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EQUIMULTIPLE COEFFICIENT IDEALS

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Abstract

Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and I an equimultiple ideal of R of analytic spread s. In this paper, we introduce the equimultiple coefficient ideals. Fix $k \in \{1, \dots, s\}$. The largest ideal L containing I such that $e_i(I_\mathfrak{p}) = e_i(L_\mathfrak{p})$ for each $i \in \{1, \dots, k\}$ and each minimal prime $\mathfrak{p}$ of I is called the k-th equimultiple coefficient ideal denoted by I_k. It is a generalization of the coefficient ideals introduced by Shah for the case of $\mathfrak{m}$-primary ideals. We also see applications of these ideals. For instance, we show that the associated graded ring $G_I(R)$ satisfies the S_1 condition if and only if $I^n = (I^n)_1$ for all n.

1. Introduction

Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring of dimension d and I an $\mathfrak{m}$-primary ideal of R. Shah [10] showed the existence of unique largest ideals I_k $(1 \le k \le d)$ lying between I and $\overline{I}$ such that the $k+1$ Hilbert coefficients of I and I_k coincide, that is, $e_i(I) = e_i(I_k)$ for $0 \le i \le k$. These ideals are called coefficient ideals. They have been studied in some articles such as [2], [5], [6] and [10]. In [10], it was found that if I contains a regular element, then the Ratliff-Rush closure I^* and the d-th coefficient ideal I_d coincide; moreover the author studied the associated primes of the associated graded ring $G_I(R)$. In [2], Ciupercă studies the relationship between the S_2-ification of the extended Rees algebra $\mathscr{R} = R[It, t^{-1}]$ and the cited ideals. In [6], when R is a domain, it is shown that the associated Ratliff-Rush ideal I^* of I is the contraction to R of the extension of I to its blowup $\mathscr{B}(I) = \{R[I/a]_P \mid a \in I - 0, P \in \operatorname{spec}(R[I/a])\}$, i.e, $I^* = \bigcap\{IS \cap R \mid S \in \mathscr{B}(I)\}$. If further R is analytically unramified, it is shown in [5], that the coefficient ideals I_k are also contracted from a blowup $\mathscr{B}(I)^{(k)}$ which is obtained from $\mathscr{B}(I)$ by a process similar to "S_2-ification".

The paper is organized as follows: in section 2, we generalize the notion of coefficient ideals (introduced by Shah); we work with an equimultiple ideal I, that is, $\operatorname{ht}(I) = s(I)$, where $s = s(I)$ is the analytic spread of I. We make

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use of Böger's theorem (on Hilbert-Samuel multiplicity) to show the existence of unique largest ideals I_k (we use the same notation as used by Shah) lying between I and $\overline{I}$, and satisfying $e_i(I_{\mathfrak{p}}) = e_i((I_k)_{\mathfrak{p}})$, for $0 \leq i \leq s$ and every minimal prime $\mathfrak{p}$ of I. We call them equimultiple coefficient ideals. Given an ideal J, we denote the unmixed part of J by J^u. We show that if I contains a regular element then $I_s = (I^*)^u$, which shows I_s is an unmixed ideal. In fact we verify that all the equimultiple coefficient ideals are unmixed (Theorem 2.12).

In section 3, we give a criterion to control the height of the associated primes of $G_I(R)$ (Theorem 3.2). As a consequence of this, we show that if $G_I(R)$ satisfies S_1, so does $G_{I^m}(R)$ for every m (Corollary 3.6). In [8], Noh and Vasconcelos showed that if R is a Cohen-Macaulay ring, the Rees algebra $R[It]$ satisfies S_2 and I is an equimultiple ideal, then all the powers I^n are unmixed ideals. In this work, we verify the same result when R is only a quasi-unmixed ring satisfying the S_2 condition. Finally, we give a way to provide associated graded rings $G_I(R)$ satisfying the S_1 condition (Corollary 3.20).

2. Equimultiple coefficient ideals

In this section, we show the existence of the equimultiple coefficient ideals and also we introduce a refined version for their existence. We show that all of them are unmixed ideals, and find their primary decompositions components. It is also seen how coefficient ideals control the height of the associated primes of $G_I(R)$. For example, the associated graded ring $G_I(R)$ satisfies the S_1 condition if and only if $(I^n)_1 = I^n$ for all n. As consequence, if $G_I(R)$ satisfies the S_1 condition then $G_{I^m}(R)$ satisfies the S_1 condition for all m. Finally, we give a way to construct associated graded rings satisfying the S_1 condition.

Let I be an ideal in a ring R. An element $r \in R$ is said to be *integral over* I if there exist an integer n and elements $a_i \in I^i$, $i = 1, \ldots, n$, such that

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0.$$

The set of all elements that are integral over I is called the *integral closure* of I, and it is denoted by $\overline{I}$.

Below, we recall the well known theorem of Böger on Hilbert coefficients. Let $\mathrm{Min}(R)$ denote the set of minimal prime ideals of the ring R. Thus, $\mathrm{Min}(R/I)$ is the set of minimal prime ideals of I.

THEOREM 2.1 (Böger [11, Corollary 11.3.2]). *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring, and let $I \subseteq J$ be two ideals such that I is equimultiple. Then $J \subseteq \overline{I}$ if and only if $e_0(I_{\mathfrak{p}}) = e_0(J_{\mathfrak{p}})$ for every $\mathfrak{p} \in \mathrm{Min}(R/I)$.*

The next two remarks may be found in [10]. They are used when we localize an equimultiple ideal I at a minimal prime P.

REMARK 2.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\dim R \geq 1$. Suppose $I \subseteq J$ are $\mathfrak{m}$-primary ideals and fix k such that $1 \leq k \leq d$. Then for all large n, $e_i(I) = e_i(J)$ with $0 \leq i \leq k$ if and only if $\ell(J^n/I^n) \leq P(n)$, where $P(n)$ is some polynomial in n of degree at most $d - (k+1)$.

PROOF. It suffices to observe that, for large n,

$$\ell(J^n/I^n) = \ell(R/I^n) - \ell(R/J^n) = \sum_{i=0}^{i=d} (-1)^i [e_i(I) - e_i(J)] \binom{n+d-i-1}{d-i}.$$

REMARK 2.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R \geq 1$. Suppose $I \subseteq I' \subseteq J$ are $\mathfrak{m}$-primary ideals and fix k such that $1 \leq k \leq d$. Then $e_i(I) = e_i(J)$ with $0 \leq i \leq k$ if and only if $e_i(I) = e_i(I') = e_i(J)$ with $0 \leq i \leq k$.

PROOF. We just use that $\ell(I'^n/I^n) \leq \ell(J^n/I^n)$ and apply Remark 2.2.

By Böger's Theorem, it is easy to see that $\overline{I}$ is the unique largest ideal L which satisfies $L \supseteq I$ and $e_0(I_\mathfrak{p}) = e_0(L_\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Min}(R/I)$. In the next result we generalize the notion of coefficient ideals, firstly introduced by Shah in [10], for a more general case which I is an equimultiple ideal.

THEOREM 2.4 (Existence of the equimultiple coefficients ideals). *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring. Assume $R/\mathfrak{m}$ is infinite and $\dim R = d \geq 1$. Let I be an equimultiple ideal with $s = s(I)$. Then there exist unique largest ideals I_k, for $1 \leq k \leq s$, containing I such that*

(1) *$e_i(I_\mathfrak{p}) = e_i((I_k)_\mathfrak{p})$, for $0 \leq i \leq k$ and every $\mathfrak{p} \in \mathrm{Min}(R/I)$, and*
(2) *$I \subseteq I_s \subseteq \cdots \subseteq I_1 \subseteq \overline{I}$.*

PROOF. Let $s = s(I)$ denote the analytic spread of I. By the known Ratliff-Rush theorem, we have $s = \dim R_\mathfrak{p}$, for any $\mathfrak{p} \in \mathrm{Min}(R/I)$. For each $k = 1, \ldots, s$, consider the set

$$V_k = \{L \mid L \text{ is an ideal of } R \text{ such that } L \supseteq I \text{ and } e_i(I_\mathfrak{p}) = e_i(L_\mathfrak{p}), \text{ for every } 0 \leq i \leq k \text{ and } \mathfrak{p} \in \mathrm{Min}(R/I)\}.$$

Firstly note that if $L \in V_k$ then $e_i(I_\mathfrak{p}) = e_i(L_\mathfrak{p})$, for every $\mathfrak{p} \in \mathrm{Min}(R/I)$, and in particular, by Böger's Theorem, $L \subseteq \overline{I}$.

Since $I \in V_k$ and R is Noetherian there exists a maximal element $J \in V_k$. We prove J is unique. Let $L \in V_k$ and $x \in L$. Since $I \subseteq (I, x) \subseteq L$, we have by Remark 2.3 that $e_i(I_\mathfrak{p}) = e_i((I, x)_\mathfrak{p}) = e_i(J_\mathfrak{p})$, for $0 \leq i \leq k$ and $\mathfrak{p} \in \mathrm{Min}(R/I)$. Then I is a reduction of (I, x), so that $(I, x)^{t+1} = (I, x)^t I$, for some t. So $x^{t+1} \in (I, x)^t I \subseteq (J, x)^t J$. Hence, $(J, x)^{t+1} = (J, x)^t J$ and then $(J,x)^n = (J,x)^t J^{n-t}$, for $n \geq t$. Fix $\mathfrak{p} \in \operatorname{Min}(R/I)$. We have, for all $n \geq t$,

$$\begin{aligned}
&\ell((J,x)^n_{\mathfrak{p}}/J^n_{\mathfrak{p}}) \\
&\quad = \ell\big(((J,x)^t J^{n-t})_{\mathfrak{p}}/J^n_{\mathfrak{p}}\big) = \ell\big((J^n_{\mathfrak{p}},\ (J^{n-1}x)_{\mathfrak{p}}, \ldots, (J^{n-t}x^t)_{\mathfrak{p}})/J^n_{\mathfrak{p}}\big) \\
&\quad \leq \sum_{i=1}^{t} \ell\big((J^{n-i}x^i)_{\mathfrak{p}} + J^n_{\mathfrak{p}}/J^n_{\mathfrak{p}}\big) \leq \sum_{i=1}^{t} \ell\big((J^{n-i}x^i)_{\mathfrak{p}} + J^n_{\mathfrak{p}}/I^n_{\mathfrak{p}}\big) \\
&\quad \leq \sum_{i=1}^{t} \big[\ell\big((J^{n-i}x^i)_{\mathfrak{p}} + I^n_{\mathfrak{p}}/I^n_{\mathfrak{p}}\big) + \ell(J^n_{\mathfrak{p}}/I^n_{\mathfrak{p}})\big] \\
&\quad \leq \sum_{i=1}^{t} \left[\ell\big((I^{n-i}x^i)_{\mathfrak{p}} + I^n_{\mathfrak{p}}/I^n_{\mathfrak{p}}\big) + \ell\left(\frac{(J^{n-i}x^i)_{\mathfrak{p}} + I^n_{\mathfrak{p}}}{(I^{n-i}x^i)_{\mathfrak{p}} + I^n_{\mathfrak{p}}}\right) + \ell(J^n_{\mathfrak{p}}/I^n_{\mathfrak{p}})\right] \\
&\quad \leq \sum_{i=1}^{t} \big[\ell(J^{n-i}_{\mathfrak{p}}/I^{n-i}_{\mathfrak{p}}) + \ell((I,x)^n_{\mathfrak{p}}/I^n_{\mathfrak{p}}) + \ell(J^n_{\mathfrak{p}}/I^n_{\mathfrak{p}})\big].
\end{aligned}$$

Since $e_i(I_{\mathfrak{p}}) = e_i((I,x)_{\mathfrak{p}})$ and $e_i(I_{\mathfrak{p}}) = e_i(J_{\mathfrak{p}})$ holds for $0 \leq i \leq k$ and every $\mathfrak{p} \in \operatorname{Min}(R/I)$, one can conclude by Remark 2.2 that $e_i(J_{\mathfrak{p}}) = e_i((J,x)_{\mathfrak{p}})$ holds for $0 \leq i \leq k$ and every $\mathfrak{p} \in \operatorname{Min}(R/I)$. But J is maximal in V_k, so $L \subseteq J$ and therefore J is the unique maximal in V_k. This ideal is denoted by I_k.

Definition 2.5. The ideals I_k above obtained will be called *equimultiple coefficient ideals*.

Let $I^* = \bigcup_{n\geq 1}(I^{n+1} : I^n)$ be the Ratliff-Rush ideal. It is known that I^* is the unique largest ideal L which satisfies $L \supseteq I$ and $L^n = I^n$ for large n. Moreover, by localizing at each $\mathfrak{p} \in \operatorname{Min}(R/I)$, we have $e_i(I_{\mathfrak{p}}) = e_i((I^*)_{\mathfrak{p}})$, for $0 \leq i \leq s$ and every $\mathfrak{p}$ minimal prime of I. By the above theorem one has $I^* \subseteq I_s$.

An ideal I is said to be *a Ratliff-Rush ideal* if $I^* = I$.

Corollary 2.6. *Assume the hypothesis of Theorem 2.4. Then* $I \subseteq J \subseteq I_k \subseteq \overline{I}$ *if and only if* $I \subseteq J$ *and* $e_i(I_{\mathfrak{p}}) = e_i(J_{\mathfrak{p}})$*, for* $1 \leq i \leq k$ *and every* $\mathfrak{p} \in \operatorname{Min}(R/I)$.

Corollary 2.7. *Assume the hypothesis of Theorem 2.4. All the coefficient ideals* I_k *are Ratliff-Rush ideals.*

Proof. We know that $(I_k)^n_{\mathfrak{p}} = ((I_k)^*)^n_{\mathfrak{p}}$, for $n \gg 0$ and any prime $\mathfrak{p}$. In particular, $e_i((I_k)_{\mathfrak{p}}) = e_i(((I_k)^*)_{\mathfrak{p}})$, for $0 \leq i \leq s$ and any $\mathfrak{p} \in \operatorname{Min}(R/I)$. So by maximality of I_k, we have $(I_k)^* \subseteq I_k$.

NOTATION 2.8. Given an ideal $J \subseteq R$, let J^u denote the unmixed part of J.

The next result shows that the coefficient ideal I_s is an unmixed ideal if I contains a regular element. Later, we will see that in fact I_s is unmixed anyway (see Theorem 2.12).

PROPOSITION 2.9. *Assume the setup of Theorem 2.4 with I containing a regular element. Then $I_s = (I^*)^u$. In particular, I_s is an unmixed ideal.*

PROOF. For simplicity of notation, let $J = (I^*)^u$ denote the unmixed part of the Ratliff-Rush closure I^*. We have $\mathrm{Min}(R/I) = \mathrm{Min}(R/I^*) = \mathrm{Min}(R/J)$. Because of the first condition in Theorem 2.4, we have $(I_s)_\mathfrak{p} \subseteq (I_\mathfrak{p})^* = I^*_\mathfrak{p}$, for every $\mathfrak{p} \in \mathrm{Min}(R/I)$. But $(I_s)_\mathfrak{p} \subseteq I^*_\mathfrak{p} = J_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Min}(R/J) = \mathrm{Ass}(R/J)$. Therefore, $I_s \subseteq J$.

Now note that by a property of the Ratliff-Rush closure, one has $I^n_\mathfrak{p} = (I^*_\mathfrak{p})^n = J^n_\mathfrak{p}$, for large n and any $\mathfrak{p} \in \mathrm{Min}(R/I)$. Thus, $e_i(I_\mathfrak{p}) = e_i(J_\mathfrak{p})$, for $0 \le i \le s$ and any $\mathfrak{p} \in \mathrm{Min}(R/I)$, so that $J \subseteq I_s$ by maximality of I_s.

PROPOSITION 2.10. *Assume the setup of Theorem 2.4 and let $J \supseteq I$ be an equimultiple ideal. Then*

(1) *if $J \subseteq I_k$ then $I_k = J_k$;*
(2) *if there exists one positive integer m such that $J^m \subseteq (I^m)_k$, then $J^n \subseteq (I^n)_k$ for all positive integers n;*
(3) *$(((I^m)_k)^n)_k = (I^{mn})_k$, for all positive integers m, n.*

PROOF. Fix k. For item (1), it suffices to use $\ell(R/I^n_\mathfrak{p}) - \ell(R/J^n_\mathfrak{p}) \le \ell(R/I^n_\mathfrak{p}) - \ell(R/(I_k)^n_\mathfrak{p})$ for each prime $\mathfrak{p} \in \mathrm{Min}(R/I)$ and the fact that the last term is, for large n, a polynomial of degree at most $s - (k+1)$.

Now we show (2). We have $e_i(I^m_\mathfrak{p}) = e_i(J^m_\mathfrak{p})$, for $0 \le i \le k$ and every $\mathfrak{p} \in \mathrm{Min}(R/I^m)$, by Corollary 2.6. By using coefficients ideals for primary case, we have, for each minimal prime $\mathfrak{p}$, that $I^m_\mathfrak{p} \subseteq J^m_\mathfrak{p} \subseteq (I^m_\mathfrak{p})_k$. By [5, Proposition 3.2], $I^n_\mathfrak{p} \subseteq J^n_\mathfrak{p} \subseteq (I^n_\mathfrak{p})_k$, for all n, so that for each minimal prime $\mathfrak{p}$, we have $e_i(I^n_\mathfrak{p}) = e_i(J^n_\mathfrak{p})$, for $0 \le i \le k$ and all n. Corollary 2.6 gives then $J^n \subseteq (I^n)_k$ for all n.

Item (3) is a combination of the two previous items.

REMARK 2.11. If I contains a regular element, by Proposition 2.10 and Corollary 2.7 we have $(I^*)_k = I_k = (I_k)^*$, since $(I^*)_k = I_k$. In particular, the unmixed part of a Ratliff-Rush ideal is also a Ratliff-Rush ideal.

THEOREM 2.12. *Assume the setup of Theorem 2.4. The coefficient ideals of I^u are $(I^u)_k = (I_k)^u$ and all the coefficient ideals of I are unmixed ideals.*

PROOF. Firstly we construct a specific chain from I^u formed by the unmixed part of the coefficient ideals of I and after show that its terms are the coefficient ideals of I^u. We have $((I_1)^u)_{\mathfrak{p}} = (I_1)_{\mathfrak{p}} \subseteq \overline{I}_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Ass}(R/\overline{I})$. So $(I_1)^u \subseteq \overline{I}$. Moreover, $((I_2)^u)_{\mathfrak{p}} = (I_2)_{\mathfrak{p}} \subseteq ((I_1)^u)_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Ass}(R/(I_1)^u)$ so that $(I_2)^u \subseteq (I_1)^u$. Inductively the desired chain is constructed.

Now let $I^u \subseteq J_s \subseteq \cdots \subseteq J_1 \subseteq \overline{I}$ be the coefficient ideals of I^u. Fix k. So J_k is the unique largest ideal for which $e_i((I^u)_{\mathfrak{p}}) = e_i((J_k)_{\mathfrak{p}})$, for $0 \leq i \leq k$ and any $\mathfrak{p} \in \mathrm{Min}(R/I^u)$. Moreover, $e_i((I^u)_{\mathfrak{p}}) = e_i(I_{\mathfrak{p}}) = e_i((I_k)_{\mathfrak{p}}) = e_i(((I_k)^u)_{\mathfrak{p}})$ for $0 \leq i \leq k$ and any $\mathfrak{p} \in \mathrm{Min}(R/I^u)$. Hence, $J_k \subseteq I_k \subseteq (I_k)^u$ and therefore $(I_k)^u = J_k$, for each k.

By Proposition 2.10, $(I^u)_k = I_k$. Therefore, I_k is an unmixed ideal for each k.

The next result expresses the primary decomposition components of the coefficient ideals I_k, besides giving another way to show they are unmixed ideals.

PROPOSITION 2.13. *Assume the setup of Theorem 2.4. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of I. Then I_k has the following primary decomposition*

$$I_k = ((I_{\mathfrak{p}_1})_k \cap R) \cap \cdots \cap ((I_{\mathfrak{p}_r})_k \cap R).$$

Furthermore, $(I_k)_{\mathfrak{p}} = (I_{\mathfrak{p}})_k$ for every prime ideal $\mathfrak{p}$.

PROOF. Fix $k \in \{1, \ldots, s\}$. To simplify notation, let J_i and H_i denote $(I_{\mathfrak{p}_i})_k$ and $(I_{\mathfrak{p}_i})_k \cap R$, respectively, where $1 \leq i \leq r$. Since J_i is $\mathfrak{p}_i R_{\mathfrak{p}_i}$-primary, H_i is $\mathfrak{p}_i$-primary. Set $H = H_1 \cap \cdots \cap H_r$. Then $H_{\mathfrak{p}_i} = (H_i)_{\mathfrak{p}_i} = J_i$ for each i, as $(H_j)R_{\mathfrak{p}_i} = R_{\mathfrak{p}_i}$ for every $j \neq i$. By definition of equimultiple coefficient ideals one may then conclude $I_k = H$.

Hence, the second part follows by observing that

$$(I_{\mathfrak{p}})_k = ((I_{\mathfrak{p}_1})_k \cap R_{\mathfrak{p}}) \cap \cdots \cap ((I_{\mathfrak{p}_t})_k \cap R_{\mathfrak{p}})$$

and

$$(I_k)_{\mathfrak{p}} = ((I_{\mathfrak{p}_1})_k \cap R)_{\mathfrak{p}} \cap \cdots \cap ((I_{\mathfrak{p}_t})_k \cap R)_{\mathfrak{p}},$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \subseteq \mathfrak{p}$ and $\mathfrak{p}_{t+1}, \ldots, \mathfrak{p}_r \not\subseteq \mathfrak{p}$

Below, we recall the definition of some blowup algebras which are important for the present paper.

DEFINITION 2.14. Let R be a ring, I an ideal and t an indeterminate over R. The *Rees algebra* of I is the subring of $R[t]$ defined as

$$R[It] := \oplus_{n \geq 0} I^n t^n.$$

The *extended Rees algebra* of I is the subring of $R[t, t^{-1}]$ defined as

$$R[It, t^{-1}] := \oplus_{n \in \mathbb{Z}} I^n t^n,$$

where, by convention, for any non-positive integer n, $I^n = R$. The *associated graded ring* of I is

$$G_I(R) := \bigoplus_{n\geq 0} \frac{I^n}{I^{n+1}} = R[It]/IR[It] = R[It, t^{-1}]/t^{-1}R[It, t^{-1}].$$

For simplicity, from now on, we also denote the extended Rees algebra of I by $\mathscr{R}$.

THEOREM 2.15. *Let* $(R, \mathfrak{m})$ *be a local ring and* I *an equimultiple ideal. If* $((I^n)^*)^u = I^n$ *for all* n*, then* $\operatorname{ht}(P) < s$ *for every* $P \in \operatorname{Ass}(G_I(R))$.

PROOF. It is easy to see that the hypothesis implies $(I^n)^* = I^n$ and $(I^n)^u = I^n$ for all n. Let $P \in \operatorname{Ass}_{\mathscr{R}}(\mathscr{R}/t^{-1}\mathscr{R})$ and $\mathfrak{p} = P \cap R$. Initially we assume R is a domain. By the Dimension Inequality, one has

$$\operatorname{ht}(P) - \operatorname{ht}(\mathfrak{p}) \leq 1 - \operatorname{tr.deg}_{R/\mathfrak{p}} \mathscr{R}/P.$$

We claim that $\operatorname{tr.deg}_{R/\mathfrak{p}} \mathscr{R}/P \neq 0$. Suppose the contrary. Note first that we can assume $\mathfrak{p}$ is maximal, since $((I^n_{\mathfrak{p}})^*)^u = I^n_{\mathfrak{p}}$ and $\operatorname{ht}(P) = \operatorname{ht}(P_{\mathfrak{p}})$. Then $\mathscr{R}/P$ is a finitely generated algebra over the field $k := R/\mathfrak{p}$. Hence, $\dim \mathscr{R}/P = \operatorname{tr.deg}_k \mathscr{R}/P = 0$. It implies $\mathscr{R}/P$ is a field, as $\mathscr{R}/P$ is a domain. Therefore, $R/\mathfrak{p}$ and $\mathscr{R}/P$ are isomorphic; whence $G_+ \subseteq P/t^{-1}\mathscr{R}$, which is a contradiction since G_+ is a regular ideal.

In conclusion, we can write $P = (t^{-1}\mathscr{R} : at^r)$, for some homogeneous element $at^r \in \mathscr{R} \setminus t^{-1}\mathscr{R}$. Hence, $Pat^r \in I^{r+1}t^r$, for some integer $r \geq 0$, so that $\mathfrak{p} = (I^{r+1} : a)$. This means $\mathfrak{p} \in \operatorname{Ass}(R/I^{r+1})$. By hypothesis, $\mathfrak{p}$ is a minimal prime of I^{r+1}, so $\operatorname{ht}(\mathfrak{p}) = s$. As $\operatorname{tr.deg}_{R/\mathfrak{p}} \mathscr{R}/P \neq 0$, through the above inequality, we obtain $\operatorname{ht}(P) \leq s$ and therefore $\operatorname{ht}(P/t^{-1}\mathscr{R}) < s$.

The case for which R is not a domain is similar. It follows by taking a minimal prime Q contained in P such that $\operatorname{ht}(P) = \operatorname{ht}(P/Q)$ and after going modulo a minimal prime.

REMARK 2.16. The equation $((I^n)^*)^u = I^n$, for all n, is equivalent to have $(I^n)^* = I^n$, for all n, and $(I^n)^u = I^n$, for all n. Moreover if I is a regular ideal and only $(I^n)^u = I^n$, for $n \gg 0$, the condition $(I^n)^* = I^n$, for all n, implies $(I^n)^u = I^n$, for all n, since $\operatorname{Ass}(R/(I^n)^*) \subseteq \operatorname{Ass}(R/(I^{n+1})^*)$, for all $n \geq 1$ (see [9, p. 14]).

3. Equimultiple coefficient ideals, associated graded ring and Serre's condition (S_n)

In this section, we see necessary and sufficient conditions for the associated graded ring $G_I(R)$ to satisfy the S_1 condition. It has a relation to the concept of equimultiple coefficient ideals. Moreover, in Theorem 3.2 and Theorem 3.17, we generalize Theorem 4 from [10], which concern the height of associated prime ideals of $G_I(R)$. In particular, we obtain that $G_I(R)$ satisfies the S_1 condition if and only if $(I^n)_1 = I^n$, for all n.

LEMMA 3.1. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and let I be a proper ideal of R. Then we have the following:*

(1) *if $\mathrm{ht}(P) < k$ for every $P \in \mathrm{Ass}(G_I(R))$, then $I^n_{\mathfrak{p}} = (I^n_{\mathfrak{p}})_k$, for all n and every $\mathfrak{p} \in \mathrm{Min}(R/I)$;*

(2) *suppose all the powers I^n are unmixed ideals, then the converse of (1) is valid.*

PROOF. To show (1), let $\mathfrak{p} \in \mathrm{Min}(R/I)$. Then for every $P_{\mathfrak{p}} \in \mathrm{Ass}_{\mathscr{R}_{\mathfrak{p}}}(\mathscr{R}_{\mathfrak{p}}/t^{-1}\mathscr{R}_{\mathfrak{p}})$ we have $\mathrm{ht}(P_{\mathfrak{p}}) = \mathrm{ht}(P) < k$. We use then the result [10, Theorem 4] to complete the assertion. For the other assertion, let $P \in \mathrm{Ass}_{\mathscr{R}}(\mathscr{R}/t^{-1}\mathscr{R})$. We have $\mathfrak{p} = P \cap R \in \mathrm{Ass}(R/I^n)$ which is minimal on I^n by assumption. Therefore, $\mathrm{ht}(P) = \mathrm{ht}(P_{\mathfrak{p}}) < k$ and the result follows.

THEOREM 3.2. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and let I be an equimultiple ideal.*

(1) *If $(I^n)_k = I^n$, for all n, then $\mathrm{ht}(P) < k$, for every $P \in \mathrm{Ass}(G_I(R))$.*

(2) *If $\mathrm{ht}(P) < k$, for every $P \in \mathrm{Ass}(G_I(R))$, then $(I^n)_k = (I^n)^u$, for all n.*

In particular, $G_I(R)$ satisfies the S_1 condition if and only if $(I^n)_1 = I^n$, for all n.

PROOF. If $(I^n)_k = I^n$, we have in particular that $(I^n)^u = I^n$, for all n. Further by localizing we obtain $(I^n_{\mathfrak{p}})_k = ((I^n)_k)_{\mathfrak{p}} = I^n_{\mathfrak{p}}$, for every minimal prime $\mathfrak{p}$ of I, by Proposition 2.13. Now one just applies Lemma 3.1 to conclude item (1).

To show (2), firstly let $\mathfrak{p}$ be a minimal prime of I and consider the localization $S^{-1}R[It, t^{-1}]$, where $S = R \setminus \mathfrak{p}$. For each associated prime

$$S^{-1}P \in \mathrm{Ass}_{S^{-1}R[It,t^{-1}]}\left(\frac{S^{-1}R[It, t^{-1}]}{t^{-1}S^{-1}R[It, t^{-1}]}\right),$$

we have $\mathrm{ht}(S^{-1}P) = \mathrm{ht}(P) < k$. Thus, by [10, Theorem 4], we obtain, for each $\mathfrak{p} \in \mathrm{Min}(R/I)$, that $I^n_{\mathfrak{p}} = (I^n_{\mathfrak{p}})_k$, for all n. In particular, $I^n_{\mathfrak{p}} = ((I^n)_k)_{\mathfrak{p}}$ by Proposition 2.13, so that $(I^n)_k = (I^n)^u$, for all positive integers n.

Now we consider the case $k = 1$. If $\mathfrak{p} \in \mathrm{Ass}(R/I^n)$, then there exists a $P \in \mathrm{Ass}_{\mathscr{R}}(\mathscr{R}/t^{-1}\mathscr{R})$ such that $\mathfrak{p} = P \cap R$. Firstly assume R is a domain. By using the Dimension Formula, we obtain $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(P) - 1 + t$, where

$$t := \mathrm{tr.deg}_{R/\mathfrak{p}} \frac{\mathscr{R}}{P} = \mathrm{tr.deg}_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}} \frac{\mathscr{R}_\mathfrak{p}}{P_\mathfrak{p}} \leq \mathrm{tr.deg}_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}} \frac{\mathscr{R}_\mathfrak{p}}{\mathfrak{p}\mathscr{R}_\mathfrak{p}} = s.$$

Therefore, each associated prime $\mathfrak{p} \in \mathrm{Ass}(R/I^n)$ is actually a minimal prime of I^n, for every n, as required. The general case may be derived by taking a minimal prime Q of $\mathscr{R}$ such that $Q \subseteq P$ and $\mathrm{ht}(P) = \mathrm{ht}(P/Q)$. The converse is immediate from (1).

COROLLARY 3.3. *Let $(R, \mathfrak{m})$ be a quasi-unmixed, analytically unramified domain satisfying the S_2 condition and let I be an equimultiple ideal such that* $\mathrm{ht}(I) \geq 2$. *If $\oplus_{n \geq 0} I_n t^n$ is the S_2-ification of the Rees algebra $R[It]$ and* $\mathrm{ht}(P) < k$, *for every $P \in \mathrm{Ass}(G_I(R))$, then $(I^n)_k \subseteq I_n$.*

PROOF. This follows directly from [2, Proposition 2.10] and Theorem 3.2.

The Theorem 3.2 derives the following result, firstly introduced by Noh and Vasconcelos [8, Theorem 2.5] for the less general case which R is a Cohen-Macaulay ring.

COROLLARY 3.4. *Let R be a quasi-unmixed ring satisfying S_2 and I an equimultiple ideal containing a regular element. If $R[It]$ satisfies S_2, then all the powers I^n are unmixed ideals.*

PROOF. We may assume R is local. It then suffices to use [1, Theorem 1.5] and later apply Theorem 3.2.

REMARK 3.5. Grothe, Hermann and Orbanz [4, Theorem 4.7] showed that if I is an equimultiple ideal of a Cohen-Macaulay local ring $(R, \mathfrak{m})$, then the Cohen-Macaulayness of $G_I(R)$ implies the Cohen-Macaulayness of $G_{I^m}(R)$, for all $m \geq 1$. Also Shah [10, Corollary 5(C)] showed the same result when R is just quasi-unmixed but I an $\mathfrak{m}$-primary ideal.

Below we see that a similar result for the S_1 condition can be obtained immediately through coefficient ideals.

COROLLARY 3.6. *Let $(R, \mathfrak{m})$ be quasi-unmixed local ring and I an equimultiple ideal. If $G_I(R)$ satisfies S_1, then so does $G_{I^m}(R)$, for all $m \geq 1$.*

Due to the above result and [1, Theorem 1.5], we obtain the following.

COROLLARY 3.7. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring satisfying S_2 and I an equimultiple ideal containing a regular element. If $R[It]$ satisfies S_2, then so does $R[I^m t]$, for all $m \geq 1$.*

THEOREM 3.8. *Let* $(R, \mathfrak{m})$ *be a quasi-unmixed local ring and* I *an equimultiple ideal of analytic spread* s. *If* $\operatorname{depth} G_I(R) \geq k$, *where* $1 \leq k \leq s$, *then* $(I^n)_j = (I^n)^u$, *for all* n *and* $s + 1 - k \leq j \leq s$.

PROOF. By using the fact that $G \otimes_R R_\mathfrak{p}$ is flat over G, one can conclude that $\operatorname{depth} G_I(R) \geq k$ implies $\operatorname{depth} G_{I_\mathfrak{p}}(R_\mathfrak{p}) \geq k$, for each $\mathfrak{p}$ prime. So by [10, Theorem 5], we have $I_\mathfrak{p}^n = (I_\mathfrak{p}^n)_j$, for all n and each minimal prime $\mathfrak{p}$ of I. Hence, $I_\mathfrak{p}^n = ((I^n)_j)_\mathfrak{p}$ and therefore $(I^n)_j = (I^n)^u$, for all positive integers n, as all coefficients ideals are unmixed ideals.

REMARK 3.9. As can be seen in the above proof, if I is an arbitrary equimultiple ideal and $\operatorname{depth} G_I(R)_+ \geq k$, one has $I_\mathfrak{p}^n = (I_\mathfrak{p}^n)_j$, for all n, $s+1-k \leq j \leq s$, and each minimal prime $\mathfrak{p}$ of I.

PROPOSITION 3.10. *Let* $(R, \mathfrak{m})$ *be a quasi-unmixed local ring satisfying the* S_{s+1} *condition and* I *an ideal with* $\operatorname{grade} I = \operatorname{ht}(I)$ *which is equimultiple. Suppose* $s(I_\mathfrak{p}) = \mu(I_\mathfrak{p})$, *for every* $\mathfrak{p} \in \operatorname{Min}(R/I)$. *Then* $G_I(R)$ *satisfies* S_1.

PROOF. By hypothesis, there exists a minimal reduction J of I generated by a regular sequence of length s and $J_\mathfrak{p} = I_\mathfrak{p}$, for each $\mathfrak{p} \in \operatorname{Min}(R/I)$, since $I_\mathfrak{p}$ has no proper reduction. Once R satisfies the S_{s+1} condition, we have J is unmixed. Hence, $I = J$ is generated by a regular sequence of length s. In particular the generating set of I form a quasi-regular sequence, thus $\operatorname{grade} G_I(R)_+ \geq s$. Moreover there is an isomorphism of graded rings

$$A = (R/I)[X_1, \ldots, X_s] \cong G_I(R),$$

where A is a polynomial ring with coefficients in R/I. We can then conclude that $\operatorname{Ass}_R(I^i/I^{i+1}) = \operatorname{Ass}_R(R/I)$, for each i. By using the exact sequence

$$0 \longrightarrow I^i/I^{i+1} \longrightarrow R/I^{i+1} \longrightarrow R/I^i \longrightarrow 0,$$

it follows by induction that $\operatorname{Ass}_R(R/I^n) = \operatorname{Ass}_R(R/I)$, for $n \geq 1$. Since I is unmixed, we conclude that $I^n = (I^n)^u$, for all n. The result follows then by Theorem 3.8 and Theorem 3.2.

An ideal I is a *locally complete intersection* if $\operatorname{ht}(I_\mathfrak{p}) = \mu(I_\mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Ass}(R/I)$.

COROLLARY 3.11. *Let* $(R, \mathfrak{m})$ *be a quasi-unmixed local ring satisfying the* S_{s+1} *condition and* I *an ideal with* $\operatorname{grade} I = \operatorname{ht}(I)$ *which is equimultiple. Suppose* I *is a locally complete intersection. Then* $G_I(R)$ *satisfies* S_1.

REMARK 3.12. In the set-up of Proposition 3.10, we obtain $(I^n)_s = \cdots = (I^n)_1 = I^n$ for all positive integers n.

DEFINITION 3.13. Let R be a local ring and I a proper ideal of R. The *reduction number* $r(I)$ of I is defined to be

$$r(I) = \min\{n \mid \text{there exists a minimal reduction } J \text{ of } I \text{ such that } I^{n+1} = JI^n\}.$$

PROPOSITION 3.14. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring satisfying the S_{s+1} condition and I an ideal with* grade $I = \mathrm{ht}(I)$ *which is equimultiple. Suppose some power I^i with $r(I^i) \leq 1$ is an unmixed ideal. If* grade $G_I(R)_+ \geq k$*, where $1 \leq k \leq s$, then $(I^n)_j = I^n$, for all n and $s+1-k \leq j \leq s$.*

PROOF. Because of Theorem 3.8, it suffices to show $(I^n)^u = I^n$, for all n. Further, by [6, (1.2)], we have $(I^n)^* = I^n$, for all n. Hence, $\mathrm{Ass}(R/I^n) \subseteq \mathrm{Ass}(R/I^{n+1})$, for all n, by [9, Lemma 6.6]. By hypothesis, we then get $\mathrm{Ass}(R/I^n) = \mathrm{Min}(R/I^n)$, for all $n \leq i$. By assumption on $r(I^i)$, there exists a minimal reduction J of I^i such that $JI^i = (I^i)^2$. Note that J may be generated by $s(I^i) = s$ elements. Thus, the hypothesis grade $I = s$ gives that grade $J = s$, so that J may be generated by a regular sequence of length s. Since R satisfies S_{s+1}, the ideal J is unmixed.

Now consider the exact sequence

$$0 \longrightarrow J/JI^i \longrightarrow R/JI^i \longrightarrow R/J \longrightarrow 0,$$

where $J/JI^i \simeq (R/I^i)^s$. Since J is an unmixed ideal one may then conclude that $(I^i)^2$ is unmixed. By considering the following exact sequence

$$0 \longrightarrow J^2/J^2I^i \longrightarrow R/(I^i)^3 \longrightarrow R/J^2 \longrightarrow 0,$$

we get $(I^i)^3$ is unmixed. We have then obtained that I^n is unmixed for infinitely many n. Therefore, all the powers I^n are unmixed ideals.

LEMMA 3.15. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring of infinite residue field and I an equimultiple ideal. For all $N \geq 1$ and all reduction $x = x_1, \ldots, x_t$ of I^N, we have*

$$(I^{N+1} : x_1, \ldots, x_k) \subseteq I_k, \quad \text{for } 1 \leq k \leq d.$$

PROOF. It is easy to see we may assume x is a minimal reduction. Fix any $N \geq 1$ and let $x_1, \ldots, x_s$ be a minimal reduction of I^N. By [10, Theorem 2], we have $(I^{N+1} : x_1, \ldots, x_k)_\mathfrak{p} \subseteq (I_\mathfrak{p})_k$, for each $\mathfrak{p} \in \mathrm{Min}(R/I)$. Hence, for each $\mathfrak{p} \in \mathrm{Min}(R/I)$, the equality $e_i((I^{N+1} : x_1, \ldots, x_k)_\mathfrak{p}) = e_i(I_\mathfrak{p})$ is true for $0 \leq i \leq k$. The result then follows by maximality of I_k.

LEMMA 3.16. *Let $(R, \mathfrak{m})$ be quasi-unmixed ring and I an equimultiple ideal with analytic spread s. Let $x_1, \dots, x_s \in I^N$, for some $N \geq 1$, and let $x'_1, \dots, x'_s$ be their images in I^N/I^{N+1}. Let $a_1, \dots, a_t \in R$ be a system of parameters module I. Then $a'_1, \dots, a'_t, x'_1, \dots, x'_s$ is a system of parameters of $G_I(R)$ if and only if $x_1, \dots, x_s$ form a minimal reduction of I^N.*

PROOF. Denote $G_I(R)$ by G and the ideal $(a'_1, \dots, a'_t)G_I(R)$ by L. If the sequence $a'_1, \dots, a'_t, x'_1, \dots, x'_s$ is a system of parameters of G, we have $\sqrt{(G/L)_+} = (x'_1, \dots, x'_s)G/L$. Then $G^n_+ \subseteq (x_1, \dots, x_s)G + L$, for some positive integer n. Thus, it is easy to see that $G^n_+ \subseteq (x_1, \dots, x_s)G + LG^n_+$. By Nakayama's Lemma, one can conclude $G^n_+ \subseteq (x_1, \dots, x_s)G$. Therefore, $x_1, \dots, x_s$ form a minimal reduction of I^N. The converse follows analogously to [4, Corollary 2.7].

The next result is also a generalization of Theorem 4 in [10] as may be easily observed.

THEOREM 3.17. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and I an equimultiple ideal. Let $a_1, \dots, a_r$ be a system of parameters modulo I and let $L = (a'_1, \dots, a'_r)$ denote the ideal generated by their images in $G_I(R)$. Fix $k \in \{1, \dots, s\}$. If $I^n = (I^n)_k$, for all n, then $\operatorname{ht}(P + L) < k + d - s$, for every $P \in \operatorname{Ass}(G_I(R))$.*

PROOF. Let P be an associated prime of $G_I(R)$. Since P is graded we have $P = (0' : y')$, where y' is the image in $G_I(R)$ of $y \in I^{n-1} - I^n$, for some $n \geq 1$. Suppose $\operatorname{ht}(P + L) \geq k + d - s$. Then we obtain $\dim((G/L)/P(G/L)) \leq s - k$. By [10, Lemma 2(E)] we can get a homogeneous system $\overline{x_1}, \dots, \overline{x_s}$ of parameters (of equal degree) for G/L such that $\overline{x_1}, \dots, \overline{x_k} \in P(G/L)$. Let $x'_1, \dots, x'_s \in G_I(R)$ be homogeneous inverse images of $\overline{x_1}, \dots, \overline{x_s}$ respectively so that $x'_1, \dots, x'_k \in P$. Since $a'_1, \dots, a'_t, x'_1, \dots, x'_s$ is a system of parameters of $G_I(R)$ ([4, Proposition 2.6]) we can then use Lemma 3.16 to obtain that $x_1, \dots, x_s$ is a minimal reduction of I^m, for some m. It is easy to see we may assume $m = nN$. Hence, $y(x_1, \dots, x_k) \subseteq I^{n-1+nN+1}$, and so $y \in ((I^n)^{N+1} : x_1, \dots, x_k)$. By Lemma 3.15, one has $y \in (I^n)_k = I^n$, which is a contradiction.

DEFINITION 3.18. Let R be a local ring and I a proper ideal of R. If for each $\mathfrak{p} \in \operatorname{Min}(R/I)$, the localization $I_\mathfrak{p}$ has reduction number $r(I_\mathfrak{p}) \leq t$, it is said that I has *generic reduction number t*.

PROPOSITION 3.19. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of infinite residue field and I an equimultiple ideal of analytic spread s. Assume R/I satisfies the S_1 condition and I has generic reduction number* 1. *Then $G_I(R)$ satisfies the S_1 condition.*

PROOF. Let J be a minimal reduction of I and consider the exact sequence

$$0 \longrightarrow J/JI \longrightarrow R/JI \longrightarrow R/J \longrightarrow 0,$$

where $J/JI \simeq (R/I)^s$. Hence, since R is Cohen-Macaulay and because I is unmixed, one may conclude JI is unmixed. In this way, $I^2R_{\mathfrak{p}} = JIR_{\mathfrak{p}}$, for all $\mathfrak{p} \in \mathrm{Ass}(R/JI) = \mathrm{Min}(R/JI)$, and then $I^2 = JI$. Consider now the following exact sequence

$$0 \longrightarrow J^2/J^2I \longrightarrow R/I^3 \longrightarrow R/J^2 \longrightarrow 0.$$

Since the associated prime ideals of J^n are the same as those of J, and J^2/J^2I is isomorphic to a power of R/I, one may conclude that I^3 is unmixed. Inductively we obtain that all powers I^n are unmixed ideals. On the other hand, if $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$, we have $R_{\mathfrak{p}}/I_{\mathfrak{p}}$ is a Cohen-Macaulay ring and hence, by [7, Proposition 26.12], we obtain that $G_{I_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is Cohen-Macaulay, for all minimal prime $\mathfrak{p}$ of I, and therefore we may use [10, Theorem 4] to obtain $I_{\mathfrak{p}}^n = (I_{\mathfrak{p}}^n)_1$, for all n and all $\mathfrak{p} \in \mathrm{Min}(R/I)$. Now one may just apply Lemma 3.1 to conclude the proof.

In [3], Corso and Polini indicated a method to provide ideals I of reduction number 1. In this way, through above proposition we may produce associated graded rings $G_I(R)$ satisfying the S_1 condition.

COROLLARY 3.20. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring, $\mathfrak{p}$ a prime ideal of height g such that $R_{\mathfrak{p}}$ is not a regular local ring and $J = (x_1, \dots, x_g) \subseteq \mathfrak{p}$ a regular sequence. Set $I = J : \mathfrak{p}$. Then $G_I(R)$ satisfies S_1.*

PROOF. Since $\mathrm{Ass}(R/I) \subseteq \mathrm{Ass}(R/J)$, the ideal I is unmixed. The result then follows from [3, Theorem 2.3].

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