MATH. SCAND. 121 (2017), 92–100

BIDUALITY AND DENSITY IN LIPSCHITZ FUNCTION SPACES

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Abstract

For pointed compact metric spaces (X, d), we address the biduality problem as to when the space of Lipschitz functions $\mathrm{Lip}_0(X, d)$ is isometrically isomorphic to the bidual of the space of little Lipschitz functions $\mathrm{lip}_0(X, d)$, and show that this is the case whenever the closed unit ball of $\mathrm{lip}_0(X, d)$ is dense in the closed unit ball of $\mathrm{Lip}_0(X, d)$ with respect to the topology of pointwise convergence. Then we apply our density criterion to prove in an alternative way the real version of a classical result which asserts that $\mathrm{Lip}_0(X, d^\alpha)$ is isometrically isomorphic to $\mathrm{lip}_0(X, d^\alpha)^{**}$ for any $\alpha \in (0, 1)$.

1. Introduction

Let (X, d) be a pointed compact metric space with the base point denoted by 0 and let $\mathbb{K}$ be the field of real or complex numbers. The Lipschitz space $\mathrm{Lip}_0(X, d)$ is the Banach space of all Lipschitz functions $f\colon X \to \mathbb{K}$ for which $f(0) = 0$, endowed with the Lipschitz norm

$$\mathrm{Lip}_d(f) = \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X,\ x \neq y \right\}.$$

A Lipschitz function $f\colon X \to \mathbb{K}$ satisfying the local flatness condition:

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{|f(x) - f(y)|}{d(x, y)} = 0,$$

is called a little Lipschitz function, and the little Lipschitz space $\mathrm{lip}_0(X, d)$ is the closed subspace of $\mathrm{Lip}_0(X, d)$ formed by all little Lipschitz functions. Furthermore, $\mathrm{Lip}_0^{\mathbb{R}}(X, d)$ and $\mathrm{lip}_0^{\mathbb{R}}(X, d)$ are the real subspaces of all real-valued functions in $\mathrm{Lip}_0(X, d)$ and $\mathrm{lip}_0(X, d)$, respectively. These spaces have

* The first author's research was partially supported by the Spanish Ministry of Economy and Competitiveness project no. MTM2014-58984-P and the European Regional Development Fund (ERDF), and Junta of Andalucía grant FQM-194. The second author's research was partially supported by Generalitat Valenciana under project GV/2015/035.

Received 10 March 2015.

DOI: https://doi.org/10.7146/math.scand.a-25987

been widely investigated for a long time. See the Weaver's book [11] for references and a complete study.

The biduality problem as to when $\mathrm{Lip}_0(X, d)$ is isometrically isomorphic to $\mathrm{lip}_0(X, d)^{**}$ has an interesting history (see [11, p. 99, Notes 3.3] and also [8, 6. Duality]). In this note, we address this question in a similar way as Bierstedt and Summers [2] do for studying the biduals of weighted Banach spaces of analytic functions, and we prove that $\mathrm{Lip}_0(X, d)$ is isometrically isomorphic to $\mathrm{lip}_0(X, d)^{**}$ if and only if the closed unit ball of $\mathrm{lip}_0(X, d)$ is dense in the closed unit ball of $\mathrm{Lip}_0(X, d)$ with respect to the topology of pointwise convergence τ_p. This density condition is equivalent to requiring that for each $f \in \mathrm{Lip}_0(X, d)$ with $\mathrm{Lip}_d(f) \le 1$, there exists a sequence $\{f_n\}$ in $\mathrm{lip}_0(X, d)$ with $\mathrm{Lip}_d(f_n) \le 1$ for all $n \in \mathbb{N}$ such that $\{f_n(x)\}$ converges to $f(x)$ as $n \to \infty$ for every $x \in X$. Then we apply our density criterion to prove in an alternative way the real version of a classical result of Johnson [7] (see also [1], [10] and [11]) which asserts that $\mathrm{Lip}_0(X, d^\alpha)$ is isometrically isomorphic to $\mathrm{lip}_0(X, d^\alpha)^{**}$ for any $\alpha \in (0, 1)$.

2. The results

Johnson [7] proved that the closed linear subspace of $\mathrm{Lip}_0(X, d)^*$ spanned by the evaluation functionals $\delta_x : \mathrm{Lip}_0(X, d) \to \mathbb{K}$, given by $\delta_x(f) = f(x)$ with $x \in X$, is a predual of $\mathrm{Lip}_0(X, d)$. The terminology Lipschitz-free Banach space of X and the notation $\mathcal{F}(X)$ for this predual of $\mathrm{Lip}_0(X, d)$ are due to Godefroy and Kalton [5]. Namely, the evaluation map $Q_X : \mathrm{Lip}_0(X, d) \to \mathcal{F}(X)^*$ defined by

$$Q_X(f)(\gamma) = \gamma(f) \qquad (f \in \mathrm{Lip}_0(X, d),\ \gamma \in \mathcal{F}(X))$$

is the natural isometric isomorphism. As usual, B_E will denote the closed unit ball of a Banach space E.

THEOREM 2.1. *Let (X, d) be a pointed compact metric space.*

(i) *The restriction map $R_X : \mathcal{F}(X) \to \mathrm{lip}_0(X, d)^*$ defined by*

$$R_X(\gamma)(f) = \gamma(f) \qquad (f \in \mathrm{lip}_0(X, d),\ \gamma \in \mathcal{F}(X)),$$

is a non-expansive linear surjective map.

(ii) *R_X is an isometric isomorphism from $\mathcal{F}(X)$ onto $\mathrm{lip}_0(X, d)^*$ if and only if $B_{\mathrm{lip}_0(X,d)}$ is dense in $B_{\mathrm{Lip}_0(X,d)}$ with respect to the topology of pointwise convergence.*

PROOF. (i) Since $\mathcal{F}(X) \subset \mathrm{Lip}_0(X, d)^*$, it is clear that R_X is a linear map from $\mathcal{F}(X)$ into $\mathrm{lip}_0(X, d)^*$ satisfying $\|R_X(\gamma)\| \le \|\gamma\|$ for all $\gamma \in \mathcal{F}(X)$.

We next prove that R_X is surjective. To this end, let us recall that de Leeuw's map $\Phi\colon \mathrm{Lip}_0(X,d) \to C_b(\widetilde{X})$ given by

$$\Phi(f)(x,y) = \frac{f(x)-f(y)}{d(x,y)} \qquad (f \in \mathrm{Lip}_0(X,d),\ (x,y) \in \widetilde{X}),$$

where $\widetilde{X} = \{(x,y) \in X^2 : x \neq y\}$, is a linear isometry of $\mathrm{Lip}_0(X,d)$ into $C_b(\widetilde{X})$, the Banach space of bounded continuous scalar-valued functions on $\widetilde{X}$ with the supremum norm, and the image of $\mathrm{lip}_0(X,d)$ is contained in $C_0(\widetilde{X})$, the closed subspace of functions which vanish at infinity. See, for example, [11, Theorem 2.1.3 and Proposition 3.1.2].

Take $\gamma \in \mathrm{lip}_0(X,d)^*$. The functional $T\colon \Phi(\mathrm{lip}_0(X,d)) \to \mathbb{K}$, defined by $T(\Phi(f)) = \gamma(f)$ for all $f \in \mathrm{lip}_0(X,d)$, is linear, continuous and $\|T\| = \|\gamma\|$. By the Hahn-Banach theorem, there exists a continuous linear functional $\widetilde{T}\colon C_0(\widetilde{X}) \to \mathbb{K}$ such that $\widetilde{T}(\Phi(f)) = T(\Phi(f))$, for all $f \in \mathrm{lip}_0(X,d)$, and $\|\widetilde{T}\| = \|T\|$. Now, by the Riesz representation theorem, there exists a finite and regular Borel measure μ on $\widetilde{X}$ with total variation $\|\mu\| = \|\widetilde{T}\|$ such that

$$\widetilde{T}(g) = \int_{\widetilde{X}} g\, d\mu \qquad (g \in C_0(\widetilde{X})),$$

and thus

$$\gamma(f) = \int_{\widetilde{X}} \Phi(f)\, d\mu \qquad (f \in \mathrm{lip}_0(X,d)).$$

If we now define

$$\widetilde{\gamma}(f) = \int_{\widetilde{X}} \Phi(f)\, d\mu \qquad (f \in \mathrm{Lip}_0(X,d)),$$

it is clear that $\widetilde{\gamma} \in \mathrm{Lip}_0(X,d)^*$ and $\widetilde{\gamma}(f) = \gamma(f)$ for all $f \in \mathrm{lip}_0(X,d)$. Finally, we show that $\widetilde{\gamma}$ is τ_p-continuous on $B_{\mathrm{Lip}_0(X,d)}$ (see [6]). Thus, let $\{f_i\}$ be a net in $B_{\mathrm{Lip}_0(X,d)}$ which converges pointwise on X to zero. Then $\{\Phi(f_i)\}$ converges pointwise on $\widetilde{X}$ to zero and, since $|\Phi(f_i)(x,y)| \leq \|\Phi(f_i)\|_\infty = \mathrm{Lip}_d(f_i) \leq 1$, for all $i \in I$ and for all $(x,y) \in \widetilde{X}$, it follows that $\{\widetilde{\gamma}(f_i)\}$ converges to 0 by the Lebesgue bounded convergence theorem. This completes the proof of (i).

(ii) Assume that $B_{\mathrm{lip}_0(X,d)}$ is τ_p-dense in $B_{\mathrm{Lip}_0(X,d)}$. Fix $\gamma \in \mathscr{F}(X)$ and let $f \in B_{\mathrm{Lip}_0(X,d)}$. Then there exists a net $\{f_i\}$ in $B_{\mathrm{lip}_0(X,d)}$ which converges to f in the topology of pointwise convergence. Since γ is τ_p-continuous on $B_{\mathrm{Lip}_0(X,d)}$ and satisfies

$$|\gamma(f_i)| = |R_X(\gamma)(f_i)| \leq \|R_X(\gamma)\|\, \mathrm{Lip}_d(f_i) \leq \|R_X(\gamma)\|,$$

for all $i \in I$, it follows that $|\gamma(f)| \leq \|R_X(\gamma)\|$ and so $\|\gamma\| \leq \|R_X(\gamma)\|$. Now, taking (i) into account we conclude that R_X is an isometric isomorphism from $\mathscr{F}(X)$ onto $\mathrm{lip}_0(X, d)^*$.

Conversely, if $B_{\mathrm{lip}_0(X,d)}$ is not τ_p-dense in $B_{\mathrm{Lip}_0(X,d)}$, by the Hahn-Banach theorem there exist a function $g \in B_{\mathrm{Lip}_0(X,d)}$ and a τ_p-continuous linear functional γ on $\mathrm{Lip}_0(X, d)$ such that $|\gamma(f)| \leq 1$, for all $f \in B_{\mathrm{lip}_0(X,d)}$, and $|\gamma(g)| > 1$. Since $\gamma \in \mathscr{F}(X)$ (see [6]) and $\|R_X(\gamma)\| = \|\gamma|_{\mathrm{lip}_0(X,d)}\| \leq 1 < |\gamma(g)| \leq \|\gamma\|$, then R_X is not an isometry.

We are now ready to obtain the main result of this note.

THEOREM 2.2. *Let (X, d) be a pointed compact metric space. Then the following are equivalent:*

(i) $\mathrm{Lip}_0(X, d)$ *is isometrically isomorphic to* $\mathrm{lip}_0(X, d)^{**}$*;*

(ii) $B_{\mathrm{lip}_0(X,d)}$ *is dense in* $B_{\mathrm{Lip}_0(X,d)}$ *with respect to the weak* topology;*

(iii) $B_{\mathrm{lip}_0(X,d)}$ *is dense in* $B_{\mathrm{Lip}_0(X,d)}$ *with respect to the topology of pointwise convergence;*

(iv) *for each* $f \in B_{\mathrm{Lip}_0(X,d)}$*, there exists a sequence* $\{f_n\}$ *in* $B_{\mathrm{lip}_0(X,d)}$ *such that* $\{f_n(x)\}$ *converges to* $f(x)$ *as* $n \to \infty$ *for every* $x \in X$*.*

PROOF. If (i) holds, then (ii) follows by the Goldstine theorem; but (ii) is the same as (iii) since the weak* topology agrees with the topology of pointwise convergence on bounded subsets of $\mathrm{Lip}_0(X, d)$ by [7, Corollary 4.4]. If (iii) is true, then R_X^* is an isometric isomorphism from $\mathrm{lip}_0(X, d)^{**}$ onto $\mathscr{F}(X)^*$ by Theorem 2.1, hence the composition $Q_X^{-1} \circ R_X^*$ is an isometric isomorphism from $\mathrm{lip}_0(X, d)^{**}$ onto $\mathrm{Lip}_0(X, d)$ and so we obtain (i).

In order to prove that (ii) is equivalent to (iv), notice that, by [7, Corollary 4.4], the family of sets

$$U(f_0; n, x_1, \ldots, x_n, \varepsilon) := \left\{f \in B_{\mathrm{Lip}_0(X,d)} : |f(x_i) - f_0(x_i)| < \varepsilon, \forall i = 1, \ldots, n\right\}$$

with $f_0 \in B_{\mathrm{Lip}_0(X,d)}$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$, is a basis of the relative weak* topology on $B_{\mathrm{Lip}_0(X,d)}$.

Suppose now that (ii) holds and let $f_0 \in B_{\mathrm{Lip}_0(X,d)}$. Given $x \in X$ and $n \in \mathbb{N}$, the set $U(f_0; 1, x, 1/n)$ is a weak* neighborhood of f_0 relative to $B_{\mathrm{Lip}_0(X,d)}$. Then, by (ii), for each $n \in \mathbb{N}$ there exists $f_n \in B_{\mathrm{lip}_0(X,d)}$ such that $f_n \in U(f_0; 1, x, 1/n)$, that is, $|f_n(x) - f_0(x)| < 1/n$. Hence $\{f_n(x)\}$ converges to $f_0(x)$ as $n \to \infty$ and we conclude that (ii) implies (iv). Conversely, assume that (iv) is valid and let $f_0 \in B_{\mathrm{Lip}_0(X,d)}$. Take $U(f_0; p, x_1, \ldots, x_p, \varepsilon)$ with $p \in \mathbb{N}$, $x_1, \ldots, x_p \in X$ and $\varepsilon > 0$. By (iv), there is a sequence $\{f_n\}$ in $B_{\mathrm{lip}_0(X,d)}$ such that $\{f_n(x)\}$ converges to $f_0(x)$ as $n \to \infty$ for every $x \in X$. In particular, for each $i \in \{1, \dots, p\}$, there is a $m_i \in \mathbb{N}$ for which it is verified $|f_n(x_i) - f_0(x_i)| < \varepsilon$ whenever $n \geq m_i$. Now, if $m = \max\{m_1, \dots, m_p\}$, then $f_m \in U(f_0; p, x_1, \dots, x_p, \varepsilon)$ and (ii) follows.

It is known that $\mathrm{Lip}_0(X, d)$ is isometrically isomorphic to $\mathrm{lip}_0(X, d)^{**}$ for a large class of metric spaces (X, d) as, for example, the Hölder spaces (X, d^α), $0 < \alpha < 1$ (see [1], [7] and [10]).

REMARK 2.3. The proof of Theorem 2.2 shows that if one of its statements holds, then the map $Q_X^{-1} \circ R_X^*$ is an isometric isomorphism from $\mathrm{lip}_0(X, d)^{**}$ onto $\mathrm{Lip}_0(X, d)$. For any $\phi \in \mathrm{lip}_0(X, d)^{**}$ and $x \in X$, an easy verification yields

$$\begin{aligned}
(Q_X^{-1} \circ R_X^*)(\phi)(x) &= \delta_x((Q_X^{-1} \circ R_X^*)(\phi)) \\
&= Q_X((Q_X^{-1} \circ R_X^*)(\phi))(\delta_x) \\
&= Q_X(Q_X^{-1}(R_X^*(\phi)))(\delta_x) \\
&= R_X^*(\phi)(\delta_x) \\
&= \phi(R_X(\delta_x)) \\
&= \phi(\delta_x|_{\mathrm{lip}_0(X,d)})
\end{aligned}$$

This identification is the same as that obtained by de Leeuw [10], Johnson [7] and Bade, Curtis and Dales [1] between the spaces $\mathrm{Lip}_0(X, d^\alpha)$ and $\mathrm{lip}_0(X, d^\alpha)^{**}$ $(0 < \alpha < 1)$.

The pointwise approximation condition given by the assertion (iv) of Theorem 2.2 can be verified to recover two classical results about the biduality problem of $\mathrm{Lip}_0(X, d^\alpha)$ $(0 < \alpha < 1)$. The former is due to Ciesielski [4] and the latter to de Leeuw [10].

EXAMPLE 2.4. Let $\alpha \in (0,1)$ and let $[0,1]$ be the unit interval with the usual metric d. Then $\mathrm{Lip}_0([0,1], d^\alpha)$ is isometrically isomorphic to $\mathrm{lip}_0([0,1], d^\alpha)^{**}$.

PROOF. Fix $f \in B_{\mathrm{Lip}_0([0,1],d^\alpha)}$ and, for each $n \in \mathbb{N}$, let $B_n(f, \cdot)$ denote the nth Bernstein polynomial for f defined by

$$B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \qquad (x \in [0, 1]).$$

Then $B_n(f, \cdot)$ also belongs to $B_{\mathrm{Lip}_0([0,1],d^\alpha)}$ (see [3] for an elementary proof) while the fact that

$$|B_n(f,x) - B_n(f,y)| \le \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) \right| \binom{n}{k} \left| x^k(1-x)^{n-k} - y^k(1-y)^{n-k} \right|$$
$$\le |x-y| \sum_{k=0}^{n} \left| f\left(\frac{k}{n}\right) \right| \binom{n}{k} 2n,$$

for all $x, y \in [0,1]$, shows that $B_n(f,\cdot) \in B_{\mathrm{lip}_0([0,1],d^\alpha)}$. Since $\{B_n(f,\cdot)\}_{n\in\mathbb{N}}$ converges to f uniformly on $[0,1]$, the example is proved by Theorem 2.2.

EXAMPLE 2.5. Let $0 < \alpha < 1$ and let $\mathbb{T}$ be the quotient additive group $\mathbb{R}/2\pi\mathbb{Z}$ with the distance

$$d(t+2\pi\mathbb{Z}, s+2\pi\mathbb{Z})$$
$$= \min\{|t-s|, |t-s-2\pi|, |t-s+2\pi|\} \qquad (t, s \in [0, 2\pi)).$$

Then $\mathrm{Lip}_0(\mathbb{T}, d^\alpha)$ is isometrically isomorphic to $\mathrm{lip}_0(\mathbb{T}, d^\alpha)^{**}$.

PROOF. We apply similar arguments to those of [10, Lemma 2.8] and use some results from harmonic analysis (see [9]). We identify each equivalence class $t + 2\pi\mathbb{Z}$ with the point $t \in [0, 2\pi)$. Let $f \in B_{\mathrm{Lip}_0(\mathbb{T},d^\alpha)}$. For each $n \in \mathbb{N}$, let K_n be the Fejér kernel defined by

$$K_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt} = \frac{1}{n+1} \left(\frac{\sin \frac{n+1}{2}}{\sin \frac{t}{2}} \right)^2 \qquad (t \in [0, 2\pi)).$$

Then the convolution

$$(K_n * f)(t) = \frac{1}{2\pi} \int_0^{2\pi} K_n(\tau) f(t-\tau)\, d\tau \qquad (t \in [0, 2\pi))$$

coincides with the Fejér mean

$$\sigma_n(f,t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \widehat{f}(j) e^{ijt} \qquad (t \in [0, 2\pi)),$$

where $\widehat{f}(j)$ is the jth Fourier coefficient of f. Given $t, s \in [0, 2\pi)$, we have

$$|\sigma_n(f,t) - \sigma_n(f,s)| \le \sum_{j=-n}^{n} \left| 1 - \frac{|j|}{n+1} \right| |\widehat{f}(j)| |e^{ijt} - e^{ijs}|$$
$$\le \sum_{j=-n}^{n} \left| 1 - \frac{|j|}{n+1} \right| \frac{\pi^\alpha |j|^{1-\alpha}}{2} (4\pi n)^n (e-1)\, d(t,s)$$

and therefore $\sigma_n(f,\cdot) \in \mathrm{lip}_0(\mathbb{T}, d^\alpha)$. Moreover,

$$\begin{aligned}|\sigma_n(f,t)-\sigma_n(f,s)| &= |(K_n * f)(t) - (K_n * f)(s)| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |K_n(\tau)||f(t-\tau)-f(s-\tau)|\,d\tau \\ &\le \mathrm{Lip}_{d^\alpha}(f)\,d(t,s)^\alpha \frac{1}{2\pi}\int_0^{2\pi} K_n(\tau)\,d\tau \\ &= \mathrm{Lip}_{d^\alpha}(f)\,d(t,s)^\alpha,\end{aligned}$$

and so $\mathrm{Lip}_{d^\alpha}(\sigma_n(f,\cdot)) \le \mathrm{Lip}_{d^\alpha}(f) \le 1$. Now take $\beta_n(f,\cdot) = \sigma_n(f,\cdot) - \sigma_n(f,0)$ which is in $B_{\mathrm{lip}_0(\mathbb{T},d^\alpha)}$. By Fejér's theorem, $\{\sigma_n(f,\cdot)\}_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{T}$ to f, and so does $\{\beta_n(f,\cdot)\}_{n\in\mathbb{N}}$. Then the desired conclusion follows from Theorem 2.2.

Our density criterion serves to give another proof of the real version of an important result by Johnson [7, Theorem 4.7] and Bade, Curtis and Dales [1, Theorem 3.5].

COROLLARY 2.6. *Let* (X,d) *be a pointed compact metric space and let* $\alpha \in (0,1)$. *Then* $\mathrm{Lip}_0^{\mathbb{R}}(X,d^\alpha)$ *is isometrically isomorphic to* $\mathrm{lip}_0^{\mathbb{R}}(X,d^\alpha)^{**}$.

PROOF. Let $f \in B_{\mathrm{Lip}_0^{\mathbb{R}}(X,d^\alpha)}$. We claim that for each $n \in \mathbb{N}$ and each finite set $F \subset X$, there exists a function $h \in \mathrm{lip}^{\mathbb{R}}(X,d^\alpha)$ such that $\mathrm{Lip}_{d^\alpha}(h) \le 1 + 1/n$ and $h(x) = f(x)$ for all $x \in F$. The notation $\mathrm{lip}^{\mathbb{R}}(X,d^\alpha)$ and later $\mathrm{Lip}^{\mathbb{R}}(X,d^\gamma)$ might be self-explanatory.

Consider $F = \{x_1,\dots,x_m\}$, for some $m \in \mathbb{N}$. There is no loss of generality in assuming that $f(x_m) \le f(x_{m-1}) \le \cdots \le f(x_1)$. If $m = 1$, we set $h(x) = f(x_1)$, for all $x \in X$. Now let $m \ge 2$ and we also may assume $f \ge 0$, for otherwise we can replace f by $f + \|f\|_\infty$. Let

$$\gamma = \min\left(\left\{\alpha + \frac{e\ln\left(1+\frac{1}{n}\right)}{\mathrm{diam}(X)} d(x_j,x_k) : j,k \in \{1,\dots,m\},\ j \ne k\right\} \cup \{1\}\right),$$

$$\rho = \max\left\{\frac{|f(x_k)-f(x_j)|}{d(x_k,x_j)^\gamma} : j,k \in \{1,\dots,m\},\ j \ne k\right\}.$$

For each $j \in \{1,\dots,m\}$, define $g_j : X \to \mathbb{R}$ by

$$g_j(x) = \max\{f(x_j) - \rho d(x_j,x)^\gamma, 0\}.$$

Notice that $0 < \alpha < \gamma \le 1$ and therefore $g_j \in \mathrm{Lip}^{\mathbb{R}}(X,d^\gamma) \subset \mathrm{lip}^{\mathbb{R}}(X,d^\alpha)$ with

$$\mathrm{Lip}_{d^\alpha}(g_j) \le \mathrm{Lip}_{d^\gamma}(g_j)\,\mathrm{diam}(X)^{\gamma-\alpha} \le \rho\,\mathrm{diam}(X)^{\gamma-\alpha}.$$

We now check that the function $h = \max\{g_1, \dots, g_m\}$ satisfies the required conditions. It is known that h is in $\mathrm{lip}^{\mathbb{R}}(X, d^\alpha)$ and it is verified $\mathrm{Lip}_{d^\alpha}(h) \le \max\{\mathrm{Lip}_{d^\alpha}(g_1), \dots, \mathrm{Lip}_{d^\alpha}(g_m)\}$. Now, given $j \in \{1, \dots, m\}$, for some $k, i \in \{1, \dots, m\}$ with $k \neq i$, we have

$$
\begin{aligned}
\mathrm{Lip}_{d^\alpha}(g_j) &\le \rho \,\mathrm{diam}(X)^{\gamma-\alpha} = \frac{|f(x_k) - f(x_i)|}{d(x_k, x_i)^\gamma} \mathrm{diam}(X)^{\gamma-\alpha} \\
&\le \mathrm{Lip}_{d^\alpha}(f) \left(\frac{\mathrm{diam}(X)}{d(x_k, x_i)} \right)^{\gamma-\alpha} \le \left(\frac{\mathrm{diam}(X)}{d(x_k, x_i)} \right)^{e \ln(1+1/n) d(x_k, x_i)/\mathrm{diam}(X)} \\
&\le 1 + \frac{1}{n}.
\end{aligned}
$$

The last inequality follows from the fact that the function

$$t \mapsto (t/\mathrm{diam}(X))^{te \ln(1+1/n)/\mathrm{diam}(X)}$$

for all $t > 0$, has a minimum value of $1/(1 + 1/n)$. Hence $\mathrm{Lip}_{d^\alpha}(h) \le 1 + 1/n$ as required. Now let $j, k \in \{1, \dots, m\}$. If $j \le k$, it is immediate that $g_k(x_j) \le f(x_k) \le f(x_j) = g_j(x_j)$, whereas that if $k < j$, we have that $|f(x_k) - f(x_j)|/d(x_k, x_j)^\gamma \le \rho$, hence $f(x_k) - \rho d(x_k, x_j)^\gamma \le f(x_j)$ and thus $g_k(x_j) \le g_j(x_j)$. Therefore $h(x_j) = g_j(x_j) = f(x_j)$ for all $j \in \{1, \dots, m\}$. The claim follows.

Now fix $n \in \mathbb{N}$ and, for each $x \in X$, let $B(x, 1/n) = \{y \in X : d(y, x)^\alpha < 1/n\}$. By the compactness of X, there is a finite subset F_n of X such that $X = \cup_{x \in F_n} B(x, 1/n)$. We can suppose that the base point $0 \in X$ is in F_n, for otherwise take the finite set $F_n \cup \{0\}$. By the claim, there exists a function $h_n \in \mathrm{lip}^{\mathbb{R}}(X, d^\alpha)$ such that $\mathrm{Lip}_{d^\alpha}(h_n) \le 1 + 1/n$ and $h_n(x) = f(x)$, for all $x \in F_n$. Hence $h_n \in \mathrm{lip}_0^{\mathbb{R}}(X, d^\alpha)$. To prove that the sequence $\{h_n\}$ converges pointwise on X to f, let $x \in X$. For each $n \in \mathbb{N}$, choose $y_n \in F_n$ such that $d(x, y_n)^\alpha < 1/n$. Note that $h_n(y_n) = f(y_n)$ and thus

$$
\begin{aligned}
|f(x) - h_n(x)| &\le |f(x) - f(y_n)| + |f(y_n) - h_n(x)| \\
&\le |f(x) - f(y_n)| + |h_n(y_n) - h_n(x)| \\
&\le (\mathrm{Lip}_{d^\alpha}(f) + \mathrm{Lip}_{d^\alpha}(h_n))\, d(x, y_n)^\alpha \\
&\le \left(2 + \frac{1}{n}\right)\frac{1}{n}.
\end{aligned}
$$

Hence the sequence $\{h_n(x)\}$ converges to $f(x)$ as $n \to \infty$. Finally, let $r_n = \max\{1, \mathrm{Lip}_{d^\alpha}(h_n)\}$ and $f_n = h_n/r_n$ for each $n \in \mathbb{N}$. It is clear that $\{f_n\}$ is a sequence in $B_{\mathrm{lip}_0^{\mathbb{R}}(X, d^\alpha)}$ that converges pointwise to f on X. Then the corollary follows from Theorem 2.2.

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