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ON THE ASYMPTOTIC EXPANSION OF THE LOGARITHM OF BARNES TRIPLE GAMMA FUNCTION II

STAMATIS KOUMANDOS and HENRIK L. PEDERSEN

Abstract

The remainders in an asymptotic expansion of the logarithm of Barnes triple gamma function give rise to completely monotonic functions of positive order.

1. Introduction and results

This paper is a continuation of the investigations in [6] of the remainders in an asymptotic expansion of the logarithm of Barnes triple gamma function, denoted by $\Gamma_3(w|1,1,1)$. The expansion, due to Ruijsenaars, is given by

$$\begin{aligned}\log\Gamma_3(w|1,1,1) = \frac{B_{3,3}(w)}{6}\log w - \frac{11}{36}B_{3,0}w^3 - \frac{3}{4}B_{3,1}w^2 - \frac{1}{2}B_{3,2}w \\ + \sum_{k=4}^{m}\frac{(-1)^k}{k!}(k-4)!\,B_{3,k}w^{3-k} + R_{3,m}(w),\end{aligned}$$

where the remainder $R_{3,m}$ of order $m \geq 3$ has the representation

$$R_{3,m}(w) = \int_0^\infty \frac{e^{-wt}}{t^4}\left(\frac{t^3}{(1-e^{-t})^3} - \sum_{k=0}^{m}\frac{(-1)^k}{k!}B_{3,k}t^k\right)dt, \quad \Re w > 0.$$

Here $B_{3,k}(x)$ denote the triple Bernoulli polynomials defined by

$$\frac{t^3e^{xt}}{(e^t-1)^3} = \sum_{k=0}^{\infty}B_{3,k}(x)\frac{t^k}{k!}, \quad |t| < 2\pi,$$

and $B_{3,k} = B_{3,k}(0)$ the triple Bernoulli numbers. (See [9, (3.13) and (3.14)].)

The main purpose of this paper is to prove the following generalization of [6, Theorem 1.3] about the even indexed remainders.

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THEOREM 1.1. *For $n \geq 6$, the remainder $(-1)^n R_{3,2n}(x)$ is a completely monotonic function of order $n-1$.*

(The definition of complete monotonicity of positive order is given below.)

Similar investigations have been carried out for Euler's Gamma function and the double gamma function of Barnes, see [5]. In the latter case it is known that also the remainder in the asymptotic expansion of order $2n$ gives rise to a completely monotonic function of order $n-1$. Theorem 1.1 states that this result still holds in the triple case. As we shall see in the next sections, the result in the triple case is much harder to obtain.

REMARK 1.2. The remainders of odd order $2n+1$ are only completely monotonic for $n \leq 5$. It follows by direct computation that for $n = 1$ the remainder is completely monotonic, for $n = 2$ it is completely monotonic of order 1, for $n = 3$ of order 3 and for $n = 4$ it is completely monotonic of order 5.

A C^∞-function $f\colon (0,\infty) \to \mathbb{R}$ is said to be completely monotonic if

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $n = 0, 1, \dots$ and for all $x > 0$. A fundamental result due to Bernstein (see [11, p. 161]) states that f is completely monotonic if and only if there exists a positive measure μ on $[0,\infty)$ such that the integral below converges for all $x > 0$ and

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t).$$

Let α be a positive number. A function $f\colon (0,\infty) \to \mathbb{R}$ is said to be completely monotonic of order α if $x^\alpha f(x)$ is a completely monotonic function. These functions were introduced and characterized in [5]. The characterization for integer values of α (Proposition 1.4) is based on the following definition.

DEFINITION 1.3. Let A_0 denote the set of positive Borel measures σ on $[0,\infty)$ such that $\int_0^\infty e^{-xs}\, d\sigma(s) < \infty$ for all $x > 0$, let A_1 denote the set of functions $t \mapsto \sigma([0,t])$, where $\sigma \in A_0$, and for $n \geq 2$, let A_n denote the set of $n-2$ times differentiable functions $\xi\colon [0,\infty) \to \mathbb{R}$ satisfying $\xi^{(j)}(0) = 0$ for $j \leq n-2$ and $\xi^{(n-2)}(t) = \int_0^t \sigma([0,s])\, ds$ for some $\sigma \in A_0$.

PROPOSITION 1.4. *Let r be a positive integer. A function $p\colon (0,\infty) \to \mathbb{R}$ is completely monotonic of order r if and only if*

$$p(x) = \int_0^\infty e^{-xt} \xi(t)\, dt$$

for some $\xi \in A_r$.

If f is completely monotonic of order α then $f(x) = g(x)x^{-\alpha}$, where g is completely monotonic. Thus, the decay of $f(x)$ as x tends to infinity (compared with that of $g(x)$) is improved by a power factor, determined by the order of complete monotonicity.

Complete monotonicity of positive order of the remainders in an asymptotic expansion thus yields more information about the behaviour of the remainders, and hence also about the accuracy of the expansion.

The proof of [6, Theorem 1.3] relied on the formula

$$(-1)^n R_{3,2n}(x) = \int_0^\infty e^{-xt} t^{2n-3}(\xi_n(t) + \eta_n(t))\,dt,$$

where

$$\xi_n(t) = t\sum_{k=1}^{\infty} \frac{1}{t^2+(2\pi k)^2}\frac{1}{(2\pi k)^{2n-2}}\left(\frac{(2n-2)(2n-1)}{(2\pi k)^2} - 2\right)$$

and

$$\begin{aligned}\eta_n(t) &\\ = \sum_{k=1}^{\infty} &\frac{1}{(t^2+(2\pi k)^2)^2}\frac{1}{(2\pi k)^{2n-2}}\left\{t\left(-3t - \frac{2(t^2-(2\pi k)^2)}{t^2+(2\pi k)^2} + 2(2n-2)\right)\right.\\ &\left.+ 2\pi k\left(6\pi k + \frac{3(2n-2)(t^2+(2\pi k)^2)}{2\pi k} + \frac{8\pi k t}{t^2+(2\pi k)^2} + \frac{2(2n-2)t}{2\pi k}\right)\right\}.\end{aligned}$$

The main ingredient in the proof was the positivity of η_n and ξ_n. This is generalized in Proposition 1.5 and Proposition 1.6. Notice that in addition to the usual notation we also use ∂_t for the derivative with respect to t.

PROPOSITION 1.5. *For any $n \geq 7$, $\partial_t^{n-1}(t^{2n-3}\eta_n(t)) \geq 0$ for $t \geq 0$.*

PROPOSITION 1.6. *For any $n \geq 6$, $\partial_t^{n-1}(t^{2n-3}\xi_n(t)) \geq 0$ for $t \geq 0$.*

PROOF OF THEOREM 1.1. For $n = 6$ the remainder can be expressed as elementary functions and it is found that $R_{3,12}$ is in fact completely monotonic of order 7. For $n \geq 7$ we argue as follows. It is clear that $\xi_n + \eta_n$ is a C^∞-function on $[0,\infty)$ and that $\partial_t^k(t^{2n-3}(\xi_n(t)+\eta_n(t)) = 0$ for $t = 0$ and $k \leq n-1$. Furthermore, from Proposition 1.5 and Proposition 1.6 we infer that $\partial_t^{n-1}(t^{2n-3}(\xi_n(t)+\eta_n(t))) \geq 0$ for $t \geq 0$. Then the proof follows from Proposition 1.4.

The proofs of Proposition 1.5 and Proposition 1.6 are based on real variable methods, including application of results on monotonicity properties of the ratio between two series. Let us state a simple version of such a result.

PROPOSITION 1.7. *Suppose that*

$$f(x) = \frac{\sum_{k=0}^{K} a_k x^k}{\sum_{k=0}^{K} b_k x^k},$$

where $\{a_k\}$ are real and $\{b_k\}$ are positive. If a_k/b_k decreases then f decreases on the positive half line.

See the survey paper [2] for information on results of this kind. We remark that a version of Proposition 1.7 for certain infinite series of functions has been obtained in [6, Lemma 2.2] and that both versions are needed in the proof of Proposition 1.6.

We shall also need a classical formula for the derivatives of $f(x^2)$ in terms of derivatives of f, see [4, 0.432.1],

$$\partial_x^n (f(x^2)) = \sum_{\ell=0}^{n} p_{\ell,n}(x) f^{(\ell)}(x^2), \tag{1}$$

where $p_{0,n}, \ldots, p_{n,n}$ are polynomials with non-negative coefficients. We note that the explicit forms of $p_{n,n}$, $p_{n-1,n}$, and $p_{n-2,n}$ are:

$$\begin{aligned} p_{n,n}(x) &= (2x)^n, \\ p_{n-1,n}(x) &= n(n-1)(2x)^{n-2}, \\ p_{n-2,n}(x) &= n(n-1)(n-2)(n-3)(2x)^{n-4}/2. \end{aligned}$$

2. The proof of Proposition 1.5

Let us begin by investigating some auxiliary functions h_n and k_n, defined as follows

$$\begin{aligned} h_n(x) &= \frac{x^{n-1}}{(1+x)^3}, \\ k_n(x) &= \frac{x^{n-2}}{(1+x)^2}. \end{aligned}$$

LEMMA 2.1. *Suppose that $n \geq 7$. Then, for $x > 0$,*

$$\begin{aligned} \partial_x^{n-1}(h_n(x^2)) &> 0, \\ \partial_x^{n-1}(k_n(x^2)) &> 0. \end{aligned}$$

PROOF. First of all, for $n = 7$ both inequalities are established by computation, so it may be assumed that $n \geq 8$. We begin by proving the assertion for h_n. It is easy to show that

$$h_n^{(n-3)}(x) = \frac{(n-1)!}{2}\frac{x^2}{(x+1)^n},$$

which readily yields

$$h_n^{(n-2)}(x) = \frac{(n-1)!}{2}\frac{1}{(x+1)^{n+1}}\big(2x + (2-n)x^2\big),$$
$$h_n^{(n-1)}(x) = \frac{(n-1)!}{2}\frac{1}{(x+1)^{n+2}}\big(2 + (4-4n)x + (n-1)(n-2)x^2\big).$$

It is evident that $\partial_x^{n-3} h_n(x) > 0$ for $x > 0$. Since $h_n^{(\ell)}(0) = 0$ for $0 \le \ell \le n-3$, it follows that $h_n^{(\ell)}(x) > 0$ for $0 \le \ell \le n-3$.

From (1) it thus follows that

$$\partial_x^{n-1}(h_n(x^2)) = \sum_{\ell=0}^{n-1} p_{\ell,n-1}(x) h_n^{(\ell)}(x^2) = \sum_{\ell=n-3}^{n-1} p_{\ell,n-1}(x) h_n^{(\ell)}(x^2) + \alpha_n(x),$$

where $\alpha_n(x) > 0$ for $x > 0$. A standard, but lengthy, computation reveals that

$$\sum_{\ell=n-3}^{n-1} p_{\ell,n-1}(x) h_n^{(\ell)}(x^2) = \frac{(n-1)!}{2}\frac{(2x)^{n-5}}{(1+x^2)^{n+2}}(a_n x^8 + b_n x^6 + c_n x^4),$$

where

$$a_n = (n-1)(n-2)\big(16 - 4(n-2) + (n-3)(n-4)/2\big),$$
$$b_n = (n-1)\big(8(n-2) - 4(n-2)^2 - 64 + (n-2)(n-3)(n-4)\big),$$
$$c_n = 32 + (n-1)(n-2)\big(8 + (n-3)(n-4)/2\big).$$

We notice that a_n and c_n are positive for all $n \ge 2$. However, b_n is positive for $n \ge 9$ and negative for $2 \le n \le 8$. An investigation of the term $a_n x^8 + b_n x^6 + c_n x^4$ reveals that it is positive for all $x > 0$ when $n \ge 8$.

Turning to k_n it can easily be shown that

$$k_n^{(n-3)}(x) = (n-2)!\frac{x}{(x+1)^{n-1}}.$$

Hence $\partial_x^{n-3} k_n(x) > 0$ for $x > 0$ and thus also $k_n^{(\ell)}(x) > 0$ for $\ell \in \{0, \dots, n-3\}$. Furthermore,

$$\partial_x^{n-1}(k_n(x^2)) = \sum_{\ell=n-3}^{n-1} p_{\ell,n-1}(x) k_n^{(\ell)}(x^2) + \beta_n(x),$$

where $\beta_n(x) > 0$ for $x > 0$. Using

$$k_n^{(n-2)}(x) = (n-2)!\frac{1}{(x+1)^n}(1+(2-n)x),$$
$$k_n^{(n-1)}(x) = (n-2)!\frac{1}{(x+1)^{n+1}}((2-2n)+(n-1)(n-2)x),$$

we find that

$$\sum_{\ell=n-3}^{n-1} p_{\ell,n-1}(x)k_n^{(\ell)}(x^2) = (n-2)!\frac{(2x)^{n-5}}{(1+x^2)^{n+1}}(d_n x^6 + e_n x^4 + f_n x^2),$$

where

$$d_n = (n-1)(n-2)\big(16-4(n-2)+(n-3)(n-4)/2\big),$$
$$e_n = (n-1)\big((n-2)(n-3)(n-4)-4(n-2)(n-3)-32\big),$$
$$f_n = (n-1)(n-2)\big(4+(n-3)(n-4)/2\big).$$

The term $d_n x^6 + e_n x^4 + f_n x^2$ is positive for all $x > 0$ when $n \geq 8$. This proves the lemma.

Proof of Proposition 1.5. We rewrite η_n as

$$\eta_n(t) = (3(2n-3)t^2 + (8n-10)t)\sum_{k=1}^{\infty}\frac{1}{(t^2+(2\pi k)^2)^2}\frac{1}{(2\pi k)^{2n-2}}$$
$$+ 3(2n-1)\sum_{k=1}^{\infty}\frac{1}{(t^2+(2\pi k)^2)^2}\frac{1}{(2\pi k)^{2n-4}}$$
$$+ 8t\sum_{k=1}^{\infty}\frac{1}{(t^2+(2\pi k)^2)^3}\frac{1}{(2\pi k)^{2n-4}}.$$

This gives, after some computations,

$$t^{2n-3}\eta_n(t) = (3(2n-3)t^3 + (8n-10)t^2)\sum_{k=1}^{\infty}\frac{1}{(2\pi k)^6}k_n(t^2/(2\pi k)^2)$$
$$+ 3(2n-1)t\sum_{k=1}^{\infty}\frac{1}{(2\pi k)^4}k_n(t^2/(2\pi k)^2)$$
$$+ 8\sum_{k=1}^{\infty}\frac{1}{(2\pi k)^4}h_n(t^2/(2\pi k)^2).$$

The assertion now follows by using Lemma 2.1.

REMARK 2.2. A function f defined on the positive half line is called a generalized Stieltjes function of order $\lambda > 0$ if

$$f(x) = \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda} + c,$$

for some positive Borel measure μ making the integrals converge and some $c \geq 0$. The class of these functions is denoted by S_λ.

The positivity of $(x^{n-1}/(x+1)^3)^{(n-3)}$ and $(x^{n-2}/(x+1)^2)^{(n-3)}$ also follows from the characterization in [10], since $1/(x+1)^3$ and $1/(x+1)^2$ are generalized Stieltjes functions of order $\lambda = 3$ and $\lambda = 2$.

Let us remark that Lemma 2.1 holds for generalized Stieltjes functions:

- if $f \in S_3$ and $n \geq 8$, then $\partial_x^j(x^{2n-2} f(x^2)) > 0$ for all $j \leq n-1$ and $x > 0$;
- if $f \in S_2$ and $n \geq 8$, then $\partial_x^j(x^{2n-4} f(x^2)) > 0$ for all $j \leq n-1$ and $x > 0$.

To see this, let $s = s(x) = x/\sqrt{t}$. Then

$$\frac{x^{2n-2}}{(x^2+t)^3} = t^{n-4} \frac{s^{2n-2}}{(s^2+1)^3},$$

so that

$$\partial_x^j \frac{x^{2n-2}}{(x^2+t)^3} = t^{n-4} \partial_x^j s(x) \partial_s^j (h_n(s^2)) = t^{n-4-j/2} \partial_s^j (h_n(s^2))$$

is positive for $t \geq 0$. (It is clearly positive for $t = 0$.) Since (for $j > 0$)

$$\partial_x^j g(x^2) = \int_0^\infty \partial_x^j \frac{x^{2n-2}}{(x^2+t)^3}\, d\mu(t) = \int_0^\infty t^{n-4-j/2} \partial_s^j (h_n(s^2))\, d\mu(t),$$

Lemma 2.1 can be applied to obtain the first assertion. The second assertion is obtained similarly.

In [7] asymptotic expansions of generalized Stieltjes functions of measures having moments of all orders are investigated.

3. The proof of Proposition 1.6

It is convenient to introduce some more notation. We let, for $n \geq 0$,

$$\theta_n(x) = \frac{x^n}{1+x},$$

and for $0 \le m \le j \le n-1$,

$$c_{m,j} = c^n_{m,j} = \binom{n}{m}\binom{n-m-1}{j-m}.$$

LEMMA 3.1. *For $j \le n-1$,*

$$\partial_x^j \theta_n(x) = j!\frac{x^{n-j}}{(1+x)^{j+1}} \sum_{m=0}^{j} c_{m,j} x^{j-m}.$$

PROOF. The idea is to use Leibniz' rule and the binomial theorem:

$$\begin{aligned}\partial_x^j \theta_n(x) &= j!\frac{x^{n-j}}{(1+x)^{j+1}} \sum_{k=0}^{j} \binom{n}{k} x^{j-k}(-1)^{j-k} \sum_{\ell=0}^{k} \binom{k}{k-\ell} x^{k-\ell} \\ &= j!\frac{(-1)^j x^{n-j}}{(1+x)^{j+1}} \sum_{m=0}^{j} x^{j-m} \sum_{k=m}^{j} (-1)^k \binom{n}{k}\binom{k}{k-m}.\end{aligned}$$

Here,

$$\begin{aligned}\sum_{k=m}^{j}(-1)^k \binom{n}{k}\binom{k}{k-m} &= (-1)^m \binom{n}{m} \sum_{k=0}^{j-m} (-1)^k \binom{n-m}{k} \\ &= (-1)^j \binom{n}{m}\binom{n-m-1}{j-m},\end{aligned}$$

and this proves the lemma.

LEMMA 3.2. *For $j \le n$, the function*

$$\frac{x\partial_x^{j+1}\theta_n(x)}{\partial_x^j \theta_n(x)}$$

decreases for $x > 0$.

PROOF. A direct computation shows that

$$\partial_x^{n+1}\theta_n(x) = -\frac{(n+1)!}{(1+x)^{n+2}}, \quad \partial_x^n \theta(x) = \frac{n!}{(1+x)^{n+1}},$$

and

$$\partial_x^{n-1}\theta_n(x) = \frac{(n-1)!}{(1+x)^n} \sum_{k=1}^{n} \binom{n}{k} x^k.$$

This gives

$$\frac{x\partial_x^n\theta_n(x)}{\partial_x^{n-1}\theta_n(x)} = \frac{n}{(1+x)\sum_{k=1}^n \binom{n}{k}x^{k-1}} \quad \text{and} \quad \frac{x\partial_x^{n+1}\theta_n(x)}{\partial_x^n\theta_n(x)} = -\frac{(n+1)x}{1+x},$$

both of which are decreasing functions.

For $j \le n-2$,

$$\begin{aligned}\frac{x\partial_x^{j+1}\theta_n(x)}{\partial_x^j\theta_n(x)} &= (j+1)\frac{\sum_{m=0}^{j+1} c_{m,j+1}x^{j+1-m}}{(1+x)\sum_{m=0}^{j} c_{m,j}x^{j-m}} \\ &= (j+1)\frac{\sum_{m=0}^{j+1} c_{m,j+1}x^{j+1-m}}{\sum_{m=0}^{j+1} d_{m,j}x^{j+1-m}},\end{aligned}$$

where

$$d_{m,j} = \begin{cases} c_{0,j}, & \text{for } m = 0, \\ c_{m-1,j} + c_{m,j}, & \text{for } 1 \le m \le j, \\ c_{j,j}, & \text{for } m = j+1. \end{cases}$$

We claim that

$$\frac{c_{j+1,j+1}}{d_{j+1,j}} > \frac{c_{j,j+1}}{d_{j,j}} > \cdots > \frac{c_{1,j+1}}{d_{1,j}} > \frac{c_{0,j+1}}{d_{0,j}}. \tag{2}$$

Once this claim is verified, the rational function $x\partial_x^{j+1}\theta_n(x)/\partial_x^j\theta_n(x)$ decreases by Proposition 1.7.

We turn to the verification of (2). We get, when $1 \le m \le j$, by a standard computation,

$$\frac{d_{m,j}}{c_{m,j+1}} = \frac{1}{n-j-1}\left(1 + j - \frac{1}{(n+1)/m - 1}\right),$$

so that $d_{m,j}/c_{m,j+1}$ decreases as m increases from 1 to j. The inequalities

$$\frac{c_{j+1,j+1}}{d_{j+1,j}} > \frac{c_{j,j+1}}{d_{j,j}} \quad \text{and} \quad \frac{c_{1,j+1}}{d_{1,j}} > \frac{c_{0,j+1}}{d_{0,j}}$$

are also verified by straightforward computation; we omit the details. The lemma is proved.

REMARK 3.3. A function $h\colon (0,\infty) \to (0,\infty)$ is called geometrically concave if

$$h(x^\lambda y^{1-\lambda}) \ge h(x)^\lambda h(y)^{1-\lambda}$$

for all $x, y > 0$ and all $\lambda \in [0,1]$. This is equivalent to $\log h$ being a concave function of $\log x$. If h is differentiable it is equivalent to the function

$xh'(x)/h(x)$ being decreasing. (For geometrical concavity and other means see e.g. [8], [1], and for an application to special functions, see [3].) This gives the following consequence of Lemma 3.2: The function $\partial_x^j \theta_n(x)$ is geometrically concave when $0 \leq j \leq n$.

PROOF OF PROPOSITION 1.6. To ease notation we replace n by $n+1$ and set out to prove

$$\partial_t^n\left(t^{2n-1}\xi_{n+1}(t)\right) > 0$$

for $t > 0$ and for any $n \geq 5$.

We put $\varphi_{k,n}(t) = t^n/(t + (2\pi k)^2)$ and notice that as before

$$\partial_t^n\left(\varphi_{k,n}(t^2)\right) = \sum_{\ell=0}^{n} p_{\ell,n}(t)\varphi_{k,n}^{(\ell)}(t^2),$$

where the polynomials $p_{\ell,n}$ have non-negative coefficients. Hence

$$\begin{aligned}\partial_t^n\left(t^{2n-1}\xi_{n+1}(t)\right) &= \sum_{\ell=0}^{n} p_{\ell,n}(t)\sum_{k=1}^{\infty}(2\pi k)^{-2n}\varphi_{k,n}^{(\ell)}(t^2)\left(\frac{2n(2n+1)}{(2\pi k)^2}-2\right) \qquad (3)\\ &= \sum_{\ell=0}^{n-2} p_{\ell,n}(t)\sum_{k=1}^{\infty}(2\pi k)^{-2n}\varphi_{k,n}^{(\ell)}(t^2)\left(\frac{2n(2n+1)}{(2\pi k)^2}-2\right)\\ &\quad + \sum_{k=1}^{\infty} u_{k,n}(t)(2\pi k)^{-2n}\left(\frac{2n(2n+1)}{(2\pi k)^2}-2\right),\end{aligned}$$

where

$$u_{k,n}(t) = p_{n-1,n}(t)\varphi_{k,n}^{(n-1)}(t^2) + p_{n,n}(t)\varphi_{k,n}^{(n)}(t^2).$$

The proof will follow if we can show that

$$\sum_{k=1}^{\infty}(2\pi k)^{-2n}\varphi_{k,n}^{(\ell)}(t^2)\left(\frac{2n(2n+1)}{(2\pi k)^2}-2\right) > 0 \qquad (4)$$

for all $\ell = 0, \ldots, n-2$ and that

$$\sum_{k=1}^{\infty}(2\pi k)^{-2n}u_{k,n}(t)\left(\frac{2n(2n+1)}{(2\pi k)^2}-2\right) > 0. \qquad (5)$$

First (4) is verified: assume $0 \leq \ell \leq n-2$. Since the functions $\varphi_{k,n}^{(\ell)}$ are positive on the positive half line, (4) is equivalent to

$$\frac{\sum_{k=1}^{\infty}(2\pi k)^{-2n-2}\varphi_{k,n}^{(\ell)}(x)}{\sum_{k=1}^{\infty}(2\pi k)^{-2n}\varphi_{k,n}^{(\ell)}(x)} > \frac{1}{n(2n+1)} \qquad (6)$$

for $x > 0$. We claim that the left hand side decreases on the positive half line as a function of x. Indeed this will follow from [6, Lemma 2.2] if it can be shown that $\{\varphi_{k,n}^{(\ell+1)}(x)/\varphi_{k,n}^{(\ell)}(x)\}$ forms an increasing sequence of functions.

We express $\varphi_{k,n}$ in terms of θ_n, as

$$\varphi_{k,n}^{(\ell)}(x) = (2\pi k)^{2n-2}(2\pi k)^{-2l}\partial_x^\ell \theta_n(x/(2\pi k)^2),$$

so that

$$\frac{\varphi_{k,n}^{(\ell+1)}(x)}{\varphi_{k,n}^{(\ell)}(x)} = \frac{1}{x}\,\frac{(x/(2\pi k)^2)\partial_x^{\ell+1}\theta_n(x/(2\pi k)^2)}{\partial_x^\ell \theta_n(x/(2\pi k)^2)}.$$

Our task amounts to showing that the function

$$u \mapsto \frac{u\partial_u^{\ell+1}\theta_n(u)}{\partial_u^\ell \theta_n(u)}$$

is decreasing on the positive half line. This is exactly the assertion in Lemma 3.2. Finally, according to Lemma 3.4, the limit of the left hand side of the relation (6) as $x \to \infty$ equals $(2\pi)^{-2}\zeta(2n+2)/\zeta(2n)$, which by [6, Lemma 3.1] is greater than $1/(n(2n+1))$, for $n \geq 5$.

Finally we verify (5): since the functions $u_{k,n}$ are positive, (5) follows if

$$\frac{\sum_{k=1}^\infty (2\pi k)^{-2n-2}u_{k,n}(x)}{\sum_{k=1}^\infty (2\pi k)^{-2n}u_{k,n}(x)} > \frac{1}{n(2n+1)}$$

for $x > 0$. The limit of the left hand side as $x \to \infty$ equals, as above, $(2\pi)^{-2}\zeta(2n+2)/\zeta(2n)$, and is thus greater than $1/(n(2n+1))$, for $n \geq 5$. The remaining problem is to show that the left hand side decreases as a function of x, for $x > 0$. To this end rewrite the function $u_{k,n}$ as follows:

$$\begin{aligned} u_{k,n}(t) &= n!\,(2t)^{n-2}\left\{\frac{4t^2/(2\pi k)^2}{(1+t^2/(2\pi k)^2)^{n+1}} + (n-1)\frac{\sum_{\ell=1}^n \binom{n}{\ell}(t^2/(2\pi k)^2)^\ell}{(1+t^2/(2\pi k)^2)^n}\right\} \\ &= n!\,(2t)^{n-2}v_n(t^2/(2\pi k)^2), \end{aligned}$$

where

$$\begin{aligned} v_n(x) &= \frac{4x}{(1+x)^{n+1}} + (n-1)\frac{\sum_{\ell=1}^n \binom{n}{\ell}x^\ell}{(1+x)^n} \\ &= \frac{4x}{(1+x)^{n+1}} + (n-1)\frac{(1+x)^n - 1}{(1+x)^n}. \end{aligned}$$

It follows that $\{u_{k,n}'(t)/u_{k,n}(t)\}$ forms an increasing sequence of functions if

$$\left\{\frac{\partial_t(v_n(t^2/(2\pi k)^2))}{v_n(t^2/(2\pi k)^2)}\right\}$$

does. Since,

$$\frac{\partial_t(v_n(t^2/(2\pi k)^2))}{v_n(t^2/(2\pi k)^2)}=\frac{2}{t}\frac{t^2/(2\pi k)^2(v_n'(t^2/(2\pi k)^2))}{v_n(t^2/(2\pi k)^2)}$$

we see that this is the case if the function $x\mapsto xv_n'(x)/v_n(x)$ decreases. The last statement can be verified without too much difficulty. By direct computations,

$$\frac{xv_n'(x)}{v_n(x)}=\frac{(n-1)(n-5)x+4+(n-1)n}{1+x}\cdot\frac{1}{4+(n-1)(1+x)(((1+x)^n-1)/x)},$$

is decreasing as a product of two decreasing functions. This proves the proposition.

LEMMA 3.4.
(a) *For* $0\le\ell\le n-1$,

$$\lim_{x\to\infty}\frac{\sum_{k=1}^{\infty}(2\pi k)^{-2n-2}\varphi_{k,n}^{(\ell)}(x)}{\sum_{k=1}^{\infty}(2\pi k)^{-2n}\varphi_{k,n}^{(\ell)}(x)}=\frac{\zeta(2n+2)}{(2\pi)^2\zeta(2n)}.$$

(b) *For* $u_{k,n}(x)=p_{n-1,n}(x)\varphi_{k,n}^{(n-1)}(x^2)+p_{n,n}(x)\varphi_{k,n}^{(n)}(x^2)$,

$$\lim_{x\to\infty}\frac{\sum_{k=1}^{\infty}(2\pi k)^{-2n-2}u_{k,n}(x)}{\sum_{k=1}^{\infty}(2\pi k)^{-2n}u_{k,n}(x)}=\frac{\zeta(2n+2)}{(2\pi)^2\zeta(2n)}.$$

We stress that the limit does not depend on ℓ.

PROOF. Assume that $\ell\le n-1$. We express again $\varphi_{k,n}$ in terms of θ_n and obtain

$$\frac{1}{x^{n-\ell-1}}\sum_{k=1}^{\infty}(2\pi k)^{-2n-2}\varphi_{k,n}^{(\ell)}(x)=\sum_{k=1}^{\infty}(2\pi k)^{-2n-2}\frac{\partial_x^\ell\theta(x/(2\pi k)^2)}{(x/(2\pi k)^2)^{n-\ell-1}}.$$

Now, by Lemma 3.1,

$$0\le\frac{\partial_u^\ell\theta_n(u)}{u^{n-\ell-1}}=\ell!\frac{u}{1+u}\sum_{m=0}^{\ell}c_{m,\ell}\frac{u^{\ell-m}}{(1+u)^\ell}\le\ell!\sum_{m=0}^{\ell}c_{m,\ell},$$

for $\ell\le n-1$, so the dominated convergence theorem can be applied. It gives us

$$\frac{1}{x^{n-\ell-1}}\sum_{k=1}^{\infty}(2\pi k)^{-2n-2}\varphi_{k,n}^{(\ell)}(x)\to\sum_{k=1}^{\infty}(2\pi k)^{-2n-2}\ell!\,c_{0,\ell}$$

as $x \to \infty$. Repeating the argument for the denominator yields

$$\frac{1}{x^{n-\ell-1}} \sum_{k=1}^{\infty} (2\pi k)^{-2n} \varphi_{k,n}^{(\ell)}(x) \to \sum_{k=1}^{\infty} (2\pi k)^{-2n} \ell!\, c_{0,\ell}$$

as $x \to \infty$ and this proves the first assertion. For the second part it is found that

$$\begin{aligned}\frac{\sum_{k=1}^{\infty}(2\pi k)^{-2n-2} u_{k,n}(x)}{\sum_{k=1}^{\infty}(2\pi k)^{-2n} u_{k,n}(x)} &= \frac{\sum_{k=1}^{\infty}(2\pi k)^{-2n-2} n!\,(2t)^{n-2} v_n(x^2/(2\pi k)^2)}{\sum_{k=1}^{\infty}(2\pi k)^{-2n} n!\,(2t)^{n-2} v_n(x^2/(2\pi k)^2)} \\ &= \frac{\sum_{k=1}^{\infty}(2\pi k)^{-2n-2} v_n(x^2/(2\pi k)^2)}{\sum_{k=1}^{\infty}(2\pi k)^{-2n} v_n(x^2/(2\pi k)^2)},\end{aligned}$$

where v_n is as in the proof of Proposition 1.6. Since v_n is positive and increasing, $0 \le v_n(x) \le \lim_{x\to\infty} v_n(x) = n-1$, and the dominated convergence theorem can be applied again. The proof is finished.

REMARK 3.5. In the proof of Proposition 1.6 it would seem more natural not to combine the two terms for $\ell = n-1$ and $\ell = n$ in the series (3). However, when investigating the ratio containing only the last term $\ell = n$, it is decreasing, but the limit as $t \to \infty$ is equal to 0, and thus positivity of the single term corresponding to $\ell = n$ in the series (3) cannot be concluded. This is the reason for combining the two terms.

REMARK 3.6. If $v\colon (0,\infty) \to (0,\infty)$ is differentiable and geometrically concave and if $\{c_n\}$ is any decreasing sequence of positive numbers the sequence $\{u_k'(x)/u_k(x)\}$ is an increasing of functions, where $u_k(x) = v(c_k x)$. This fact was used in the proof of Proposition 1.6, for $c_k = (2\pi k)^{-2}$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
THE UNIVERSITY OF CYPRUS
P. O. BOX 20537
1678 NICOSIA
CYPRUS
E-mail: skoumand@ucy.ac.cy

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF COPENHAGEN
UNIVERSITETSPARKEN 5
DK-2100
DENMARK
E-mail: henrikp@math.ku.dk