains. We note that an analytic disk in the boundary of Ω is the image of a holomorphic function $F\colon \mathbb{D} \to b\Omega$.

Before we state our main result we define $\Gamma_{b\Omega}$, the set of all linear parametrizations of "circular" affine non-trivial analytic disks in $b\Omega$, as follows:

$$\Gamma_{b\Omega} = \left\{F\colon \mathbb{D} \to b\Omega : F(\xi) = \xi z + p \text{ for some } p \in b\Omega, z \in \mathbb{C}^n \setminus \{0\}\right\}.$$

We note that in case where there are no non-trivial affine disks in the boundary of Ω, the set $\Gamma_{b\Omega}$ is empty.

In the main result below and the rest of the paper, f_z and $f_{\overline{z}}$ denote the derivative of f with respect to z and $\overline{z}$ respectively.

THEOREM 1. *Let Ω be a C^1-smooth bounded convex domain in $\mathbb{C}^2$, let τ_Ω denote the diameter of Ω, and let $\varphi \in C^1(\overline{\Omega})$ be such that $\varphi \circ F$ is harmonic for every holomorphic $F\colon \mathbb{D} \to b\Omega$. Then the Hankel operator H_φ satisfies the following essential norm estimate:*

$$\sup_{F\in\Gamma_{b\Omega}} \left\{ \frac{|F'(0)|}{\sqrt{2}\,\tau_\Omega} \inf_{\xi\in\mathbb{D}} \left\{ |(\varphi\circ F)_{\overline{\xi}}(\xi)| \right\} \right\} \leq \|H_\varphi\|_e \leq \sup_{F\in\Gamma_{b\Omega}} \left\{ \frac{\sqrt{e}\,\tau_\Omega}{|F'(0)|} \sup_{\xi\in\mathbb{D}} \left\{ |(\varphi\circ F)_{\overline{\xi}}(\xi)| \right\} \right\}.$$

REMARK 1. Both estimates in the theorem above are defined to be zero in the case $\Gamma_{b\Omega} = \emptyset$. That is, when there are no non-trivial analytic disks in $b\Omega$ we get $\|H_\varphi\|_e = 0$. This is in accordance with the fact that, in this case, H_φ is compact.

REMARK 2. $F'(0)$ measures the size of the disk $F(\mathbb{D}) \subset b\Omega$. So it is interesting that the essential norm depends on the "bar" derivatives of φ on the disks in the boundary as well as the size of these disks.

In case of the bidisk we get a better estimate for the lower bound as in the following theorem.

THEOREM 2. *Let $\varphi \in C^1(\overline{\mathbb{D}^2})$ such that the functions $z \to \varphi(z, e^{i\theta})$ and $w \to \varphi(e^{i\theta}, w)$ are harmonic on $\mathbb{D}$ for all $\theta \in [0, 2\pi]$. Then the Hankel operator H_φ satisfies the following essential norm estimate:*

$$\|H_\varphi\|_e \geq \sup_{F\in\Gamma_{b\mathbb{D}^2}} \left\{ \frac{|F'(0)|}{\sqrt{2}} \inf_{\xi\in\mathbb{D}} \left\{ |(\varphi \circ F)_{\overline{\xi}}(\xi)| \right\} \right\}$$

REMARK 3. The diameter of the bidisk $\tau_{\mathbb{D}^2} = 2\sqrt{2}$ is the distance between $(-1, -1)$ and $(1, 1)$. Hence $\sqrt{2}\,\tau_{\mathbb{D}^2} = 4 > \sqrt{2}$. Thus the lower bound in Theorem 2 is better than the one in Theorem 1.

Proofs of Theorem 1 and Theorem 2

LEMMA 3. *Let $\gamma \in C^1_0(U)$, where $U \subset \mathbb{D}$ is a domain. Then $\|\gamma_\xi\| = \|\gamma_{\overline{\xi}}\|$.*

PROOF. Since γ is compactly supported in U there are no boundary terms in the following integration by parts formula:

$$\begin{aligned}\|\gamma_\xi\|^2 &= \int_U \gamma_\xi(\xi)\overline{\gamma}_{\overline{\xi}}(\xi)\,dV(\xi) = \int_U \gamma(\xi)\overline{\gamma}_{\overline{\xi}\xi}(\xi)\,dV(\xi)\\ &= \int_U \gamma_{\overline{\xi}}(\xi)\overline{\gamma}_\xi(\xi)\,dV(\xi) = \|\gamma_{\overline{\xi}}\|^2.\end{aligned}$$

Therefore, $\|\gamma_\xi\| = \|\gamma_{\overline{\xi}}\|$.

We note that a unitary affine mapping F on $\mathbb{C}^n$ is of the form $F(z) = Az + p$, where A is a $n \times n$ unitary matrix and $p \in \mathbb{C}^n$.

LEMMA 4. *Let V be a bounded domain in $\mathbb{C}^n$, F a unitary affine mapping, and $\phi \in L^\infty(V)$. Then $\|H_\phi\|_e = \|H_{\phi\circ F}\|_e$, where $H_{\phi\circ F}$ is the Hankel operator (with symbol $\phi \circ F$) on $A^2(F^{-1}(V))$.*

PROOF. Let $U = F^{-1}(V)$ and let the pull-back $F^*\colon A^2(V) \to A^2(U)$ be defined as $F^*(f) = f \circ F$ for $f \in A^2(V)$. Then one can check that F^* is an isometry. Furthermore, the Bergman kernel transformation formula of Bell (see, [6, Proposition 6.1.7]) gives $P^V = (F^{-1})^* P^U F^*$, where P^U, P^V are the Bergman projections on U and V, respectively. Then for $f \in A^2(V)$, we have

$$\begin{aligned}
(F^{-1})^* H_{\phi\circ F} F^*(f) &= (F^{-1})^* H_{\phi\circ F}(f \circ F) \\
&= (F^{-1})^* \big(\phi(F) f(F) - P^U(\phi(F) f(F))\big) \\
&= \phi f - (F^{-1})^* P^U F^*(\phi f) \\
&= \phi f - P^V(\phi f) \\
&= H_\phi(f).
\end{aligned}$$

Also $T^V\colon A^2(V) \to L^2(V)$ is a compact linear operator if and only if $T^U\colon A^2(U) \to L^2(U)$ is compact where $T^V = (F^{-1})^* T^U F^*$. Furthermore,

$$\|H_\phi - T^V\| = \|(F^{-1})^* H_{\phi\circ F} F^* - (F^{-1})^* T^U F^*\| = \|H_{\phi\circ F} - T^U\|.$$

One can check that, the equality above implies that $\|H_\phi\|_e = \|H_{\phi\circ F}\|_e$.

We will use the $\overline{\partial}$-Neumann problem to obtain the upper bound in Theorem 1. The $\overline{\partial}$-Neumann operator, denoted by N, is defined as the solution operator for the complex Laplacian $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on square integrable (0, 1)-forms on Ω, denoted by $L^2_{(0,1)}(\Omega)$. We refer the reader to the books [3], [8] and references therein, for more information about the $\overline{\partial}$-Neumann problem. In the following theorem we list the properties we need about N (see [3, Theorem 4.4.1]).

THEOREM. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$. There exists a bounded self-adjoint operator $N\colon L^2_{(0,1)}(\Omega) \to L^2_{(0,1)}(\Omega)$ such that*

(i) $(\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*)N = I$ *on* $L^2_{(0,1)}(\Omega)$,

(ii) $\overline{\partial}^* N$ *is the solution operator to* $\overline{\partial}u = v$ *that produces solutions orthogonal to* $A^2(\Omega)$,

(iii) *the Bergman projection P satisfies the following equality*

$$P = I - \overline{\partial}^* N \overline{\partial},$$

where I is the identity mapping,

(iv) *the operators $\overline{\partial}N$, $\overline{\partial}^*N$, $\overline{\partial}\overline{\partial}^*N$ and $\overline{\partial}^*\overline{\partial}N$ are bounded, and*

$$\|N\| \leq e\tau_\Omega^2, \quad \|\overline{\partial}N\| \leq \sqrt{e}\,\tau_\Omega, \quad \|\overline{\partial}^*N\| \leq \sqrt{e}\,\tau_\Omega,$$

where τ_Ω is the diameter of Ω.

We note that (ii) and (iii) in the theorem above imply that $\overline{\partial} H_\varphi f = f \overline{\partial} \varphi$ for $\varphi \in C^1(\overline{\Omega})$ and $f \in A^2(\Omega)$.

REMARK 4. Before we start the proof of Theorem 1, we note that even though [4, Corollary 2] is stated for $\varphi \in C^\infty(\overline{\Omega})$, observation of the proof reveals that it is enough to have a C^1-smooth domain Ω and $\varphi \in C^1(\overline{\Omega})$.

PROOF OF THEOREM 1. First we will prove the lower bound. Since Ω is a C^1-smooth bounded convex domain in $\mathbb{C}^2$, [4, Corollary 2] implies that H_φ is compact if and only if $\varphi \circ F$ is holomorphic for any holomorphic $F\colon \mathbb{D} \to b\Omega$. Thus, in order to find the essential norm estimate, without loss of generality, we assume that there exists holomorphic $F\colon \mathbb{D} \to b\Omega$ such that $(\varphi \circ F)_{\overline{\xi}} \not\equiv 0$. Since φ is C^1-smooth, this means that $\varphi_{\overline{z}}(F) \neq 0$ on some open set. But the domain Ω is convex which implies that the disk $F(\mathbb{D})$ is an affine disk (see [5] and [4]). Using Lemma 4 we can thus assume that there exists $\tau_0 \in (0, \tau_\Omega)$ such that

(i) $\varphi_{\overline{z}}(z, 0) \neq 0$ for all $|z| \leq \tau_0$,

(ii) $\{(z, w) \in \mathbb{C}^2 : |z| \leq \tau_0, w = 0\} \subset b\Omega$.

Since Ω is bounded we can also deduce that

(iii) $\Omega \subset \{z \in \mathbb{C} : |z| < \tau_\Omega\} \times \{w \in \mathbb{C} : |w| < \tau_\Omega, \operatorname{Re}(w) > 0\}$.

With this setup, we can now put a wedge W in Ω perpendicular to $D = \{z \in \mathbb{C} : |z| < \tau_0\}$. Furthermore, W can be chosen as close to flat as we want if we are willing to choose its radius very small. That is, for any $\varepsilon_1 > 0$ there exists $r_0 > 0$ so that $D \times W \subset \Omega$, where

$$W = \left\{ re^{i\theta} \in \mathbb{C} : 0 \leq r < r_0, |\theta| < \frac{\pi - \varepsilon_1}{2} \right\}.$$

Let us choose

$$\chi(z) = \frac{2}{\pi \tau_0^2}\left(1 - \frac{|z|^2}{\tau_0^2}\right), \quad \text{for } z \in \overline{D}.$$

Then $\chi \in C^\infty(\overline{D})$, $\chi \geq 0$ with $\chi(z) = 0$ for $|z| = \tau_0$. Then we have

$$\int_D \chi(z)\, dV(z) = \frac{2}{\pi \tau_0^2} 2\pi \int_0^{\tau_0} \left(\rho - \frac{\rho^3}{\tau_0^2}\right) d\rho = \frac{4}{\tau_0^2}\left(\frac{\tau_0^2}{2} - \frac{\tau_0^4}{4\tau_0^2}\right) = 1,$$

and

$$\|\chi_z\|^2 = \frac{4}{\pi^2 \tau_0^4} \int_D \frac{|z|^2}{\tau_0^4}\, dV(z) = \frac{4}{\pi^2 \tau_0^8} 2\pi \int_0^{\tau_0} \rho^3\, d\rho = \frac{8}{\pi \tau_0^8} \frac{\tau_0^4}{4} = \frac{2}{\pi \tau_0^4}.$$

Hence
$$\frac{\int_D \chi(z)\, dV(z)}{\|\chi_z\|} = \tau_0^2\sqrt{\frac{\pi}{2}} = \frac{V(D)}{\sqrt{2\pi}}.$$

Let us first restrict φ onto D and extend the restriction as a C^1-smooth function ϕ_1 defined on $\mathbb{C}\times\{0\}$. Finally, we extend the function ϕ_1 trivially as a C^1-smooth function φ_1 on $\mathbb{C}^2$. That is, $\varphi_1(z,w) = \varphi(z,0)$. Let us define $\varphi_2 = \varphi - \varphi_1$ and
$$\gamma(z) = \frac{\chi(z)}{\varphi_{1\overline{z}}(z,0)}, \quad \text{for } z \in \overline{D},$$

where $\varphi_{1\overline{z}}$ denotes $\partial\varphi_1/\partial\overline{z}$. We will continue to use this notation below when appropriate.

We note that, in the rest of the proof $\|\,.\,\|$ and $\|\,.\,\|_U$ denote the L^2 norm on Ω and on open set U, respectively.

Let us define $\alpha_j = 1 - 2^{-2j-1}$ and
$$f_j(z,w) = \frac{1}{2^j w^{\alpha_j}}, \quad \text{for } (z,w)\in\Omega.$$

Using polar coordinates one can show that
$$\|f_j\|_W = \sqrt{\pi-\varepsilon_1}\, r_0^{1-\alpha_j} \quad \text{and} \quad \|f_j\| \le \pi\tau_\Omega^{2-\alpha_j}. \tag{1}$$

We will use the following equality in the second equality in (2) below.
$$\begin{aligned}\frac{\partial H_{\varphi_1} f_j}{\partial\overline{z}} d\overline{z} + \frac{\partial H_{\varphi_1} f_j}{\partial\overline{w}} d\overline{w} &= \overline{\partial} H_{\varphi_1} f_j = \overline{\partial}(\varphi_1 f_j - P(\varphi_1 f_j)) \\ &= f_j\overline{\partial}\varphi_1 = f_j\frac{\partial\varphi_1}{\partial\overline{z}} d\overline{z}.\end{aligned}$$

Then, for $w\in W$ we have
$$\begin{aligned}\frac{1}{2^j w^{\alpha_j}}\int_D \chi(z)\, dV(z) &= \int_D f_j(z,w)\frac{\partial\varphi_1}{\partial\overline{z}}(z,w)\gamma(z)\, dV(z) \\ &= \int_D \frac{\partial H_{\varphi_1} f_j}{\partial\overline{z}}(z,w)\gamma(z)\, dV(z) \\ &= -\int_D H_{\varphi_1} f_j(z,w)\gamma_{\overline{z}}(z)\, dV(z).\end{aligned} \tag{2}$$

We note that in the last equality above we used integration by parts and the fact that $\gamma(z) = 0$ for $|z| = \tau_0$.

Now we take the absolute values of both sides of (2) and then apply the Cauchy-Schwarz inequality to the right hand side to get

$$|f_j(0,w)|\int_D \chi(z)\,dV(z) \le \|H_{\varphi_1} f_j\|_D \|\gamma_{\overline{z}}\|_D.$$

After integrating over the wedge W and dividing by $\|\gamma_{\overline{z}}\|_D$ we get

$$\|f_j(0,\,.\,)\|_W \frac{\int_D \chi(z)\,dV(z)}{\|\gamma_{\overline{z}}\|_D} \le \|H_{\varphi_1} f_j\|_{D\times W} \le \|H_{\varphi_1} f_j\|.$$

We remind the reader that $\varphi_{1\overline{z}}$ and $\varphi_{1z\overline{z}}$ below will denote $\partial\varphi_1/\partial\overline{z}$ and $\partial^2\varphi_1/\partial z\partial\overline{z}$, respectively. Since we assumed that φ is harmonic on D, Lemma 3 implies that

$$\|\gamma_{\overline{z}}\|_D = \|\gamma_z\|_D = \left\| \frac{\chi_z}{\varphi_{1\overline{z}}} - \chi \frac{\varphi_{1z\overline{z}}}{(\varphi_{1\overline{z}})^2} \right\|_D = \left\| \frac{\chi_z}{\varphi_{1\overline{z}}} \right\|_D \le \frac{\|\chi_z\|_D}{\inf_D |\varphi_{1\overline{z}}|}.$$

Then

$$\begin{aligned} \|H_{\varphi_1} f_j\| &\ge \frac{\int_D \chi(z)\,dV(z)}{\|\gamma_{\overline{z}}\|_D} \|f_j(0,\,.\,)\|_W \\ &\ge \frac{\int_D \chi(z)\,dV(z)}{\|\chi_z\|_D} \big(\inf_D |\varphi_{1\overline{z}}|\big) \|f_j(0,\,.\,)\|_W. \end{aligned} \tag{3}$$

Therefore, inequality (3) and the fact that $\|f_j(0,\,.\,)\|_W = \sqrt{\pi-\varepsilon_1}\, r_0^{1-\alpha_j}$ imply that

$$\|H_{\varphi_1} f_j\| \ge r_0^{1-\alpha_j} \sqrt{\frac{\pi-\varepsilon_1}{2\pi}}\, V(D)\big(\inf_D |\varphi_{1\overline{z}}|\big). \tag{4}$$

Now we turn to φ_2. Since $\varphi_2(z,0)=0$, for every $\varepsilon>0$ there exists $\delta>0$ and j_ε so that

(i) $|\varphi_2(z,w)|<\varepsilon$, for $(z,w)\in\overline{\Omega}$ and $|w|\le\delta$, and

(ii) $|f_j(z,w)|<\varepsilon$, for $(z,w)\in\overline{\Omega}$, $|w|\ge\delta$ and $j\ge j_\varepsilon$.

Let us denote $\Omega_{1,\delta}=\{(z,w)\in\Omega : |w|<\delta\}$ and $\Omega_{2,\delta}=\{(z,w)\in\Omega : |w|>\delta\}$. Then for $j\ge j_\varepsilon$ we have

$$\begin{aligned} \|H_{\varphi_2} f_j\| &\le \|\varphi_2 f_j\| = \|\varphi_2 f_j\|_{\Omega_{1,\delta}} + \|\varphi_2 f_j\|_{\Omega_{2,\delta}} \\ &\le \varepsilon(\|f_j\| + \|\varphi_2\|) \le \varepsilon(\pi\tau_\Omega^{2-\alpha_j} + \|\varphi_2\|). \end{aligned}$$

Then, $\limsup_{j\to\infty}\|H_{\varphi_2} f_j\| \le \varepsilon(\pi\tau_\Omega + \|\varphi_2\|)$. Since ε is arbitrary, we get

$$\lim_{j\to\infty}\|H_{\varphi_2} f_j\| = 0. \tag{5}$$

By the definition of essential norms for Hankel operators, for any $\varepsilon_2 > 0$ there exists a compact operator $K_{\varepsilon_2}\colon A^2(\Omega) \to L^2(\Omega)$ such that

$$\|H_\varphi\|_e \geq \|H_\varphi - K_{\varepsilon_2}\| - \varepsilon_2.$$

Then

$$\begin{aligned}
\|H_\varphi\|_e &\geq \limsup_{j\to\infty} \frac{\|H_\varphi f_j - K_{\varepsilon_2} f_j\|}{\pi \tau_\Omega^{2-\alpha_j}} - \varepsilon_2 \\
&\geq \limsup_{j\to\infty} \frac{\|H_{\varphi_1} f_j\| - \|H_{\varphi_2} f_j\| - \|K_{\varepsilon_2} f_j\|}{\pi \tau_\Omega^{2-\alpha_j}} - \varepsilon_2 \\
&= \limsup_{j\to\infty} \frac{\|H_{\varphi_1} f_j\|}{\pi \tau_\Omega^{2-\alpha_j}} - \varepsilon_2.
\end{aligned} \tag{6}$$

In the last equality we used (5), compactness of K_{ε_2}, and the fact that $f_j \to 0$ weakly. Therefore, combining (4) and (6) together with the fact that the constants $\varepsilon_1, \varepsilon_2 > 0$ are arbitrary we get

$$\|H_\varphi\|_e \geq \frac{1}{\sqrt{2}\pi\tau_\Omega} \sup_{D\subset b\Omega} \Big\{ V(D) \inf_{\xi\in D} \{|\varphi_{\overline{z}}(\xi)|\} \Big\}.$$

We note that there is a one-to-one correspondence between the (affine) disks in $b\Omega$ and $F \in \Gamma_{b\Omega}$. Since we need $F\colon \mathbb{D} \to D$ to be a surjection, we must have $F(\xi) = (\tau_0 \xi, 0)$. Then one can show that

$$V(D) \inf_{\xi\in D} \{|\varphi_{\overline{z}}(\xi)|\} = \pi |F'(0)| \inf_{\xi\in\mathbb{D}} \{|(\varphi\circ F)_{\overline{\xi}}(\xi)|\}.$$

Therefore, we have

$$\|H_\varphi\|_e \geq \sup_{F\in\Gamma_{b\Omega}} \left\{ \frac{|F'(0)|}{\sqrt{2}\,\tau_\Omega} \inf_{\xi\in\mathbb{D}} \Big\{ |(\varphi\circ F)_{\overline{\xi}}(\xi)| \Big\} \right\}.$$

Now we turn to the upper estimate. Let ρ be a defining function for Ω. That is, ρ is a C^1-smooth function in a neighborhood of $\overline{\Omega}$ such that $\rho < 0$ on Ω, $\rho > 0$ on $\mathbb{C}^2 \setminus \overline{\Omega}$, and $|\nabla\rho| \neq 0$ on $b\Omega$. Then we define the complex tangential and complex normal vector fields as

$$L_1 = \frac{2\sqrt{2}}{\|\nabla\rho\|} \left(\frac{\partial\rho}{\partial w} \frac{\partial}{\partial z} - \frac{\partial\rho}{\partial z} \frac{\partial}{\partial w} \right),$$

$$L_2 = \frac{2\sqrt{2}}{\|\nabla\rho\|} \left(\frac{\partial\rho}{\partial \overline{z}} \frac{\partial}{\partial z} + \frac{\partial\rho}{\partial \overline{w}} \frac{\partial}{\partial w} \right).$$

One can check that $\{L_1, L_2\}$ form a continuous orthonormal basis for the space of $(1,0)$ type vector fields on a neighborhood on $b\Omega$. Let ω_1 and ω_2 be the differential forms of type $(1,0)$ that are dual to L_1 and L_2, respectively. That is,

$$\omega_1 = \frac{\sqrt{2}}{\|\nabla\rho\|}\left(\frac{\partial\rho}{\partial\overline{w}}dz - \frac{\partial\rho}{\partial\overline{z}}dw\right),$$

$$\omega_2 = \frac{\sqrt{2}}{\|\nabla\rho\|}\left(\frac{\partial\rho}{\partial z}dz + \frac{\partial\rho}{\partial w}dw\right).$$

One can check that $\|\omega_1\| = \|\omega_2\| = 1$ and $\overline{\partial} f = \overline{L}_1(f)\overline{\omega}_1 + \overline{L}_2(f)\overline{\omega}_2$, for any $f \in C^1(\overline{\Omega})$ (see special boundary charts in [8, p. 12]).

Using the method in the first part of the proof of Theorem 3 in [4, p. 3739–3740] ($\widetilde{\beta}$ and $\widehat{\beta}$ in [4] correspond to φ_3 and φ_4 below, respectively), we define $\varphi_3, \varphi_4 \in C^1(\overline{\Omega})$ such that

(i) $\varphi = \varphi_3 + \varphi_4$,

(ii) $\varphi_3 = \varphi$ and $\overline{L}_2(\varphi_3) = 0$ on $b\Omega$,

(iii) $\varphi_4 = 0$ on $b\Omega$.

We note that φ_4 is a uniform limit of compactly supported smooth functions on Ω. This fact together with Montel's Theorem imply that H_{φ_4} is a limit of compact operators in the operator norm. Hence H_{φ_4} is compact and $\|H_\varphi\|_e = \|H_{\varphi_3}\|_e$.

Let

$$\Pi = \overline{\bigcup_{F\in\Gamma_{b\Omega}} F(\mathbb{D})}$$

and let $\chi_\varepsilon \in C^\infty(\overline{\Omega})$ be such that $0 \le \chi_\varepsilon \le 1$, $\chi_\varepsilon = 1$ on $\Pi_\varepsilon = \{z \in \overline{\Omega} : d(z, \Pi) \le \varepsilon\}$, and $\chi_\varepsilon = 0$ on $\overline{\Omega} \setminus \Pi_{2\varepsilon}$. Then for $f \in A^2(\Omega)$, we have

$$H_{\varphi_3} = \overline{\partial}^* N M_{\overline{\partial}\varphi_3} = \overline{\partial}^* N M_{\chi_\varepsilon\overline{\partial}\varphi_3} + \overline{\partial}^* N M_{(1-\chi_\varepsilon)\overline{\partial}\varphi_3},$$

where M_h denotes multiplication by h. First we will show that $\overline{\partial}^* N M_{(1-\chi_\varepsilon)\overline{\partial}\varphi_3}$ is compact on $A^2(\Omega)$. Let $f \in A^2(\Omega)$.

$$\begin{aligned}\|\overline{\partial}^* N f(1-\chi_\varepsilon)\overline{\partial}\varphi_3\|^2 &= \langle\overline{\partial}^* N f(1-\chi_\varepsilon)\overline{\partial}\varphi_3, \overline{\partial}^* N f(1-\chi_\varepsilon)\overline{\partial}\varphi_3\rangle \\ &= \langle f\overline{\partial}\varphi_3, (1-\chi_\varepsilon) N\overline{\partial}\overline{\partial}^* N f(1-\chi_\varepsilon)\overline{\partial}\varphi_3\rangle \\ &\lesssim \|f\|\|(1-\chi_\varepsilon) N\overline{\partial}\overline{\partial}^* N f(1-\chi_\varepsilon)\overline{\partial}\varphi_3\|.\end{aligned}$$

Now we will use the fact that $(1-\chi_\varepsilon)N$ is compact. This is essentially done on pages 3740–3741 in the proof of Theorem 3 in [4]. The idea is to use compactness of the $\overline{\partial}$-Neumann operator locally to get the following compactness estimate: for every $\varepsilon_1 > 0$ there exists a compact operator K_{ε_1} on $L^2_{(0,1)}(\Omega)$ so that

$$\|(1-\chi_\varepsilon)Nh\| \leq \varepsilon\|h\| + \|K_{\varepsilon_1}h\|.$$

Then using the fact that $\overline{\partial\partial}^* N$ is a bounded operator in the second inequality below, we get

$$\begin{aligned}\|(1-\chi_\varepsilon)N\overline{\partial\partial}^* Nf(1-\chi_\varepsilon)\overline{\partial}\varphi_3\| &\leq \varepsilon_1\|\overline{\partial\partial}^* Nf(1-\chi_\varepsilon)\overline{\partial}\varphi_3\| \\ &\quad + \|K_{\varepsilon_1}\overline{\partial\partial}^* Nf(1-\chi_\varepsilon)\overline{\partial}\varphi_3\| \\ &\lesssim \varepsilon_1\|f\| + \|\widetilde{K}_{\varepsilon_1}f\|_{(0,1)},\end{aligned}$$

where $\widetilde{K}_{\varepsilon_1} = K_{\varepsilon_1}\overline{\partial\partial}^* NM_{(1-\chi_\varepsilon)\overline{\partial}\varphi_3}$ is a compact operator. Therefore, $\overline{\partial}^* NM_{(1-\chi_\varepsilon)\overline{\partial}\varphi_3}$ satisfies a compactness estimate and hence it is compact. Then

$$\overline{\partial}\varphi_3 = \overline{L}_1(\varphi_3)\overline{\omega}_1 + \overline{L}_2(\varphi_3)\overline{\omega}_2.$$

Using the facts that $\overline{L}_2\varphi_3 = 0$ and $\varphi = \varphi_3$ on $b\overline{\Omega}$, we get

$$|\overline{\partial}\varphi_3| = |\overline{L}_1(\varphi_3)| = |\overline{L}_1(\varphi)| \quad \text{on } b\Omega.$$

Therefore, we have

$$\|\overline{\partial}^* Nf\chi_\varepsilon\overline{\partial}\varphi_3\| \leq \|\overline{\partial}^* N\|\|f\chi_\varepsilon\overline{\partial}\varphi_3\| \leq \|\overline{\partial}^* N\|\sup\{|\overline{L}_1(\varphi)(z)| : z \in \Pi_{2\varepsilon}\}\|f\|.$$

So if we let ε go to zero and use the fact that $\|\overline{\partial}^* N\| \leq \sqrt{e}\,\tau_\Omega$, we get

$$\|H_\varphi\|_e \leq \sqrt{e}\,\tau_\Omega \sup\{|\overline{L}_1(\varphi)(z)| : z \in \Pi\}.$$

On the other hand, for $p \in \Pi$ there exist $p_j \in \Pi$, $\xi_j \in \mathbb{D}$ and $F_j \in \Gamma_{b\Omega}$ such that $F_j(\xi_j) = p_j$ and $\lim p_j = p$. We note that if p is not on the boundary of a disk then we can choose $p_j = p$ for all j.

Let $F_j(\xi) = (F_{j1}(\xi), F_{j2}(\xi))$, for $\xi \in \mathbb{D}$. Since Ω is convex in $\mathbb{C}^2$ and we assume that p_j is in a horizontal disk, F_{j1} is linear and F_{j2} is constant. The chain rule and the fact that L_1 is the complex tangential derivative imply that

$$(\varphi\circ F_j)_{\overline{\xi}}(\xi_j) = \varphi_{\overline{z}}(p_j)\overline{F_{j1\xi}(\xi_j)} = \overline{L}_1(\varphi)(p_j)\overline{F'_{j1}(\xi_j)} = \overline{L}_1(\varphi)(p_j)\overline{F'_{j1}(0)}.$$

Hence

$$|\overline{L}_1(\varphi)(p_j)| = \frac{|(\varphi\circ F_j)_{\overline{\xi}}(\xi_j)|}{\left|\overline{F'_j(0)}\right|}.$$

Then, if we take supremum over j we get

$$|\overline{L}_1(\varphi)(p)| \leq \sup_j\sup_{\xi\in\mathbb{D}}\left\{\frac{|(\varphi\circ F_j)_{\overline{\xi}}(\xi)|}{|F'_j(0)|}\right\}.$$

Hence, we have

$$\|H_\varphi\|_e \le \sup_{F\in\Gamma_{b\Omega}} \left\{ \frac{\sqrt{e}\,\tau_\Omega}{|F'(0)|} \sup_{\xi\in\mathbb{D}}\big\{|(\varphi\circ F)_{\overline{\xi}}(\xi)|\big\} \right\}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. The proof of Theorem 2 is very similar to the first part of the proof of Theorem 1. So instead of running through the whole argument again we will point out where they differ and the modifications needed for this proof. Without of loss of generality we may assume that there exists $z_0 \in \mathbb{D}$, $p \in b\mathbb{D}$ such that $\varphi_{\overline{z}}(z_0, p) \neq 0$. In this case wedge W is replaced by the disk $\mathbb{D}$ in w. Let us choose a sequence $\{p_j\} \subset \mathbb{D}$ such that $\lim_{j\to\infty} p_j = p$. Let us define $f_j(w) = k_{p_j}(w)$ where k_{p_j} is the normalized Bergman kernel of $\mathbb{D}$ centered at p_j. Then instead of (1), we have

$$\|f_j\|_{\mathbb{D}} = 1 \quad \text{and} \quad \|f_j\| = \sqrt{\pi}.$$

The decomposition of φ is unnecessary in the case of the bidisk. Or simply we decompose $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 = \varphi$ and $\varphi_2 = 0$. We choose $D \subset \mathbb{D}\times\{p\}$ such that $(z_0, p) \in D$ and $\varphi_{\overline{z}}$ does not vanish on D. In a similar fashion to the proof of Theorem 1, we get the following inequality.

$$\|f_j\|_{\mathbb{D}} \frac{\int_D \chi(z)\,dV(z)}{\|\gamma_{\overline{z}}\|_D} \le \|H_\varphi f_j\|_{D\times\mathbb{D}} \le \|H_\varphi f_j\|.$$

Then

$$\|H_\varphi f_j\| \ge \frac{V(D)}{\sqrt{2\pi}} \big(\inf_D |\varphi_{\overline{z}}|\big).$$

We can estimate the essential norm as in (6)

$$\|H_\varphi\|_e \ge \limsup_{j\to\infty} \frac{\|H_\varphi f_j\|}{\sqrt{\pi}} - \varepsilon$$

for an arbitrary $\varepsilon > 0$. Furthermore, we choose $r > 0$ so that $F(\xi) = (r(\xi - z_0), p)$ and $D = F(\mathbb{D})$. Then

$$V(D) \inf_{\xi\in D}\{|\varphi_{\overline{z}}(\xi)|\} = \pi |F'(0)| \inf_{\xi\in\mathbb{D}}\{|(\varphi\circ F)_{\overline{\xi}}(\xi)|\}.$$

Hence

$$\|H_\varphi\|_e \ge \limsup_{j\to\infty} \frac{\|H_{\varphi_1} f_j\|}{\sqrt{\pi}} - \varepsilon \ge \frac{|F'(0)|}{\sqrt{2}} \inf_{\xi\in\mathbb{D}}\{|(\varphi\circ F)_{\overline{\xi}}(\xi)|\} - \varepsilon.$$

Now we take supremum over F and let ε go to zero, to get

$$\|H_\varphi\|_e \geq \sup_{F\in\Gamma_{b\mathbb{D}^2}} \left\{ \frac{|F'(0)|}{\sqrt{2}} \inf_{\xi\in\mathbb{D}} \left\{ |(\varphi\circ F)_{\overline{\xi}}(\xi)| \right\} \right\}.$$

This completes the proof of Theorem 2.

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