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HAAGERUP APPROXIMATION PROPERTY VIA BIMODULES

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(In memory of Uffe Haagerup)

Abstract

The Haagerup approximation property (HAP) is defined for finite von Neumann algebras in such a way that the group von Neumann algebra of a discrete group has the HAP if and only if the group itself has the Haagerup property. The HAP has been studied extensively for finite von Neumann algebras and it was recently generalized to arbitrary von Neumann algebras by Caspers-Skalski and Okayasu-Tomatsu. One of the motivations behind the generalization is the fact that quantum group von Neumann algebras are often infinite even though the Haagerup property has been defined successfully for locally compact quantum groups by Daws-Fima-Skalski-White. In this paper, we fill this gap by proving that the von Neumann algebra of a locally compact quantum group with the Haagerup property has the HAP. This is new even for genuine locally compact groups.

1. Introduction

The notion of the Haagerup property for locally compact groups is introduced after the celebrated work of U. Haagerup [14] on the reduced group C^*-algebras of the free groups. This notion is a very useful generalization of amenability and has been extensively studied in various settings (see [8]). Like the case of amenability, it is only natural to capture this property through operator algebras. Indeed, M. Choda [9] has defined a property now called the Haagerup approximation property (we will abbreviate it as HAP) for *finite tracial* von Neumann algebras and proved that the group von Neumann algebra LG of a discrete group G has the HAP if and only if G has the Haagerup property. The HAP (or its relative version) has been exploited extensively in the study of finite von Neumann algebras as means of deformations in Pops's deformation-vs.-rigidity strategy [24].

The HAP was recently generalized to general von Neumann algebras independently by Caspers-Skalski [7], [6] and by Okayasu-Tomatsu [20], [21].

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Their definitions vary, but turn out to be equivalent (see [5]) and seem to lay a satisfactory foundation for the study of the HAP for general von Neumann algebras. One of the motivations behind the generalization is the fact that quantum group von Neumann algebras are often infinite even though the Haagerup property has been defined successfully for locally compact quantum groups by Daws-Fima-Skalski-White [11]. In this paper, we fill this gap by proving that the von Neumann algebra of a locally compact quantum group with the Haagerup property has the HAP, and that the converse also holds true for *strongly inner amenable* locally compact quantum groups. This extends the same result obtained by Daws-Fima-Skalski-White [11] for the case of discrete quantum groups. Another motivation would be to incorporate Popa's deformation-vs.-rigidity strategy to the study of general von Neumann algebras.

To pursue the latter motivation and to deal with locally compact (quantum) group von Neumann algebras, we take Connes's view [10, V.B] that the theory of bimodules is to von Neumann algebras what the theory of unitary representations is to groups. So, we will give yet another characterization of the HAP in terms of bimodules, which is preceded by the work of Bannon-Fang [3] for finite von Neumann algebras. Thus, we introduce the *strict mixing* property for bimodules and prove that a von Neumann algebra has the HAP if and only if it admits a strictly mixing bimodule which is amenable and that the von Neumann algebra of a locally compact quantum group with the Haagerup property admits such a bimodule.

CONVENTIONS. The inner product of a Hilbert space is linear in the first variable. By (M, φ), etc., we will mean a pair of von Neumann algebra M and a distinguished fns (faithful normal semifinite) weight φ on it. The symbol $\odot$ means the algebraic tensor product, while $\otimes$ means the von Neumann algebraic, Hilbert space, or the spatial C*-algebraic tensor product. All $*$-representations are assumed to be non-degenerate.

2. Preliminary on bimodules

In this section, we will review the theory of bimodules over von Neumann algebras. See [10, V.B], [23], or [25, XI.3] for a comprehensive treatment. In the literature, bimodules are also called *correspondences*.

Let M and N be von Neumann algebras. A Hilbert space $\mathscr{H}$ is said to be an *M-N bimodule* if it comes together with a $*$-representation $\pi_{\mathscr{H}}$ of $M \odot N^{\mathrm{op}}$ that is normal in each variable. Here N^{op} denotes the opposite von Neumann algebra of N. We refer to $\pi_{\mathscr{H}}|_M$ as the left M-action and to $\pi_{\mathscr{H}}|_{N^{\mathrm{op}}}$ as the right N-action, and simply write $a\xi x = \pi_{\mathscr{H}}(a \otimes x^{\mathrm{op}})\xi$ for $a \in M$, $x \in N$, and

$\xi \in \mathscr{H}$. The complex conjugate $\overline{\mathscr{H}}$ of an M-N bimodule is naturally an N-M bimodule. The notation ${}_M\mathscr{H}_N$ will indicate that $\mathscr{H}$ is an M-N bimodule.

An M-N bimodule $\mathscr{H}$ is *weakly contained* in another M-N bimodule $\mathscr{K}$ (denoted by $\mathscr{H} \preceq \mathscr{K}$) if the identity map on $M \odot N^{\mathrm{op}}$ extends to a continuous $*$-homomorphism from $\mathrm{C}^*(\pi_{\mathscr{K}}(M \odot N^{\mathrm{op}}))$ to $\mathrm{C}^*(\pi_{\mathscr{H}}(M \odot N^{\mathrm{op}}))$, that is to say, if for any $\xi \in \mathscr{H}$, any finite subsets $E \subset M$ and $F \subset N$, and any $\varepsilon > 0$, one can find $\eta_1, \dots, \eta_n \in \mathscr{K}$ such that $\left|\langle a\xi x, \xi\rangle - \sum_i \langle a\eta_i x, \eta_i\rangle\right| < \varepsilon$ for all $(a, x) \in E \times F$.

The identity bimodule over M is the M-M bimodule $L^2(M)$, given by $a\xi x = aJx^*J\xi$, where $L^2(M)$ is the standard form for M and J is the modular conjugation. When a fns weight φ on M is fixed, we identify $L^2(M)$ with $L^2(M, \varphi)$ and J with J_φ. The vector in $L^2(M)$ that corresponds to $x \in \mathfrak{n}_\varphi := \{x \in M : \varphi(x^*x) < \infty\}$ is denoted by $x\varphi^{1/2}$. Also, $\varphi^{1/2}x := Jx^*\varphi^{1/2}$ for $x \in \mathfrak{n}_\varphi^*$.

For M-N bimodules (or just right N-modules) $\mathscr{H}$ and $\mathscr{K}$, the Banach space of bounded right N-module maps from $\mathscr{H}$ into $\mathscr{K}$ is denoted by $\mathbb{B}(\mathscr{H}_N, \mathscr{K}_N)$. In the case when $\mathscr{H}$ and $\mathscr{K}$ coincide, we simply denote it by $\mathbb{B}(\mathscr{H}_N)$. Thus, $\mathbb{B}(L^2(N)_N)$ coincides with N acting on $L^2(N)$ from the left.

Let us fix fns weights φ on M and ψ on N. A vector $\xi \in {}_M\mathscr{H}_N$ is said to be *left ψ-bounded* if $L_\psi(\xi)\colon \psi^{1/2}x \mapsto \xi x$, $x \in \mathfrak{n}_\psi^*$, is bounded and hence defines an element in $\mathbb{B}(L^2(N, \psi)_N, \mathscr{H}_N)$. The subspace $\mathscr{D}(\mathscr{H}, \psi)$ of left ψ-bounded vectors is dense in $\mathscr{H}$ [25, Lemma IX.3.3]. We note that if $\xi \in \mathscr{D}(\mathscr{H}, \psi)$ and $a \in M$, then $a\xi \in \mathscr{D}(\mathscr{H}, \psi)$ and $L_\psi(a\xi) = aL_\psi(\xi)$. For $\xi_1, \xi_2 \in \mathscr{D}(\mathscr{H}, \psi)$, we denote by $L_\psi(\xi_2^* \times \xi_1)$ the element in N that corresponds to $L_\psi(\xi_2)^*L_\psi(\xi_1) \in \mathbb{B}(L^2(N)_N) = N$. It in fact belongs to $\mathfrak{n}_\psi^*\mathfrak{n}_\psi$ and satisfies $\psi(L_\psi(\xi_2^* \times \xi_1)) = \langle \xi_1, \xi_2\rangle$ (see [25, p. 199]). Similarly, ξ is said to be *right φ-bounded* if $R_\varphi(\xi)\colon a\varphi^{1/2} \mapsto a\xi$, $a \in \mathfrak{n}_\varphi$, is bounded. It is (φ, ψ)*-bounded* (or simply *bounded*) if it is simultaneously left ψ- and right φ-bounded. The bounded vectors are dense in $\mathscr{H}$ (see Theorem 1).

Let $\theta\colon M \to N$ be a normal completely positive map. Associated to it is the M-N bimodule $L^2(M \otimes_\theta N)$ which is defined to be the Hilbert space completion of $M \odot L^2(N)$ under the semi-inner product $\langle b_1 \otimes_\theta \eta_1, b_2 \otimes_\theta \eta_2\rangle = \langle \theta(b_2^*b_1)\eta_1, \eta_2\rangle$ on simple tensors. Here the symbol $\otimes_\theta$ is used for mnemonic reason. The left M-action and the right N-action are given by $a(b \otimes_\theta \eta)x = (ab) \otimes_\theta (\eta x)$. Suppose now that $\psi \circ \theta \le C\varphi$ for some constant $C > 0$, and fix analytic elements $b \in \mathfrak{n}_\varphi$ and $y \in \mathfrak{n}_\psi$. Then, the vector $\xi = b \otimes_\theta (y\psi^{1/2})$ is (φ, ψ)-bounded. Indeed, one has $\|\xi x\| \le \|y^*\theta(b^*b)y\|^{1/2}\|\psi^{1/2}x\|$ and

$$\begin{aligned}\|a\xi\| = \|\theta(b^*a^*ab)^{1/2}\psi^{1/2}\sigma^\psi_{i/2}(y)\| &\le C^{1/2}\|\sigma^\psi_{i/2}(y)\|\|ab\varphi^{1/2}\| \\ &\le C^{1/2}\|\sigma^\varphi_{i/2}(b)\|\|\sigma^\psi_{i/2}(y)\|\|a\varphi^{1/2}\|.\end{aligned}$$

Let (M,φ), (N,ψ), and (P,ω) be von Neumann algebras. The *relative tensor product* ${}_M\mathscr{H}_N \otimes_N \mathscr{K}_P$ of the bimodules ${}_M\mathscr{H}_N$ and ${}_N\mathscr{K}_P$ is the M-P bimodule which is defined to be the Hilbert space completion of $\mathscr{D}(\mathscr{H},\psi)\odot\mathscr{K}$ under the semi-inner product

$$\langle \xi_1 \otimes_\psi \eta_1, \xi_2 \otimes_\psi \eta_2\rangle = \langle L_\psi(\xi_2^* \times \xi_1)\eta_1, \eta_2\rangle$$

on simple tensors. If moreover the η_i's are right ψ-bounded, then one also has $\langle \xi_1 \otimes_\psi \eta_1, \xi_2 \otimes_\psi \eta_2\rangle = \langle \xi_1, \xi_2 \bar{R}_\psi(\eta_2^* \times \eta_1)\rangle$, where $\bar{R}_\psi(\eta_2^* \times \eta_1) \in N$ is given by $J_\psi R_\psi(\eta_2)^* R_\psi(\eta_1) J_\psi$ [25, Proposition 3.15]. The left M-action and the right P-action are given by $a(\xi \otimes_\psi \eta)x = (a\xi) \otimes_\psi (\eta x)$. We note that if ξ and η are left ψ- and left ω-bounded, then $\xi \otimes_\psi \eta$ is left ω-bounded with $\|L_\omega(\xi \otimes_\psi \eta)\| \le \|L_\psi(\xi)\|\|L_\omega(\eta)\|$. Also, if ξ and η are (φ,ψ)- and (ψ,ω)-bounded, then $\xi\otimes_\psi\eta$ is (φ,ω)-bounded with $\|R_\varphi(\xi\otimes_\psi\eta)\| \le \|R_\varphi(\xi)\|\|R_\psi(\eta)\|$. The relative tensor product construction is associative and continuous with respect to the weak containments, i.e., ${}_N\mathscr{K}_P \preceq {}_N\mathscr{K}'_P$ implies ${}_M\mathscr{H}_N\otimes_N\mathscr{K}_P \preceq {}_M\mathscr{H}_N\otimes_N\mathscr{K}'_P$, and likewise for the first variable. We note that there is a canonical isomorphism ${}_M\mathscr{H}_N\otimes_N L^2(N)_N \cong {}_M\mathscr{H}_N$ via $\xi \otimes_\psi \psi^{1/2}x \leftrightarrow \xi x$ for $\xi \in \mathscr{D}(\mathscr{H},\psi)$ and $x \in \mathfrak{n}_\psi^*$. Likewise ${}_M L^2(M)_M \otimes_M \mathscr{H}_N \cong {}_M\mathscr{H}_N$ via $a\varphi^{1/2} \otimes_\varphi \xi \leftrightarrow a\xi$ for $a \in \mathfrak{n}_\varphi$ and $\xi \in \mathscr{H}$.

The following fact is well-known in the case of finite von Neumann algebras and the general case is probably also known to the specialists, but the authors did not find it in literature.

THEOREM 1 (cf. [23, 1.2.2]). *Let (M,φ) and (N,ψ) be von Neumann algebras and ${}_M\mathscr{H}_N$ be an M-N bimodule. Then the (φ,ψ)-bounded vectors are dense in $\mathscr{H}$.*

PROOF. We first prove the theorem assuming that M is semifinite (but φ need not be a trace). For this, we claim that for every $\xi \in \mathscr{H}$ there is a net $(c_i)_i$ of contractions in M such that $c_i\xi$ are right φ-bounded and $c_i\xi \to \xi$. Indeed, fix an fns trace τ on M and view normal states on M as τ-measurable operators on $L^2(M,\tau)$ affiliated with M (see [25, XI.2] for an account of measurable operators). Thus the vector functional ω_ξ on M corresponds to $h \in L^1(M,\tau)_+$ in such a way that $\langle a\xi,\xi\rangle = \tau(ha)$ for $a \in M$. Similarly, take an increasing net $(\varphi_j)_j$ of normal positive functionals on M such that $\varphi = \sup\varphi_j$, and denote by k_j the element in $L^1(M,\tau)_+$ that corresponds to φ_j. Let $c_{n,j} = \chi_{[n^{-1},\infty)}(k_j)(1+n^{-1}h)^{-1/2} \in M$. Since φ is faithful, one has $c_{n,j} \to 1$ ultrastrongly. Moreover, the inequality

$$c_{n,j}hc_{n,j}^* = \chi_{[n^{-1},\infty)}(k_j)\frac{h}{1+n^{-1}h}\chi_{[n^{-1},\infty)}(k_j) \le n^2 k_j$$

implies that $\|ac_{n,j}\xi\| = \tau(hc_{n,j}^* a^* ac_{n,j})^{1/2} \le n\varphi_j(a^*a)^{1/2} \le n\|a\varphi^{1/2}\|$ for every $a \in \mathfrak{n}_\varphi$. Thus we obtain the claim. Since the space $\mathscr{D}(\mathscr{H}, \psi)$ of left ψ-bounded vectors is dense and is a left M-module, we see by the claim that the bounded vectors are dense in $\mathscr{H}$.

Now let (M, φ) be an arbitrary von Neumann algebra and denote the modular action by σ. Let $\tilde{M} = M \rtimes_\sigma \mathbb{R}$ and $\tilde{\varphi}$ denote the corresponding crossed product and the dual weight. We will denote by $\pi: M \to \tilde{M}$ the canonical inclusion and by $\{u_s : s \in \mathbb{R}\}$ the unitary elements in $\tilde{M}$ that implement the modular action. Thus $u_s\pi(a)u_s^* = \pi(\sigma_s(a))$ for $a \in M$ and $\tilde{M} = (\pi(M) \cup u_\mathbb{R})''$. Since $\tilde{M}$ is semifinite, the result of the previous paragraph says that $(\tilde{\varphi}, \psi)$-bounded vectors are dense in the relative tensor product $\tilde{\mathscr{H}} := {}_{\tilde{M}}L^2(\tilde{M}, \tilde{\varphi})_M \otimes_M \mathscr{H}_N$. We identify $\tilde{\mathscr{H}}$ with $L^2(\mathbb{R}, \mathscr{H})$ where the left $\tilde{M}$-action is given by $(u_s\xi)(t) = \xi(t-s)$ for $s \in \mathbb{R}$ and $(\pi(a)\xi)(t) = \sigma_t^{-1}(a)\xi(t)$ for $a \in M$; and the right N-action is given by $(\xi x)(t) = \xi(t)x$ for $x \in N$. For every $f \in K(\mathbb{R})$ (compactly supported continuous functions), we define $L_f \in \mathbb{B}(\mathscr{H}_N, L^2(\mathbb{R}, \mathscr{H})_N)$ by $L_f\xi = f \otimes \xi$, where $(f \otimes \xi)(t) = f(t)\xi$. Since $\{f^* * g : f, g \in K(\mathbb{R})\}$ has dense span in $L^2(\mathbb{R})$, the proof of the theorem will be done once we prove that $L_{f^**g}^*\zeta \in \mathscr{H}$ is (φ, ψ)-bounded for every $f, g \in K(\mathbb{R})$ and every $(\tilde{\varphi}, \psi)$-bounded $\zeta \in \tilde{\mathscr{H}}$. That $L_{f^**g}^*\zeta$ is left ψ-bounded is obvious. A direct computation shows that $aL_{f^**g}^*\zeta = L_g^*\tilde{a}\zeta$ for every $a \in M$. Here $\tilde{a} = \int f(s)au_s\,ds \in \tilde{M}$. Indeed, for every $\xi \in \mathscr{H}$ one has

$$
\begin{aligned}
\langle aL_{f^**g}^*\zeta, \xi\rangle &= \int \Big\langle \zeta(t), \int \bar{f}(s)g(t+s)\,ds\,a^*\xi \Big\rangle dt \\
&= \iint f(s)\langle a\zeta(t-s), g(t)\xi\rangle\,dt\,ds = \langle \tilde{a}\zeta, L_g\xi\rangle.
\end{aligned}
$$

It follows that

$$
\|aL_{f^**g}^*\zeta\| \le \|L_g\|\|R_{\tilde{\varphi}}(\zeta)\|\|\tilde{a}\tilde{\varphi}^{1/2}\| = \|g\|_2\|R_{\tilde{\varphi}}(\zeta)\|\|f\|_2\|a\varphi^{1/2}\|
$$

for every $a \in \mathfrak{n}_\varphi$, and hence $L_{f^**g}^*\zeta$ is right φ-bounded also.

LEMMA 2. *Let (M, φ) and (N, ψ) be von Neumann algebras and $\xi \in {}_M\mathscr{H}_N$ be a (φ, ψ)-bounded vector. Then, the completely positive map*

$$
\theta_\xi : M \ni a \mapsto L_\psi(\xi)^* a L_\psi(\xi) = L_\psi(\xi^* \times a\xi) \in N
$$

satisfies $\psi \circ \theta_\xi \le \|R_\varphi(\xi)\|^2\varphi$. The corresponding operator

$$
T_{\varphi,\psi}(\xi): L^2(M, \varphi) \ni a\varphi^{1/2} \mapsto \theta_\xi(a)\psi^{1/2} \in L^2(N, \psi), \quad a \in \mathfrak{n}_\varphi,
$$

is equal to $L_\psi(\xi)^* R_\varphi(\xi)$. *Moreover, for every* $a \in \mathfrak{n}_\varphi$ *and* $x \in \mathfrak{n}_\psi$, *one has*

$$\langle \theta_\xi(a)\psi^{1/2}, \psi^{1/2}x^*\rangle = \langle a\xi, \xi x^*\rangle = \langle L_\psi(\xi)^* R_\varphi(\xi) a\varphi^{1/2}, \psi^{1/2}x^*\rangle.$$

In particular, the M-N *bimodules* $\overline{M\xi N}$ *and* $L^2(M \otimes_{\theta_\xi} N)$ *are isomorphic via the correspondence* $a\xi x \leftrightarrow a \otimes_{\theta_\xi} \psi^{1/2}x$ *for* $a \in M$ *and* $x \in \mathfrak{n}_\psi^*$.

Proof. That the map is completely positive is clear. For $a \in M$, one has

$$\psi(\theta_\xi(a^*a)) = \psi(L_\psi((a\xi)^* \times (a\xi))) = \|a\xi\|^2 \le \|R_\varphi(\xi)\|^2 \varphi(a^*a).$$

This proves the first assertion. It follows that for every $a \in \mathfrak{n}_\varphi$ and $x, y \in \mathfrak{n}_\psi$, one has $\theta_\xi(a) \in \mathfrak{n}_\psi$ and

$$\begin{aligned}\langle \theta_\xi(a)\psi^{1/2}, \psi^{1/2}x^*y\rangle &= \langle a\xi y^*, \xi x^*\rangle = \langle a\xi, \xi x^* y\rangle \\ &= \langle R_\varphi(\xi)a\varphi^{1/2}, L_\psi(\xi)\psi^{1/2}x^*y\rangle.\end{aligned}$$

Note that $\mathfrak{n}_\psi^*\mathfrak{n}_\psi$ is L^2-dense in $\mathfrak{n}_\psi$. The proof of the last assertion is routine.

Hence, every M-N bimodule is isomorphic to a direct sum of bimodules of the form $L^2(M \otimes_\theta N)$. To complete the picture, recall from a previous paragraph that if $\psi \circ \theta \le C\varphi$ for some constant $C > 0$ and $b \in \mathfrak{n}_\varphi$ and $y \in \mathfrak{n}_\psi$ are analytic elements, then $\xi = b \otimes_\theta (y\psi^{1/2})$ is (φ, ψ)-bounded. The completely positive map θ_ξ arising from Lemma 2 is related to the original θ by the relation $\theta_\xi(a) = y^*\theta(b^*ab)y$. Indeed, one has

$$\langle \theta_\xi(a)\psi^{1/2}x_1, \psi^{1/2}x_2\rangle = \langle a\xi x_1, \xi x_2\rangle = \langle \theta(b^*ab)y\psi^{1/2}x_1, y\psi^{1/2}x_2\rangle$$

for every $x_1, x_2 \in \mathfrak{n}_\psi^*$. Also, it is not difficult to see that the maps arising in Lemma 2 are compatible with the relative tensor product, as follows.

Lemma 3. *Let* (M, φ), (N, ψ), *and* (P, ω) *be von Neumann algebras, and let* $\xi \in {}_M\mathscr{H}_N$ *and* $\eta \in {}_N\mathscr{K}_P$ *be bounded vectors. Then,* $\theta_{\xi\otimes_\psi\eta} = \theta_\eta \circ \theta_\xi$ *and* $T_{\varphi,\omega}(\xi \otimes_\psi \eta) = T_{\psi,\omega}(\eta)T_{\varphi,\psi}(\xi)$.

3. Mixing bimodules and the Haagerup approximation property

Recall that a von Neumann algebra M is amenable (or injective) if and only if the identity bimodule ${}_M L^2(M)_M$ is weakly contained in the *coarse bimodule* ${}_M L^2(M) \otimes L^2(M)_M$. The coarse bimodule has a very strong mixing property. The precise notion of mixing property for bimodules has been introduced in [3] in the case of finite von Neumann algebras under the name of "C_0-correspondences". There it is proved that a finite von Neumann algebra M has the HAP if and only if the identity correspondence $L^2(M)$ is weakly contained in a C_0-correspondence [3, Theorem 3.4]. In this section, we extend this to the general setting.

DEFINITION 4. Let (M, φ) and (N, ψ) be von Neumann algebras with distinguished fns weights. An M-N bimodule $\mathscr{H}$ is said to be *strictly mixing* if the family

$$\mathscr{H}_{\mathrm{mix}} = \{\xi \in \mathscr{H} : \xi \text{ is } (\varphi, \psi)\text{-bounded and } T_{\varphi,\psi}(\xi) \text{ is compact}\}$$

is M-N cyclic in $\mathscr{H}$ (i.e., $M\mathscr{H}_{\mathrm{mix}}N$ has a dense span in $\mathscr{H}$). As we will see in Corollary 8, the strict mixing property of $\mathscr{H}$ does not depend on the choices of fns weights (but the subset $\mathscr{H}_{\mathrm{mix}}$ does).

It is not clear whether $\mathscr{H}_{\mathrm{mix}}$ is always a linear subspace, or even that it contains a linear subspace with the same closed linear span. However, the infinite multiple of $\mathscr{H}$ has the latter property, as we will prove below. Hence, we may assume that $\mathscr{H}$ has this property in all cases which are dealt with in this paper.

LEMMA 5 (cf. [3, Definition 3.1]). *Let $\theta\colon M \to N$ be a normal completely positive map such that $\psi \circ \theta \le C\varphi$ for some constant $C > 0$ and such that $T_\theta\colon L^2(M, \varphi) \to L^2(N, \psi)$ defined by $T_\theta a\varphi^{1/2} = \theta(a)\psi^{1/2}$ is compact. Then, the M-N bimodule $L^2(M \otimes_\theta N)$ is strictly mixing. Conversely, if $\mathscr{H}$ is a strictly mixing M-N bimodule such that ${}_M\mathscr{H}_N \cong \bigoplus_\kappa {}_M\mathscr{H}_N$ for the density character κ of $\mathscr{H}$, then $\mathscr{H}$ is isomorphic to a direct sum $\bigoplus_i L^2(M \otimes_{\theta_i} N)$ of bimodules as above. In particular, the subset $\mathscr{H}_{\mathrm{mix}}$ contains a dense linear subspace of $\mathscr{H}$.*

PROOF. We will prove that $L^2(M \otimes_\theta N)_{\mathrm{mix}}$ contains a dense linear subspace. We stick to the notations used in the previous section and put $\xi = \sum_k b_k \otimes_\theta (y_k\psi^{1/2})$ for finite sequences $b_k \in \mathfrak{n}_\varphi$, $y_k \in \mathfrak{n}_\psi$ of analytic elements. Then, by the remark following Lemma 2, one has

$$\begin{aligned}\theta_\xi(a)\psi^{1/2} &= \sum_{k,l} y_k^*\theta(b_k^* a b_l) y_l \psi^{1/2} = \sum_{k,l} y_k^* J_\psi \sigma^\psi_{i/2}(y_l)^* J_\psi \theta(b_k^* a b_l)\psi^{1/2} \\ &= \sum_{k,l} y_k^* J_\psi \sigma^\psi_{i/2}(y_l)^* J_\psi T_\theta b_k^* J_\varphi \sigma^\varphi_{i/2}(b_l)^* J_\varphi a \varphi^{1/2}.\end{aligned}$$

Hence $T_{\varphi,\psi}(\xi) = \sum_{k,l} y_k^* J_\psi \sigma^\psi_{i/2}(y_l)^* J_\psi T_\theta b_k^* J_\varphi \sigma^\varphi_{i/2}(b_l)^* J_\varphi$ and it is compact because T_θ is so.

Now let $\{\xi_i : i \in \kappa\}$ be an M-N cyclic family in $\mathscr{H}_{\mathrm{mix}}$ and $\mathscr{K}_i := \overline{M\xi_i N} \subset \mathscr{H}$. Then, $\mathscr{H}$ is isomorphic to a subbimodule of $\bigoplus \mathscr{K}_i$, and $\mathscr{K}_i \cong L^2(M \otimes_{\theta_{\xi_i}} N)$ for each i, by Lemma 2. Since $\mathscr{H}$ has infinite multiplicity, $\mathscr{H} \cong \mathscr{H} \oplus \mathscr{K}_i$ for each i and $\mathscr{H} \oplus \bigoplus_i \mathscr{K}_i \cong \bigoplus_i \mathscr{K}_i$. It follows that $\mathscr{H} \cong \bigoplus_i (\mathscr{H} \oplus \mathscr{K}_i) \cong \bigoplus_i \mathscr{K}_i$. Note that for any finite sequence $\xi_i \in (\mathscr{K}_i)_{\mathrm{mix}}$, one has $\sum_i \xi_i \in (\bigoplus_i \mathscr{K}_i)_{\mathrm{mix}}$, since $T_{\varphi,\psi}(\sum_i \xi_i) = \sum_i T_{\varphi,\psi}(\xi_i)$.

Although it will not be used, we note the following fact. The converse is not clear.

LEMMA 6 (cf. [22, Definition 2.3]). *A strictly mixing M-N bimodule $\mathscr{H}$ is mixing in the sense that $\langle a_n \xi x_n, \xi \rangle \to 0$ for any $\xi \in \mathscr{H}$ and any bounded nets $(a_n)_n$ in M and $(x_n)_n$ in N, one of which is ultraweakly null.*

PROOF. Since $(a_n)_n$ and $(x_n)_n$ are bounded nets, we may assume that $\xi \in \mathscr{H}_{\mathrm{mix}}$. Given $\varepsilon > 0$, take $p \in \mathfrak{n}_\varphi$ and $q \in \mathfrak{n}_\psi$ such that $\|p\xi - \xi\| + \|\xi - \xi q^*\| < \varepsilon$. Then, $(a_n p \varphi^{1/2})_n$ and $(\psi^{1/2} q^* x_n^*)_n$ are bounded nets respectively in $L^2(M)$ and $L^2(N)$, one of which is weakly null. Hence $\langle a_n p \xi x_n, \xi q^* \rangle = \langle T_{\varphi,\psi}(\xi) a_n p \varphi^{1/2}, \psi^{1/2} q^* x_n^* \rangle \to 0$, by the strict mixing property. Since $\varepsilon > 0$ was arbitrary, one concludes that $\langle a_n \xi x_n, \xi \rangle \to 0$.

PROPOSITION 7. *Let (M, φ), (N, ψ), and (P, ω) be von Neumann algebras. Then the relative tensor product bimodule ${}_M\mathscr{H}_N \otimes_N \mathscr{K}_P$ is strictly mixing if one of ${}_M\mathscr{H}_N$ and ${}_N\mathscr{K}_P$ is strictly mixing.*

PROOF. This follows from Theorem 1 and Lemma 3.

COROLLARY 8. *The strict mixing property of an M-N bimodule $\mathscr{H}$ does not depend on the choices of fns weights.*

PROOF. Let $\mathscr{H}$ be a strictly mixing (M, φ)-(N, ψ) bimodule, and let φ_0 be another fns weight on M. Let us view $L^2(M)$ as an (M, φ_0)-(M, φ) bimodule. Then by Proposition 7, the relative tensor product ${}_M L^2(M)_M \otimes_M \mathscr{H}_N$ is a strictly mixing (M, φ_0)-(N, ψ) bimodule, but it is canonically isomorphic to $\mathscr{H}$ as an M-N bimodule. So the strict mixing property does not depend on the choice of φ, and similarly neither on ψ.

There are several equivalent definitions of the Haagerup approximation property (HAP) for a von Neumann algebra M, but in the end they are equivalent to that the finite von Neumann algebra $p(M \rtimes_\sigma \mathbb{R})p$ has the HAP for a modular action σ and a finite projection p with full central support (see [5], [7], [6], [15], [20], [21]). We recall that such $p(M \rtimes_\sigma \mathbb{R})p$ is *amenably equivalent* (in the sense of Anantharaman-Delaroche [1]) to the original M. Here we say an M-N bimodule $\mathscr{H}$ is *left amenable* if the identity bimodule ${}_M L^2(M)_M$ is weakly contained in ${}_M\mathscr{H}_N \otimes_N \overline{\mathscr{H}}_M$; and M and N are amenably equivalent if there exist a left amenable M-N bimodule and a left amenable N-M bimodule. See [1] for the details. In fact, it is not too difficult to see that, for any crossed product $\tilde{M} = M \rtimes_\sigma G$ of a von Neumann algebra M by an amenable locally compact group G, both ${}_M L^2(\tilde{M})_{\tilde{M}}$ and ${}_{\tilde{M}} L^2(\tilde{M})_M$ are left amenable. We note that the relative tensor product of two left amenable bimodules is again left amenable by continuity [1, 2.13]. The following theorem is reminiscent of the fact [12] that a von Neumann algebra M is *semi-discrete* if and only if it is *amenable*: ${}_M L^2(M)_M \preceq {}_M L^2(M) \otimes L^2(M)_M$.

THEOREM 9. *For a von Neumann algebra M, the following are equivalent.*

(1) *M has the HAP.*

(2) *The identity bimodule $L^2(M)$ is weakly contained in a strictly mixing M-M bimodule.*

(3) *There are a von Neumann algebra N and a strictly mixing M-N bimodule which is left amenable.*

PROOF. Proposition 7 implies the equivalence (2) $\Leftrightarrow$ (3), as well as invariance of (3) under the amenable equivalence. Now the proof of (1) $\Leftrightarrow$ (3) reduces to the case of finite von Neumann algebras, but in which case it is done by [3, Theorem 3.4].

4. Locally compact quantum group von Neumann algebras

In this section, we study the relationship between the unitary representations of a locally compact quantum group G and the bimodules of the group von Neumann algebra LG.

For a locally compact group G, there are the function algebras $C_0(G) \subset L^\infty(G)$ and the group operator algebras $\mathrm{C}^*_\lambda(G) \subset LG$. A *locally compact quantum group G* also comes with pairs of an ultraweakly dense C^*-subalgebra of a von Neumann algebra, $C_0(G) \subset L^\infty(G)$ and $\mathrm{C}^*_\lambda(G) \subset LG$. The latter is often written as $C_0(\hat{G}) \subset L^\infty(\hat{G})$. See [17], [18] for a comprehensive treatment of theory of locally compact quantum groups. Recall that the comultiplication $\Delta\colon L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$ is given by $\Delta(x) = W^*(1 \otimes x)W$, where W is the multiplicative unitary of G [17, 5.2]. Then, $W \in M(C_0(G) \otimes \mathrm{C}^*_\lambda(G))$ and Δ maps $C_0(G)$ into $M(C_0(G) \otimes C_0(G))$ (see the remarks after Theorem 8.2 in [18]). We denote by φ the left Haar weight (which is unique up to a scalar multiple) on $L^\infty(G)$ and by $\hat{\varphi}$ the dual weight on LG. The modular conjugations are written respectively by J and $\hat{J}$. The map $\hat{R}\colon a \mapsto Ja^*J$ defines an anti-$*$-automorphism on $\mathrm{C}^*_\lambda(G)$, called the *unitary antipode* [17, 5.3].

The map λ from $L^1(G) := L^\infty(G)_*$ to $\mathrm{C}^*_\lambda(G)$, given by $\lambda(\omega) = (\omega \otimes \mathrm{id})(W)$, has a dense range and is called the *left regular representation*. More generally, a *unitary representation* of G (or a *unitary corepresentation* of $C_0(G)$) on $\mathscr{H}_U$ is a unitary element $U \in M(C_0(G) \otimes \mathbb{K}(\mathscr{H}_U))$ which satisfies $(\Delta \otimes \mathrm{id})(U) = U_{13}U_{23}$ (see [17, Definition 3.5]). This identity is equivalent to Fell's absorption principle for locally compact quantum groups:

$$W ⓣ U := W_{12}U_{13} = W_{12}(\Delta \otimes \mathrm{id})(U)U_{23}^* = U_{23}W_{12}U_{23}^*.$$

There exist the *universal group* C*-*algebra* $C^*_u(G)$ and the universal unitary representation $W_u \in M(C_0(G) \otimes C^*_u(G))$, with $\lambda_u\colon L^1(G) \to C^*_u(G)$ given by $\lambda_u(\omega) = (\omega \otimes \mathrm{id})(W_u)$ (see [16]). Thus, there is a bijective correspondence between the unitary representations U of G and the $*$-representations ϕ_U of $C^*_u(G)$ on $\mathscr{H}_U$, which is given by $\phi_U(\lambda_u(\omega)) = (\omega \otimes \mathrm{id})(U)$, or equivalently $U = (\mathrm{id} \otimes \phi_U)(W_u)$ (see [16, Proposition 5.2]). A *coefficient* of U is defined to be $f_\omega \in L^\infty(G)$, which is given by $f_\omega = (\mathrm{id} \otimes \omega)(U)$ for $\omega \in \mathbb{B}(\mathscr{H}_U)_*$. When $\omega = \omega_\eta$ is the vector functional associated with $\eta \in \mathscr{H}_U$, we simply write $f_\eta = f_\omega$. The *LG-LG bimodule associated with a unitary representation* U is the bimodule $\mathscr{H} := L^2(G) \otimes \mathscr{H}_U$ which is given by the tensor product representation $W ⓣ U$ of G on $L^2(G) \otimes \mathscr{H}$. Namely, $\lambda(\omega) \in LG$ acts on $\mathscr{H}$ from the left by

$$\phi_{W ⓣ U}(\lambda_u(\omega)) = (\omega \otimes \mathrm{id} \otimes \mathrm{id})(W ⓣ U) = U(\lambda(\omega) \otimes 1)U^*$$

and from the right by $\hat{J}\lambda(\omega)^*\hat{J} \otimes 1$. Let V be another unitary representation of G. Then, U is said to be *weakly contained* in V (denoted by $U \preceq V$) if ϕ_U is weakly contained in ϕ_V, i.e., if the identity map extends to a continuous $*$-homomorphism σ from $\phi_V(C^*_u(G))$ to $\phi_U(C^*_u(G))$, which will satisfy $(\mathrm{id} \otimes \sigma)(V) = U$.

Proposition 10. *Let U and V be unitary representations of G such that $U \preceq V$. Then one has ${}_{LG}(L^2(G) \otimes \mathscr{H}_U)_{LG} \preceq {}_{LG}(L^2(G) \otimes \mathscr{H}_V)_{LG}$.*

The following lemma will be used in the proof and later. The operator $T_{\hat{\varphi},\hat{\varphi}}(\xi)$ appearing in Lemma 2 is simply written as $T_{\hat{\varphi}}(\xi)$.

Lemma 11. *Let $\zeta = \xi \otimes \eta \in {}_{LG}(L^2(G) \otimes \mathscr{H}_U)_{LG}$ be a simple tensor such that ξ is $(\hat{\varphi}, \hat{\varphi})$-bounded. Then, ζ is $(\hat{\varphi}, \hat{\varphi})$-bounded and satisfies $T_{\hat{\varphi}}(\zeta) = T_{\hat{\varphi}}(\xi) f_\eta$.*

Proof. We recall here the definition of the dual weight $\hat{\varphi}$. Let $\mathscr{I}$ be the collection of $\omega \in L^1(G)$ such that there is a $C > 0$ satisfying $|\omega(x^*)| \leq C\|x\varphi^{1/2}\|$ for $x \in \mathfrak{n}_\varphi$. By Riesz representation theorem, there is $\xi(\omega) \in L^2(G)$ such that $\omega(x^*) = \langle \xi(\omega), x\varphi^{1/2} \rangle$ for $x \in \mathfrak{n}_\varphi$. We define $a\omega \in L^1(G)$ by $(a\omega)(x) = \omega(xa)$ for $a, x \in L^\infty(G)$ and $\omega \in L^1(G)$. This makes $\mathscr{I}$ a left $L^\infty(G)$-module and ξ a module map. Namely, $\xi(a\omega) = a\xi(\omega)$ for every $a \in L^\infty(G)$ and $\omega \in \mathscr{I}$. The dual weight $\hat{\varphi}$ is defined in such a way that every $\omega \in \mathscr{I}$ satisfies $\lambda(\omega) \in \mathfrak{n}_{\hat{\varphi}}$ and $\lambda(\omega)\hat{\varphi}^{1/2} = \xi(\omega)$ [17, Proposition 5.22]. It follows that, for every $x \in \mathfrak{n}_{\hat{\varphi}}$, one has

$$\begin{aligned}\langle \lambda(\omega)\zeta, \zeta x^* \rangle &= \langle (\omega \otimes \mathrm{id} \otimes \mathrm{id})(W_{12}U_{13})(\xi \otimes \eta), \xi x^* \otimes \eta \rangle \\ &= \langle (\omega \otimes \mathrm{id})(W(f_\eta \otimes 1))\xi, \xi x^* \rangle \\ &= \langle \lambda(f_\eta \omega)\xi, \xi x^* \rangle = \langle R_{\hat{\varphi}}(\xi)\xi(f_\eta \omega), L_{\hat{\varphi}}(\xi)\hat{\varphi}^{1/2} x^* \rangle \\ &= \langle T_{\hat{\varphi}}(\xi) f_\eta \lambda(\omega)\hat{\varphi}^{1/2}, \hat{\varphi}^{1/2} x^* \rangle.\end{aligned}$$

Since $\lambda(\mathscr{I})$ is dense in $\mathfrak{n}_{\hat{\varphi}}$, this proves the lemma.

Proof of Proposition 10. Let a simple tensor $\zeta = \xi \otimes \eta \in L^2(G) \otimes \mathscr{H}_U$ such that ξ is $(\hat{\varphi}, \hat{\varphi})$-bounded and $\|\xi\| = \|\eta\| = 1$ be given. Since ϕ_U is weakly contained in ϕ_V, there is a net $(\omega_i)_i$ of normal states on $\mathbb{B}(\mathscr{H}_V)$ such that $\omega_i \circ \phi_V \to \omega_\eta \circ \phi_U$ pointwise on $\mathrm{C}^*_{\mathrm{u}}(G)$. Then, one has $f_{\omega_i} \to f_\omega$ ultraweakly, since for every $\xi, \eta \in L^2(G)$, one has

$$\langle f_\omega \xi, \eta \rangle = (\omega_{\xi,\eta} \otimes (\omega \circ \phi_U))(W_{\mathrm{u}}) = \lim_i (\omega_{\xi,\eta} \otimes (\omega_i \circ \phi_V))(W_{\mathrm{u}}) = \lim_i \langle f_{\omega_i} \xi, \eta \rangle.$$

Here note that $(\omega_{\xi,\eta} \otimes \mathrm{id})(W_{\mathrm{u}}) \in \mathrm{C}^*_{\mathrm{u}}(G)$. We may assume that $\omega_i = \sum_{j=1}^{n(i)} \omega_{\eta_{i,j}}$ for some $\eta_{i,j} \in \mathscr{H}_V$. Let $\zeta_{i,j} = \xi \otimes \eta_{i,j} \in L^2(G) \otimes \mathscr{H}_V$. Then, by Lemmas 2 and 11, for every $a, x \in \mathfrak{n}_{\hat{\varphi}}$, one has

$$\begin{aligned}\langle a\zeta, \zeta x^* \rangle &= \langle T_{\hat{\varphi}}(\zeta) a\hat{\varphi}^{1/2}, \hat{\varphi}^{1/2} x^* \rangle = \langle T_{\hat{\varphi}}(\xi) f_\eta a\hat{\varphi}^{1/2}, \hat{\varphi}^{1/2} x^* \rangle \\ &= \lim_i \sum_{j=1}^{n(i)} \langle T_{\hat{\varphi}}(\xi) f_{\eta_{i,j}} a\hat{\varphi}^{1/2}, \hat{\varphi}^{1/2} x^* \rangle = \lim_i \sum_{j=1}^{n(i)} \langle a\zeta_{i,j}, \zeta_{i,j} x^* \rangle.\end{aligned}$$

Since $\max\left\{|\langle a\zeta, \zeta x^* \rangle|, \left|\sum_{j=1}^{n(i)} \langle a\zeta_{i,j}, \zeta_{i,j} x^* \rangle\right|\right\} \leq \|T_{\hat{\varphi}}(\xi)\| \|a\hat{\varphi}^{1/2}\| \|\hat{\varphi}^{1/2} x^*\|$, the above equality in fact holds for all $a, x \in LG$. This means that $\omega_\zeta \circ \pi_{L^2(G)\otimes\mathscr{H}_U}$ is continuous on $\pi_{L^2(G)\otimes\mathscr{H}_V}(LG \odot LG^{\mathrm{op}})$. Since such states ω_ζ form a cyclic family, we conclude that $L^2(G) \otimes \mathscr{H}_U \preceq L^2(G) \otimes \mathscr{H}_V$.

We will prove a partial converse to Proposition 10. For this, we have to consider the comultiplication $\hat{\Delta}_{\max}\colon \mathrm{C}^*_{\mathrm{u}}(G) \to M(\mathrm{C}^*_{\mathrm{u}}(G) \otimes_{\max} \mathrm{C}^*_{\mathrm{u}}(G))$ with respect to the maximal tensor product. Let $\pi_i\colon \mathrm{C}^*_{\mathrm{u}}(G) \to M(\mathrm{C}^*_{\mathrm{u}}(G) \otimes_{\max} \mathrm{C}^*_{\mathrm{u}}(G))$ be the embeddings given by $\pi_1(a) = a \otimes 1$ and $\pi_2(a) = 1 \otimes a$. Then, we consider the unitary representation

$$X := (\mathrm{id} \otimes \pi_2)(W_{\mathrm{u}})(\mathrm{id} \otimes \pi_1)(W_{\mathrm{u}}) \in M(C_0(G) \otimes (\mathrm{C}^*_{\mathrm{u}}(G) \otimes_{\max} \mathrm{C}^*_{\mathrm{u}}(G))).$$

Since the second variables of $(\mathrm{id} \otimes \pi_1)(W_{\mathrm{u}})$ and $(\mathrm{id} \otimes \pi_2)(W_{\mathrm{u}})$ commute, X is indeed a unitary representation. We put $\hat{\Delta}_{\max} := \phi_X$. Namely, $\hat{\Delta}_{\max}$ is the

$*$-homomorphism that satisfies $(\mathrm{id}\otimes\hat{\Delta}_{\max})(W_{\mathrm{u}})=X$. The coassociativity of $\hat{\Delta}_{\max}$ follows from

$$\begin{aligned}(\mathrm{id}\otimes(&\hat{\Delta}_{\max}\otimes\mathrm{id}))(X)\\ &=(\mathrm{id}\otimes(\hat{\Delta}_{\max}\otimes\mathrm{id}))((\mathrm{id}\otimes\pi_2)(W_{\mathrm{u}})\cdot(\mathrm{id}\otimes\pi_1)(W_{\mathrm{u}}))\\ &=(\mathrm{id}\otimes\pi_3')(W_{\mathrm{u}})\cdot(\mathrm{id}\otimes\pi_2')(W_{\mathrm{u}})(\mathrm{id}\otimes\pi_1')(W_{\mathrm{u}})\\ &=(\mathrm{id}\otimes(\mathrm{id}\otimes\hat{\Delta}_{\max}))(X).\end{aligned}$$

Here $\pi_i'\colon \mathrm{C}^*_{\mathrm{u}}(G)\to M(\mathrm{C}^*_{\mathrm{u}}(G)\otimes_{\max}\mathrm{C}^*_{\mathrm{u}}(G)\otimes_{\max}\mathrm{C}^*_{\mathrm{u}}(G))$ denote the obvious embeddings. Moreover, for the quotient map $q\colon \mathrm{C}^*_{\mathrm{u}}(G)\otimes_{\max}\mathrm{C}^*_{\mathrm{u}}(G)\to \mathrm{C}^*_{\mathrm{u}}(G)\otimes\mathrm{C}^*_{\mathrm{u}}(G)$, the map $q\circ\hat{\Delta}_{\max}$ is equal to the usual comultiplication $\hat{\Delta}_{\mathrm{u}}$ on $\mathrm{C}^*_{\mathrm{u}}(G)$ (see [17, p. 311]).

Let $\mathscr{H}$ be an *LG*-*LG* bimodule with the $*$-representation $\pi_{\mathscr{H}}\colon LG\odot LG^{\mathrm{op}}\to\mathbb{B}(\mathscr{H})$. We define the *unitary representation $U_{\mathscr{H}}$ associated with $\mathscr{H}$* to be the one given by

$$\phi_{U_{\mathscr{H}}}=\pi_{\mathscr{H}}\circ(\lambda\otimes\lambda^{\mathrm{op}})\circ\hat{\Delta}_{\max}\colon \mathrm{C}^*_{\mathrm{u}}(G)\to\mathbb{B}(\mathscr{H}).$$

Here we identify the $*$-homomorphisms from $\mathrm{C}^*_{\mathrm{u}}(G)$ with the corresponding representations from $L^1(G)$, and define $\lambda^{\mathrm{op}}\colon \mathrm{C}^*_{\mathrm{u}}(G)\to\mathrm{C}^*_{\lambda}(G)^{\mathrm{op}}$ by $\lambda^{\mathrm{op}}(\omega):=\hat{R}(\lambda(\omega))^{\mathrm{op}}$, where $\hat{R}$ is the unitary antipode. Let us define $\pi^{(i)}_{\mathscr{H}}\colon\mathrm{C}^*_{\mathrm{u}}(G)\to\mathbb{B}(\mathscr{H})$ by $\pi^{(1)}_{\mathscr{H}}(\lambda_{\mathrm{u}}(\omega))=\pi_{\mathscr{H}}(\lambda(\omega)\otimes 1)$ and $\pi^{(2)}_{\mathscr{H}}(\lambda_{\mathrm{u}}(\omega))=\pi_{\mathscr{H}}(1\otimes\lambda^{\mathrm{op}}(\omega))$. Then, it follows from the definition that

$$\begin{aligned}U_{\mathscr{H}}=(\mathrm{id}\otimes\pi^{(2)}_{\mathscr{H}})(W_{\mathrm{u}})&(\mathrm{id}\otimes\pi^{(1)}_{\mathscr{H}})(W_{\mathrm{u}})\\ &\in M(C_0(G)\otimes\mathrm{C}^*(\pi_{\mathscr{H}}(\mathrm{C}^*_{\lambda}(G)\odot\mathrm{C}^*_{\lambda}(G)^{\mathrm{op}})).\end{aligned}$$

PROPOSITION 12. *If $\mathscr{H}$ and $\mathscr{K}$ are LG-LG bimodules such that $\mathscr{H}\preceq\mathscr{K}$, then $U_{\mathscr{H}}\preceq U_{\mathscr{K}}$.*

PROOF. This is obvious from the definition.

The *conjugation representation* $V_{\mathrm{c}}:=U_{L^2(G)}$ is the one that is associated with the identity bimodule of *LG* and is given by $V_{\mathrm{c}}=(1\otimes K)W(1\otimes K)^*W$, where $K=\hat{J}J$. Indeed, one has $(\mathrm{id}\otimes\pi^{(1)}_{L^2(G)})(W_{\mathrm{u}})=W$ and $(\mathrm{id}\otimes\pi^{(2)}_{L^2(G)})(W_{\mathrm{u}})=(1\otimes K)W(1\otimes K)^*$, since

$$\begin{aligned}(\omega\otimes\pi^{(2)}_{L^2(G)})(W_{\mathrm{u}})&=\hat{J}\hat{R}(\lambda(\omega))^*\hat{J}=K\lambda(\omega)K^*\\ &=(\omega\otimes\mathrm{id})((1\otimes K)W(1\otimes K)^*)\end{aligned}$$

for every $\omega \in L^1(G)$. We say a locally compact quantum group G is *strongly inner amenable* (in the locally compact setting, see [19]) if the trivial representation 1 is weakly contained in V_c. This property is formally stronger than the *inner amenability* as introduced in [13]. All inner amenable locally compact groups, strongly amenable locally compact quantum groups, and unimodular discrete quantum groups are inner amenable. Since it is irrelevant to the present work, we omit the rather routine proofs of "strong amenability $\Rightarrow$ strong inner amenability $\Rightarrow$ inner amenability".

Proposition 10 implies the well-known fact [4] that if G is strongly amenable (i.e., $1 \preceq W$), then LG is amenable (i.e., ${}_{LG}L^2(G)_{LG} \preceq {}_{LG}(L^2(G) \otimes L^2(G))_{LG}$). Conversely, if LG is amenable, then $V_c \preceq W$ by Proposition 12. Hence if G is moreover strongly inner amenable, then G is strongly amenable (cf. [19]). It would be interesting to know whether every discrete quantum group is strongly inner amenable (cf. [26]), and whether the property $V_c \preceq W$ is equivalent to amenability of LG.

PROPOSITION 13. *The following hold.*

(1) *Let U be a unitary representation of G on $\mathscr{H}$ and $\mathscr{H} = L^2(G) \otimes \mathscr{H}_U$ be the associated LG-LG bimodule. Then, the unitary representation $U_{\mathscr{H}}$ associated with $\mathscr{H}$ is equal to $V_c \textcircled{\scriptsize T} U$. In particular, if G is strongly inner amenable, then $U \preceq U_{\mathscr{H}}$.*

(2) *Let $\mathscr{H}$ be an LG-LG bimodule and $U_{\mathscr{H}}$ be the associated unitary representation of G on $\mathscr{H}$. Then, the LG-LG bimodule $L^2(G) \otimes \mathscr{H}$ associated with $U_{\mathscr{H}}$ is unitarily equivalent to ${}_{\hat{\Delta}(LG)}(\mathscr{H} \otimes L^2(G))_{\hat{\Delta}(LG)}$.*

PROOF. Ad (1): a routine computation shows

$$\begin{aligned} U_{\mathscr{H}} &= (\mathrm{id} \otimes \pi_{\mathscr{H}}^{(2)})(W_u)(\mathrm{id} \otimes \pi_{\mathscr{H}}^{(1)})(W_u) \\ &= ((1 \otimes K)W(1 \otimes K)^*)_{12} \cdot (W \textcircled{\scriptsize T} U) \\ &= ((1 \otimes K)W(1 \otimes K)^* W)_{12} U_{13} = V_c \textcircled{\scriptsize T} U. \end{aligned}$$

This proves the first assertion. If $1 \preceq V_c$, then $U = 1 \textcircled{\scriptsize T} U \preceq V_c \textcircled{\scriptsize T} U$.

Ad (2): to ease the notation, write $U_i = (\mathrm{id} \otimes \pi_{\mathscr{H}}^{(i)})(W_u)$ and $\pi_{\mathscr{H}}(a \otimes x^{\mathrm{op}}) = \pi_1(a)\pi_2(x^{\mathrm{op}})$, and denote by Σ the flip either on $L^2(G) \otimes L^2(G)$ or on $\mathscr{H} \otimes L^2(G)$. Note that $U_{\mathscr{H}} = U_2 U_1$ and that $\hat{\Delta}(a) = \Sigma W(a \otimes 1)W^*\Sigma$ for $a \in LG$ [17, Theorem 5.17]. Thus for the unitary operator $Y := U_2\Sigma$ from $\mathscr{H} \otimes L^2(G)$ onto $L^2(G) \otimes \mathscr{H}$, one has

$$\begin{aligned} Y^*\pi_{L^2(G)\otimes\mathscr{H}}(a \otimes 1)Y &= \Sigma^* U_2^* U_{\mathscr{H}}(a \otimes 1)U_{\mathscr{H}}^* U_2 \Sigma \\ &= \Sigma^* U_1(a \otimes 1)U_1^* \Sigma = (\pi_1 \otimes \mathrm{id})(\hat{\Delta}(a)). \end{aligned}$$

We abuse the notation and view $\pi^{(2)}_{\mathscr{H}}$ as a $*$-homomorphism from LG into $\mathbb{B}(\mathscr{H})$, which is given by $\pi^{(2)}_{\mathscr{H}}(x) = \pi_2(\hat{R}(x)^{\mathrm{op}})$. The right action of LG on $L^2(G)$ is denoted by $\rho(y^{\mathrm{op}}) = \hat{J}y^*\hat{J}$. Note that

$$\begin{aligned}(\pi_2 \otimes \rho)(x^{\mathrm{op}} \otimes y^{\mathrm{op}}) &= \pi^{(2)}_{\mathscr{H}}(Jx^*J) \otimes \hat{J}y^*\hat{J} \\ &= (\pi^{(2)}_{\mathscr{H}} \otimes \mathrm{id})((J \otimes \hat{J})(x \otimes y)^*(J \otimes \hat{J})).\end{aligned}$$

Since $(\hat{J} \otimes J)W(\hat{J} \otimes J) = W^*$ [17, Section 5.3], it follows that

$$\begin{aligned}Y^*\pi_{L^2(G)\otimes\mathscr{H}}(1 \otimes x^{\mathrm{op}})Y &= \Sigma^* U_2^*(\hat{J}x^*\hat{J} \otimes 1)U_2\Sigma \\ &= (\pi^{(2)}_{\mathscr{H}} \otimes \mathrm{id})(\Sigma W^*(\hat{J}x^*\hat{J} \otimes JJ)W\Sigma) \\ &= (\pi^{(2)}_{\mathscr{H}} \otimes \mathrm{id})((J \otimes \hat{J})\hat{\Delta}(x^*)(J \otimes \hat{J})) \\ &= (\pi_2 \otimes \rho)(\hat{\Delta}(x)^{\mathrm{op}}).\end{aligned}$$

This proves the assertion.

Recall from [11] that a unitary representation U is said to be *mixing* if the coefficient $f_\omega = (\mathrm{id} \otimes \omega)(U)$ belongs to $C_0(G)$ for every $\omega \in \mathbb{B}(\mathscr{H}_U)_*$.

Proposition 14. *If U is a mixing unitary representation of a locally compact quantum group G, then the LG-LG bimodule $L^2(G) \otimes \mathscr{H}_U$ is strictly mixing. Conversely, if $\mathscr{H}$ is a strictly mixing LG-LG bimodule, then the unitary representation $U_{\mathscr{H}}$ is mixing.*

Proof. Let $\omega \in \mathscr{I}$ (see Proof of Lemma 11) be an element which is analytic with respect to $t \mapsto \rho_t(\omega) := \omega(\delta^{-it}\tau_{-t}(\,\cdot\,))$ (see [17, 5.22] or [18, 8.7] for the notation). Then, the vector $\xi := \lambda(\omega)\hat{\varphi}^{1/2}$ is bounded with $L_{\hat{\varphi}}(\xi) = \lambda(\omega) \in \mathrm{C}^*_\lambda(G)$ and $R_{\hat{\varphi}}(\xi) = \hat{J}\lambda(\rho_{i/2}(\omega))^*\hat{J} \in \hat{J}\mathrm{C}^*_\lambda(G)\hat{J}$. Hence, by Lemma 11, the vector $\zeta := \xi \otimes \eta$ is bounded for every $\eta \in \mathscr{H}_U$ and satisfies

$$T_{\hat{\varphi}}(\zeta) = L_{\hat{\varphi}}(\xi)^* R_{\hat{\varphi}}(\xi) f_\eta \in \mathrm{C}^*_\lambda(G) \cdot \hat{J}\mathrm{C}^*_\lambda(G)\hat{J} \cdot C_0(G) \subset \mathbb{K}(L^2(G)),$$

where the last inclusion is by [2, Lemma 5.5]. Since such ζ's have a dense span, this proves that $L^2(G) \otimes \mathscr{H}_U$ is strictly mixing.

For the converse, it suffices to show $(\mathrm{id} \otimes \omega_{a\eta,\eta x^*})(U_{\mathscr{H}}) \in C_0(G)$ for every $\eta \in \mathscr{H}_{\mathrm{mix}}$ and $a, x \in n_\varphi$. Let us fix $\xi \in L^2(G)$ and we will compute

$$\begin{aligned}&\langle(\mathrm{id} \otimes \omega_{a\eta,\eta x^*})(U_{\mathscr{H}})\xi, \xi\rangle \\ &\qquad = \langle(\mathrm{id} \otimes \pi^{(1)}_{\mathscr{H}})(W_{\mathrm{u}})(\xi \otimes a\eta), (\mathrm{id} \otimes \pi^{(2)}_{\mathscr{H}})(W_{\mathrm{u}}^*)(\xi \otimes \eta x^*)\rangle.\end{aligned}$$

Let $\zeta \in L^2(G)$ be given and consider $L_\zeta : \mathscr{H} \ni \eta' \mapsto \zeta \otimes \eta' \in L^2(G) \otimes \mathscr{H}$. Then,

$$\begin{aligned} L_\zeta^*(\mathrm{id} \otimes \pi_{\mathscr{H}}^{(2)})(W_{\mathrm{u}}^*)(\xi \otimes \eta x^*) &= \pi_{\mathscr{H}}^{(2)}((\omega_{\xi,\zeta} \otimes \mathrm{id})(W_{\mathrm{u}}^*))\eta x^* \\ &= \eta x^* \hat{R}((\omega_{\xi,\zeta} \otimes \mathrm{id})(W^*)) \\ &= L_{\hat{\varphi}}(\eta) \hat{J} \hat{R}((\omega_{\xi,\zeta} \otimes \mathrm{id})(W^*))^* \hat{J} \hat{\varphi}^{1/2} x^* \\ &= L_{\hat{\varphi}}(\eta) K (\omega_{\xi,\zeta} \otimes \mathrm{id})(W^*) K^* \hat{\varphi}^{1/2} x^* \\ &= L_\zeta^* (1 \otimes L_{\hat{\varphi}}(\eta) K) W^* (1 \otimes K)^* (\xi \otimes \hat{\varphi}^{1/2} x^*). \end{aligned}$$

Since $\zeta \in L^2(G)$ was arbitrary, it follows that

$$(\mathrm{id} \otimes \pi_{\mathscr{H}}^{(2)})(W_{\mathrm{u}}^*)(\xi \otimes \eta x^*) = (1 \otimes L_{\hat{\varphi}}(\eta) K) W^* (1 \otimes K)^* (\xi \otimes \hat{\varphi}^{1/2} x^*).$$

A similar but much easier computation shows

$$(\mathrm{id} \otimes \pi_{\mathscr{H}}^{(1)})(W_{\mathrm{u}})(\xi \otimes a\eta) = (1 \otimes R_{\hat{\varphi}}(\eta)) W (\xi \otimes a\hat{\varphi}^{1/2}).$$

Therefore, by Lemma 2,

$$\begin{aligned} &\langle (\mathrm{id} \otimes \omega_{a\eta,\eta x^*})(U_{\mathscr{H}})\xi, \xi \rangle \\ &\qquad = \langle (1 \otimes K) W (1 \otimes K^* T_{\hat{\varphi}}(\eta)) W (\xi \otimes a\hat{\varphi}^{1/2}), (\xi \otimes \hat{\varphi}^{1/2} x^*) \rangle. \end{aligned}$$

Since $\xi \in L^2(G)$ was arbitrary, this implies

$$(\mathrm{id} \otimes \omega_{a\eta,\eta x^*})(U_{\mathscr{H}}) = (\mathrm{id} \otimes \omega_{\hat{\varphi}^{1/2} x^*, a\hat{\varphi}^{1/2}})((1 \otimes K) W (1 \otimes K^* T_{\hat{\varphi}}(\eta)) W).$$

Since $T_{\hat{\varphi}}(\eta)$ is compact, we conclude that $(\mathrm{id} \otimes \omega_{a\eta,\eta x^*})(U_{\mathscr{H}}) \in \mathrm{C}^*\{(\mathrm{id} \otimes \omega)(W) : \omega\} = C_0(G)$. This finishes the proof of the converse.

Recall from [11] that a locally compact quantum group is said to have the *Haagerup property* if the trivial unitary representation 1 is weakly contained in a mixing unitary representation. The following theorem extends the same result for discrete quantum groups in [11, Theorems 7.4 and 7.7].

THEOREM 15. *Let G be a locally compact quantum group. If G has the Haagerup property, then LG has the HAP. Conversely, if G is strongly inner amenable and LG has the HAP, then G has the Haagerup property.*

PROOF. First suppose that G has the Haagerup property, i.e., there is a mixing unitary representation U such that $1 \preceq U$. Then, ${}_{LG}L^2(G)_{LG} \preceq {}_{LG}(L^2(G) \otimes \mathscr{H}_U)_{LG}$ by Proposition 10, but the latter bimodule is strictly mixing by Proposition 14. Now, Theorem 9 applies and we conclude that LG has the HAP. Conversely, suppose that LG has the HAP. Then by Theorem 9 there is a strictly mixing bimodule $\mathscr{H}$ such that ${}_{LG}L^2(G)_{LG} \preceq {}_{LG}\mathscr{H}_{LG}$. By Proposition 14, the associated unitary representation $U_{\mathscr{H}}$ is mixing and, by Proposition 12, it satisfies $V_{\mathrm{c}} \preceq U_{\mathscr{H}}$. Hence, if G is moreover strongly inner amenable, then the mixing representation $U_{\mathscr{H}}$ weakly contains the trivial representation.

While it may be true that the HAP (resp. amenability) of LG implies the Haagerup property (resp. amenability) of G for discrete quantum groups, this need not be true for general locally compact quantum groups. For example, a simple connected higher rank Lie group, such as $\mathrm{SL}(3, \mathbb{R})$, has a type I and hence amenable group von Neumann algebra, but it does not have the Haagerup property because of Kazhdan's property (T) (see [8]).

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