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A SIMPLE SUFFICIENT CONDITION FOR TRIVIALITY OF OBSTRUCTIONS IN THE ORBIFOLD CONSTRUCTION FOR SUBFACTORS

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Abstract

We present a simple sufficient condition for triviality of obstructions in the orbifold construction. As an application, we can show the existence of subfactors with principal graph D_{2n} without full use of Ocneanu's paragroup theory.

1. Introduction

In the subfactor theory initiated by V. F. R. Jones [15], one of the fundamental constructions of subfactors is the orbifold construction. It was introduced by D. E. Evans and Y. Kawahigashi [6] as a method of producing new subfactors. The origin of their work is [16], where Kawahigashi showed the existence of subfactors whose principal graphs are Dynkin diagram D_{2n}. Roughly speaking, the orbifold construction is to take a "quotient" by an internal symmetry of subfactors, which is realized by taking a crossed product construction by an abelian group. The orbifold construction has been further studied by [21], [22], [9], [10].

A typical example of the orbifold construction is the case of a Jones subfactor with principal graph A_{2n-1} [15], which possesses certain $\mathbb{Z}/2\mathbb{Z}$-symmetry. By the orbifold construction, graph change occurs only for A_{4n-3}-subfactors, and the orbifold construction produces subfactors with principal graph D_{2n}. This is because some obstruction, which prevents graph change, appears for an A_{4n-1}-subfactor. Thus the most important problem is to determine the triviality of obstructions in the construction. In general, this problem requires complicated combinatorial computations of connections.

In this paper, we present a simple sufficient condition for triviality of obstructions appearing in the orbifold construction. Namely, we show that an obstruction vanishes if a tensor category arising from a subfactor has a nice

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fusion rule. As an application, we can show the existence of subfactors with principal graph D_{2n} without full use of Ocneanu's paragroup theory, and the proof is easier than that of [16]. Our argument is inspired by the computation presented in [12], and we use a sector technique for the proof. The sector approach has been used in the theory of α-induction [23], [1], [2], [3], [4] in an effective manner. For example, Böckenhauer and Evans [2, Section 3] studied details of the orbifold construction and D_{2n}-subfactors by using the theory of α-induction in the framework of conformal field theory. Also see [23, pp. 381, Examples 1, 2] about E_6 subfactors and E_8 subfactors arising from conformal inclusions. We mention that our approach in this paper does not require conformal field theory and is simpler when compared with above cited papers.

This paper is organized as follows. In §2, we recall the definition of the Loi invariant [18], and construct half braidings in the sense of Izumi [14]. This allows us to extend an endomorphism to a crossed product factor, which is a special case of α-induction. In §3, we explain basic properties of this extension, and show the main theorem. Then we present examples of applications of our theorem.

2. Half braidings

We refer to [7] for the fundamentals of subfactor theory, and [11], [13] for the basics of sector theory. In this paper, we mainly treat type III factors. Here we recall some notation for sectors. For von Neumann algebras $\mathscr{A}, \mathscr{B}$, let $\mathrm{Mor}(\mathscr{A}, \mathscr{B})$ be the set of all unital continuous injective morphisms from $\mathscr{A}$ into $\mathscr{B}$, and $\mathrm{Sect}(\mathscr{B}, \mathscr{A}) = \mathrm{Mor}(\mathscr{A}, \mathscr{B})/\mathrm{Int}(\mathscr{B})$. When $\mathscr{A} = \mathscr{B}$, we write $\mathrm{Mor}(\mathscr{A}, \mathscr{A}) = \mathrm{End}(\mathscr{A})$ and $\mathrm{Sect}(\mathscr{A}, \mathscr{A}) = \mathrm{Sect}(\mathscr{A})$. For $\rho, \sigma \in \mathrm{Mor}(\mathscr{A}, \mathscr{B})$, the space of intertwiners is defined by $(\sigma, \rho) := \{a \in \mathscr{B} \mid \rho(x)a = a\sigma(x), x \in \mathscr{A}\}$. When σ is irreducible, i.e., $(\sigma, \sigma) = \sigma(\mathscr{A})' \cap \mathscr{B} = \mathbb{C}1$, (σ, ρ) becomes a Hilbert space via the inner product $\langle a, b\rangle 1 = b^*a$.

Let $\mathscr{N} \subset \mathscr{M}$ be an irreducible subfactor of type III with finite index, and $\iota\colon \mathscr{N} \hookrightarrow \mathscr{M}$ an inclusion map. Then $\gamma = \iota\bar{\iota}$ is a canonical endomorphism for $\mathscr{N} \subset \mathscr{M}$. Set

$$\mathscr{M} \supset \iota(\mathscr{N}) \supset \iota\bar{\iota}(\mathscr{M}) \supset \iota\bar{\iota}\iota(\mathscr{N}) \supset \cdots = \mathscr{M} \supset \mathscr{N}_1 \supset \mathscr{N}_2 \supset \mathscr{N}_3 \supset \cdots.$$

Set

$$\begin{aligned}
{}_{\mathscr{M}}\Delta_{\mathscr{M}} &= \{[\sigma] \in \mathrm{Sect}(\mathscr{M}) \mid \sigma \prec (\iota\bar{\iota})^n, n \in \mathbb{N}, \sigma \text{ is irreducible}\},\\
{}_{\mathscr{M}}\Delta_{\mathscr{N}} &= \{[\sigma] \in \mathrm{Sect}(\mathscr{M}, \mathscr{N}) \mid \sigma \prec (\iota\bar{\iota})^n\iota, n \in \mathbb{N}, \sigma \text{ is irreducible}\},\\
{}_{\mathscr{N}}\Delta_{\mathscr{M}} &= \{[\sigma] \in \mathrm{Sect}(\mathscr{N}, \mathscr{M}) \mid \sigma \prec (\bar{\iota}\iota)^n\bar{\iota}, n \in \mathbb{N}, \sigma \text{ is irreducible}\},\\
{}_{\mathscr{N}}\Delta_{\mathscr{N}} &= \{[\sigma] \in \mathrm{Sect}(\mathscr{N}) \mid \sigma \prec (\bar{\iota}\iota)^n, n \in \mathbb{N}, \sigma \text{ is irreducible}\}.
\end{aligned}$$

and $\Delta := {}_{\mathscr{M}}\Delta_{\mathscr{M}} \sqcup {}_{\mathscr{M}}\Delta_{\mathscr{N}} \sqcup {}_{\mathscr{N}}\Delta_{\mathscr{M}} \sqcup {}_{\mathscr{N}}\Delta_{\mathscr{N}}$.

We recall the definition of the Loi invariant [18]. Fix isometries $R \in (\mathrm{id}, \iota\bar{\iota})$ and $\bar{R} \in (\mathrm{id}, \bar{\iota}\iota)$ such that $R^*\iota(\bar{R}) = \bar{R}^*\bar{\iota}(R) = 1/d(\iota)$. Let $\mathrm{Aut}(\mathcal{M}, \mathcal{N})$ be a set of automorphisms of $\mathcal{M}$ which preserve $\mathcal{N}$ globally. Take $\alpha \in \mathrm{Aut}(\mathcal{M}, \mathcal{N})$. Since α preserves $\iota(\mathcal{N})$, we have $\alpha\iota = \iota\alpha$. Thus $[\bar{\iota}][\alpha] = [\alpha][\bar{\iota}] \in \mathrm{Sect}(\mathcal{N}, \mathcal{M})$, and we can take $u \in U(\mathcal{N})$ such that $\alpha\bar{\iota} = \mathrm{Ad}(u)\circ\bar{\iota}\alpha$. The choice of u is not unique, but we can easily see $\iota(u)^*\alpha(R) \in (\mathrm{id}, \iota\bar{\iota})$, $u^*\alpha(\bar{R}) \in (\mathrm{id}, \bar{\iota}\iota)$ and

$$(u^*\alpha(\bar{R}))^*\bar{\iota}(\iota(u)\alpha(R)) = (\iota(u)\alpha(R))^*\iota(u^*\alpha(\bar{R})) = \frac{1}{d(\iota)}.$$

Thus we can choose a unique $u_\alpha \in U(\mathcal{N})$ by $\iota(u_\alpha^*)\alpha(R) = R$, and $u_\alpha^*\alpha(\bar{R}) = \bar{R}$.

LEMMA 2.1.
(1) *We have* $\iota(u_{\alpha\beta}) = \alpha\iota(u_\beta)\iota(u_\alpha)$ *for* $\alpha, \beta \in \mathrm{Aut}(\mathcal{M}, \mathcal{N})$.
(2) *We have* $u_\alpha = v\bar{\iota}(v^*)$ *for* $\alpha = \mathrm{Ad}(v)$, $v \in U(\mathcal{N})$.

PROOF. Statement (1) follows from the uniqueness of u_α. Statement (2) can be verified by direct computation.

Define $v_\alpha^{(0)} := 1$ and $v_\alpha^{(k+1)} := v_\alpha^{(k)}(\iota\bar{\iota})^k\iota(u_\alpha)$. Then we have $v_\alpha^{(k)} \in \mathcal{N}_{2k-1}$, $\alpha(\iota\bar{\iota})^k = \mathrm{Ad}(v_\alpha^{(k)}) \circ (\iota\bar{\iota})^k\alpha$, and $v_\alpha^{(k)*}$ satisfies a 1-cocycle identity for α.

Let $\alpha^{(k)} := \mathrm{Ad}(v_\alpha^{(k)*}) \circ \alpha$. Then $\alpha^{(k)}$ preserves the Jones projections for a tunnel $\mathcal{M} \supset \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_{2k} \supset \mathcal{N}_{2k+1}$. (Note the Jones projections are given by $\{(\iota\bar{\iota})^n(RR^*)\}_{n\geq 0} \cup \{(\bar{\iota}\iota)^n(\bar{R}\bar{R}^*)\}_{n\geq 0}$.) Hence $\alpha^{(k)}$ preserves this tunnel globally, and $\alpha^{(k)}(\mathcal{N}_\ell' \cap \mathcal{M}) = \mathcal{N}_\ell' \cap \mathcal{M}$ hold for all $0 \leq \ell \leq 2k+1$. We can see $\alpha^{(k)}|_{\mathcal{N}_{2\ell+1}' \cap \mathcal{M}} = \alpha^{(\ell)}|_{\mathcal{N}_{2\ell+1}' \cap \mathcal{M}}$ for all $\ell \leq k$.

DEFINITION 2.2 ([18]). The *Loi invariant* $\Phi(\alpha)$ of α is defined by $\Phi(\alpha) = \{\alpha^{(k)}|_{\mathcal{N}_{2k+1}' \cap \mathcal{M}}\}_k$.

By using the triviality of the Loi-invariant, we can construct a half braiding unitary $\mathcal{E}(\sigma, \alpha) \in (\sigma\alpha, \alpha\sigma)$ for $[\sigma] \in \Delta$, $\alpha \in \mathrm{Ker}(\Phi)$. (The notion of a half braiding was introduced by Izumi [14], inspired by the work of Xu [23].) Namely, we have the following theorem.

THEOREM 2.3 ([19, Theorem 2.1]). *Let* $\alpha \in \mathrm{Aut}(\mathcal{M}, \mathcal{N})$. *If* $\Phi(\alpha)$ *is trivial, then there exists a unitary* $\mathcal{E}(\sigma, \alpha)$ *for* $[\sigma] \in \Delta$ *such that*

(i) $\mathcal{E}(\sigma, \alpha) \in (\sigma\alpha, \alpha\sigma)$,
(ii) $\mathcal{E}(\sigma_1, \alpha)\sigma_1\big(\mathcal{E}(\sigma_2, \alpha)\big)T = \alpha(T)\mathcal{E}(\sigma_3, \alpha)$ *for any* $[\sigma_i] \in \Delta$, $i = 1, 2, 3$, *and* $T \in (\sigma_3, \sigma_1\sigma_2)$,
(iii) $\mathcal{E}(\sigma, \alpha\beta) = \alpha(\mathcal{E}(\sigma, \beta))\mathcal{E}(\sigma, \alpha)$, $\mathcal{E}(\sigma, \mathrm{Ad}(v)) = v\sigma(v^*)$ *for* $v \in U(\mathcal{N})$.

The second condition is a braiding fusion equation (BFE), and the third condition means that $\mathcal{E}(\sigma, \alpha)^*$ is a 1-cocycle for α.

Here we only explain how to construct $\mathscr{E}(\sigma,\alpha)$, and outline of the proof. See [19] for the details of the proof.

Let $[\sigma]\in {}_{\mathscr{M}}\Delta_{\mathscr{M}}$. Fix $n\in\mathbb{N}$ and an isometry $T\in(\sigma,(\iota\bar{\iota})^n)$. Define $W_T=\alpha(T^*)v_\alpha^{(n)}T$. It is clear that $W_T\in(\sigma\alpha,\alpha\sigma)$. By using $\Phi(\alpha)=1$, we can show that the definition of W_T does not depend on the choice of $T\in(\sigma,(\iota\bar{\iota})^n)$, and that W_T is a unitary.

Next we show that W_T does not depend on n. Take any $\pi\prec\sigma\iota$, and an isometry $S\in(\pi,\sigma\iota)$. Then $\tilde{S}=\sqrt{d(\sigma)d(\iota)d(\pi)^{-1}}S^*\sigma(R)\in(\sigma,\pi\bar{\iota})$ is an isometry [13, Proposition 2.2]. We can easily verify $TS\tilde{S}\in(\sigma,(\iota\bar{\iota})^{n+1})$. Again by the triviality of $\Phi(\alpha)$, we can show $W_T=W_{TS\tilde{S}}$.

Combining these ideas, we know that W_T does not depend on n and T. Hence $\mathscr{E}(\sigma,\alpha):=W_T$ is well-defined. The condition (iii) follows from Lemma 2.1.

To show (ii), fix n,m and isometries $S_1\in(\sigma_1,(\iota\bar{\iota})^n)$, $S_2\in(\sigma_2,(\iota\bar{\iota})^m)$. Then $S_3:=S_1\sigma_1(S_2)T\in(\sigma_3,(\iota\bar{\iota})^{n+m})$ is an isometry. We can show $W_{S_1}\sigma_1(W_{S_2})T=\alpha(T)W_{S_3}$ by using $\Phi(\alpha)=1$. In a similar way, we can construct a half braiding $\mathscr{E}(\sigma,\alpha)$ for each $[\sigma]\in{}_{\mathscr{A}}\Delta_{\mathscr{B}}$, $\mathscr{A},\mathscr{B}\in\{\mathscr{N},\mathscr{M}\}$.

Remark 2.4. We can extend $\mathscr{E}(\sigma,\alpha)$ for a reducible σ as follows. Let $\sigma=\sum_{i=1}^n w_i\sigma_i(x)w_i^*$ and set $\mathscr{E}(\sigma,\alpha):=\sum_i\alpha(w_i)\mathscr{E}(\sigma_i,\alpha)w_i^*$. Then BFE implies $\mathscr{E}(\rho\sigma,\alpha)=\mathscr{E}(\rho,\alpha)\rho(\mathscr{E}(\sigma,\alpha))$.

3. Vanishing of obstructions in the orbifold construction

In this section, we make the following assumption.

Assumption 3.1.

(A1) there exists $\alpha\in\mathrm{Aut}(\mathscr{M},\mathscr{N})$ such that $[\alpha]\in{}_{\mathscr{M}}\Delta_{\mathscr{M}}$, and α gives an outer action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathscr{N}\subset\mathscr{M}$,

(A2) the Loi-invariant of α is trivial,

(A3) there exists a self-conjugate $[\rho]\in{}_{\mathscr{M}}\Delta_{\mathscr{M}}$ such that $[\alpha][\rho]=[\rho]$ and $[\rho]^2\succ[\rho]$.

In (A3), we can choose representatives of $[\alpha]$ and $[\rho]$ such that $\alpha\rho=\rho$ as in [12, Example 3.2]. We remark that this condition yields the triviality of the Connes obstruction of α, and hence (A1) holds. In general, we only have $\mathbb{Z}/n\mathbb{Z}$-kernel if we do not assume (A3). In what follows, we fix this choice of α and ρ.

The crossed product inclusion $\mathscr{N}\rtimes_\alpha\mathbb{Z}/n\mathbb{Z}\subset\mathscr{M}\rtimes_\alpha\mathbb{Z}/n\mathbb{Z}$ is called an orbifold subfactor for $\mathscr{N}\subset\mathscr{M}$, and this construction is called the orbifold construction [6].

As seen in the previous section, we have a half braiding $\mathscr{E}(\sigma, \alpha) \in (\sigma\alpha, \alpha\sigma)$, $[\sigma] \in \Delta$. Once we get half braidings, we can define an extension

$$\sigma \in \mathrm{Mor}(\mathscr{A}, \mathscr{B}) \to \tilde{\sigma} \in \mathrm{Mor}(\mathscr{A} \rtimes_\alpha \mathbb{Z}/n\mathbb{Z}, \mathscr{B} \rtimes_\alpha \mathbb{Z}/n\mathbb{Z})$$

by setting

$$\tilde{\sigma}(\lambda) = \mathscr{E}(\sigma, \alpha)^* \lambda,$$

where λ is an implementing unitary for α, and $\mathscr{A}, \mathscr{B} \in \{\mathscr{N}, \mathscr{M}\}$. The conditions (i) and (iii) in Theorem 2.3 imply that $\tilde{\sigma}$ indeed gives a morphism. The condition (ii) implies $(\sigma_3, \sigma_1\sigma_2) \subset (\tilde{\sigma}_3, \tilde{\sigma}_1\tilde{\sigma}_2)$. Thus the extension $\sigma \to \tilde{\sigma}$ preserves the sector operation, and it is a special case of α-induction studied in [23], [1], [2], [3], [4]. It is easy to see $\hat{\alpha}\tilde{\sigma} = \tilde{\sigma}\hat{\alpha}$, where $\hat{\alpha}$ is the dual action given by $\hat{\alpha}(\lambda) = \omega\lambda$, $\omega = e^{2\pi i/n}$.

It is trivial that $\mathscr{E}(\alpha, \alpha)$ is a scalar with $\mathscr{E}(\alpha, \alpha)^n = 1$, but it may be nontrivial. We say $\mathscr{E}(\alpha, \alpha)$ is an obstruction in the orbifold construction. This notion comes from the following theorem.

THEOREM 3.2. *Assume $\mathscr{E}(\alpha, \alpha) = 1$. Then we have the following.*

(1) *We have $\tilde{\alpha} = \mathrm{Ad}(\lambda)$. Thus $[\tilde{\sigma}] = [\widetilde{\alpha\sigma}]$ holds as sectors. If we have $[\sigma] \neq [\alpha^k\sigma]$ for all $k = 1, 2, \ldots, n-1$, then $\tilde{\sigma}$ is irreducible.*

(2) *We have $(\tilde{\rho}, \tilde{\rho}) = \left\{\sum_{k=0}^{n-1} a_k\lambda^k \mid a_k \in \mathbb{C}\right\} \cong \ell^\infty(\mathbb{Z}/n\mathbb{Z})$. Therefore we have an irreducible decomposition $[\tilde{\rho}] = \bigoplus_{k=0}^{n-1}[\pi_i]$. Here π_k is an irreducible sector corresponding to a minimal projection $p_k = n^{-1}\sum_{\ell=0}^{n-1} \omega^{k\ell}\lambda^\ell$.*

(3) *Let $\hat{\alpha}$ be the dual action. Then $[\hat{\alpha}][\pi_k][\hat{\alpha}^{-1}] = [\pi_{k+1}]$. Thus $d(\pi_k) = d(\rho)/n$.*

(4) *If n is odd, then all $[\pi_k]$ are self-conjugate. If n is even, then $\overline{[\pi_k]} = [\pi_k]$, or $\overline{[\pi_k]} = \left[\pi_{k+\frac{n}{2}}\right]$ hold.*

PROOF. (1) For $a \in \mathscr{M}$, $\tilde{\alpha}(a) = \alpha(a) = \mathrm{Ad}(\lambda)(a)$ holds. For λ, $\tilde{\alpha}(\lambda) = \mathscr{E}(\alpha, \alpha)^*\lambda = \lambda = \mathrm{Ad}(\lambda)(\lambda)$ by the assumption. Hence we have $\tilde{\alpha} = \mathrm{Ad}(\lambda)$.

We show the latter statement for $[\sigma] \in {}_{\mathscr{M}}\Delta_{\mathscr{M}}$. (Other cases can be verified in the same way.) Take $a = \sum_{k=0}^{n-1} a_k\lambda^k \in (\tilde{\sigma}, \tilde{\sigma})$. For $x \in \mathscr{M}$,

$$\sum_{k=0}^{n-1} \sigma(x)a_k\lambda^k = \tilde{\sigma}(x)a = a\tilde{\sigma}(x) = \sum_{k=0}^{n-1} a_k\lambda^k\sigma(x) = \sum_{k=0}^{n-1} a_k\alpha^k(\sigma(x))\lambda^k.$$

Thus $a_k \in (\alpha^k\sigma, \sigma)$. By the assumption, $a_k = 0$ for $1 \leq k \leq n-1$, and $a_0 \in \mathbb{C}1$. Hence $\tilde{\sigma}$ is irreducible. (For a proof of this fact, the condition $\mathscr{E}(\alpha, \alpha) = 1$ is unnecessary.)

(2) Let $a = \sum_{k=0}^{n-1} a_k \lambda^k \in (\tilde{\rho}, \tilde{\rho})$. In a similar way as above, we get $a_k \in (\alpha^k \rho, \rho)$. Since we have chosen α and ρ so that $\alpha\rho = \rho$, $a_k \in \mathbb{C}1$.

If we apply BFE for $\sigma_1 = \alpha$, $\sigma_2 = \sigma_3 = \rho$ and $T = 1 \in (\rho, \alpha\rho) = (\rho, \rho)$, we get $\alpha(\mathscr{E}(\rho, \alpha)) = \mathscr{E}(\alpha, \alpha)^* \mathscr{E}(\rho, \alpha)$. Thus we have $\alpha(\mathscr{E}(\rho, \alpha)) = \mathscr{E}(\rho, \alpha)$ by the assumption. Then we have

$$\tilde{\rho}(\lambda)a = \sum_{k=0}^{n-1} \mathscr{E}(\rho, \alpha)^* a_k \lambda^{k+1},$$

and

$$\begin{aligned} a\tilde{\rho}(\lambda) = \left(\sum_{k=0}^{n-1} a_k \lambda^k\right) \mathscr{E}(\rho, \alpha)^* \lambda &= \sum_{k=0}^{n-1} a_k \alpha^k (\mathscr{E}(\rho, \alpha)^*) \lambda^{k+1} \\ &= \sum_{k=0}^{n-1} \mathscr{E}(\rho, \alpha)^* a_k \lambda^{k+1}. \end{aligned}$$

Thus $a \in (\tilde{\rho}, \tilde{\rho})$, and we obtain $(\tilde{\rho}, \tilde{\rho}) = \left\{ \sum_{k=0}^{n-1} a_k \lambda^k \mid a_k \in \mathbb{C} \right\}$.

(3) Fix an isometry v_k with $v_k v_k^* = p_k$, and set $u = \hat{\alpha}(v_k^*) v_{k+1}$. Since $\hat{\alpha}(p_k) = p_{k+1}$, we can easily see u is a unitary. Again by $\hat{\alpha}(p_k) = p_{k+1}$, we have

$$\begin{aligned} \hat{\alpha}\pi_k \hat{\alpha}^{-1}(x) &= \hat{\alpha}(v_k^*) \hat{\alpha} \tilde{\rho} \hat{\alpha}^{-1}(x) \hat{\alpha}(v_k) = \hat{\alpha}(v_k^*) \tilde{\rho}(x) \hat{\alpha}(v_k) \\ &= \hat{\alpha}(v_k^* p_k) \tilde{\rho}(x) \hat{\alpha}(p_k v_k) = \hat{\alpha}(v_k^*) p_{k+1} \tilde{\rho}(x) p_{k+1} \hat{\alpha}(v_k) \\ &= u v_{k+1}^* \tilde{\rho}(x) v_{k+1} u = \mathrm{Ad}(u) \pi_{k+1}(x). \end{aligned}$$

(4) Since ρ is self-conjugate, there exists k with $\overline{[\pi_0]} = [\pi_k]$. By considering the conjugate of $[\pi_k] = \hat{\alpha}^k [\pi_0] \hat{\alpha}^{-k}$, we get

$$[\pi_0] = \overline{[\pi_k]} = \hat{\alpha}^k \overline{[\pi_0]} \hat{\alpha}^{-k} = \hat{\alpha}^k [\pi_k] \hat{\alpha}^{-k}$$

and hence $[\pi_0] = [\pi_{2k}]$ holds. Thus $k = 0$ if n is odd. In this case, all $[\pi_i]$ are self-conjugate. If $n = 2n'$, then k must be 0 or n'. If $k = 0$, then all $[\pi_i]$ is self-conjugate. If $k = n'$, then we have $\overline{[\pi_i]} = [\pi_{i+n'}]$ for all i.

Remark 3.3. (1) It is easy to see $\tilde{\iota}$ is an inclusion map $\mathscr{N} \rtimes_\alpha \mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathscr{M} \rtimes_\alpha \mathbb{Z}/n\mathbb{Z}$, and $\widetilde{\iota\bar{\iota}}$ is a canonical endomorphism for this inclusion.

(2) Even if we do not assume $\mathscr{E}(\alpha, \alpha) = 1$, we get similar results. For example, the statement corresponding to (2) is the following:

Let $\mathscr{E}(\alpha, \alpha)$ be a primitive ℓ-th root, and set $n/\ell = m$. Then the irreducible decomposition of $\tilde{\rho}$ is $\tilde{\rho} = \bigoplus_{k=0}^{m-1} [\pi_k]$.

(3) As we mentioned before Theorem 3.2, $\mathscr{E}(\alpha, \alpha)^n = 1$ follows from the triviality of the Connes obstruction of α. In general, $\mathscr{E}(\alpha, \alpha)^n$ is equal to the Connes obstruction of α [17, Lemma 2.3].

By Theorem 3.2, the graph change occurs by the orbifold construction under the assumption $\mathscr{E}(\alpha, \alpha) = 1$. (See also the following examples.)

We would like to determine when $\mathscr{E}(\alpha, \alpha) = 1$. We have the following sufficient condition.

THEOREM 3.4. *Put* $m := \dim(\rho, \rho^2)$. *If* m *and* n *are relatively prime, then we have* $\mathscr{E}(\alpha, \alpha) = 1$.

PROOF. This proof is inspired by the computation in [12]. As explained in the proof of Theorem 3.2, we have $\alpha(\mathscr{E}(\rho, \alpha)) = \mathscr{E}(\alpha, \alpha)^*\mathscr{E}(\rho, \alpha)$. Put $U = \mathscr{E}(\rho, \alpha)^*$, and $a = \mathscr{E}(\alpha, \alpha)$. Of course we have $a^n = 1$.

Since $(\rho\alpha, \alpha\rho) = (\rho\alpha, \rho)$, we have $\rho\alpha = \mathrm{Ad}(U)\rho$, and hence $U \in (\rho, \rho\alpha)$.

We also have $U \in (\rho^2, \rho^2)$ due to $\alpha\rho = \rho$. Let $z \in (\rho^2, \rho^2)$ be a minimal central projection corresponding to the irreducible component ρ of ρ^2. Fix an orthonormal basis $\{T_i\}_{i=1}^m \subset (\rho, \rho^2)$. We have $\sum_{i=1}^m T_i T_i^* = z$, and

$$U = \sum_{i=1}^{m} d_{ij} T_i T_j^* + (1 - z)U = (T_1, \dots, T_m) D \begin{pmatrix} T_1^* \\ \vdots \\ T_m^* \end{pmatrix} + (1 - z)U,$$

for some unitary matrix $D = (d_{ij}) \in M_m(\mathbb{C})$.

Since α acts on (ρ, ρ^2) as a unitary and $\alpha(z) = z$, there exists a unitary matrix $V \in M_m(\mathbb{C})$ such that

$$(\alpha(T_1), \dots, \alpha(T_m)) = (T_1, \dots, T_m)V.$$

By the condition $\alpha(U) = aU$, we get $VDV^* = aD$. By taking determinants, we get $\det(D) = a^m \det(D)$. Hence we get $a^m = 1$. Since m and n are relatively prime, we have $a = 1$.

REMARK 3.5. (1) In the above proof, if we have $\mathrm{Tr}(D) \neq 0$, then $a = 1$ follows immediately. However it does not seem easy to determine $\mathrm{Tr}(D) \neq 0$ in the general case.

(2) Let us assume that ${}_{\mathcal{M}}\Delta_{\mathcal{M}}$ is a near group category, i.e., ${}_{\mathcal{M}}\Delta_{\mathcal{M}} = \{[\alpha_g]\}_{g \in G} \cup \{[\rho]\}$ for some action α of a finite group G. Gannon and Evans show that $\dim(\rho, \rho^2)$ is $|G| - 1$ or multiple of $|G|$ in [5]. (In the former case, $|G|$ is prime.)

EXAMPLE 3.6. Let $\mathcal{N} \subset \mathcal{M}$ be a Jones subfactor with principal graph A_{4n-3} [15]. Sectors of $\mathcal{N} \subset \mathcal{M}$ appear as follows:

$$[\rho_0] - [\rho_1] - \cdots - [\rho_{2n-2}] - \cdots - [\rho_{4n-4}]$$

Here $\rho_0 = \mathrm{id}_{\mathcal{M}}$, and $\rho_1 = \iota$. Then $\alpha := \rho_{4n-4}$ is an automorphism of $\mathcal{N} \subset \mathcal{M}$ with period 2. It is well known that all $[\rho_{2k}]$ are self-conjugate, and $[\alpha][\rho_k] = [\rho_{4n-4-k}]$ hold for $0 \leq k \leq 2n-2$. (See [11].) In particular, the sector $[\rho_{2n-2}]$ is self-conjugate, and $[\alpha][\rho_{2n-2}] = [\rho_{2n-2}]$ holds. Moreover, we have $\dim(\rho_{2n-2}, \rho_{2n-2}^2) = 1$. Thus the assumption of Theorem 3.4 is satisfied, and the obstruction vanishes. By Theorem 3.2, we have $[\tilde{\rho}_k] = [\tilde{\rho}_{4n-4-k}]$ for $0 \leq k < 2n-2$, and $[\tilde{\rho}_{2n-2}] = [\pi_0] \oplus [\pi_1]$. Hence the principal graph of the orbifold subfactor is a Dynkin diagram D_{2n}.

Since there exists essentially only one biunitary connection on D_n (see [16, Section 3] for details of the proof), the above argument shows the uniqueness of the flat connection on D_{2n}.

EXAMPLE 3.7. Let $\mathcal{N} \subset \mathcal{M}$ be a subfactor with principal graph $E_6^{(1)}$. Then ${}_{\mathcal{M}}\Delta_{\mathcal{M}}$ is $\{[\mathrm{id}], [\alpha], [\alpha]^2, [\rho]\}$ with the following fusion rule:

$$[\alpha]^3 = [\mathrm{id}], \quad [\alpha][\rho] = [\rho][\alpha] = [\rho], \quad [\rho]^2 = [\mathrm{id}] \oplus [\alpha] \oplus [\alpha^2] \oplus 2[\rho]$$

We can take α as $\alpha^3 = \mathrm{id}$, and α gives an outer action of $\mathbb{Z}/3\mathbb{Z}$ on $\mathcal{N} \subset \mathcal{M}$ with trivial Loi invariant. Thus Theorem 3.4 can be applied. In this case, we have

$$[\tilde{\mathrm{id}}] = [\tilde{\alpha}] = [\tilde{\alpha}^2], \quad [\tilde{\rho}] = [\pi_0] \oplus [\pi_1] \oplus [\pi_2].$$

Therefore, the principal graph of an orbifold subfactor is $D_4^{(1)}$. (Note the statistical dimension of ρ is $d(\rho) = 3$.) There exist two subfactors with principal graph $D_4^{(1)}$, which arise as the crossed product by $\mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, respectively. Condition (4) of Theorem 3.2 implies that the orbifold subfactor is the crossed product by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The subfactor treated above is an $SU(3)_3$ subfactor. (See [20], [6] for $SU(N)_\ell$ subfactors.) We can apply our main theorem to $SU(3)_{3k}$-subfactors for $k+1 \not\equiv 0 \pmod 3$. Indeed, there exist $\alpha \in \mathrm{Aut}(\mathcal{M}, \mathcal{N})$ with $\alpha^3 = \mathrm{id}$, $\Phi(\alpha) = \mathrm{id}$, and a unique self-conjugate sector $[\rho]$ fixed by α for an $SU(3)_{3k}$ subfactor $\mathcal{N} \subset \mathcal{M}$. (The sector $[\rho]$ corresponds to a young diagram $(2k, k, 0)$.) We have $\dim(\rho, \rho^2) = k+1$ by applying the Littlewood-Richardson rule for $SU_q(3)_{3k}$ [8]. (When $k+1 \equiv 0 \pmod 3$, we can not apply Theorem 3.4. However, the obstruction vanishes in this case [6].)

If the assumption of Theorem 3.4 is not satisfied, $\mathscr{E}(\alpha, \alpha)$ may be fail to be 1.

EXAMPLE 3.8 ([12, Example 3.4]). Let $\mathscr{N} \subset \mathscr{M}$ be a subfactor with principal graph E_6. Then ${}_{\mathscr{M}}\Delta_{\mathscr{M}} = \{[\mathrm{id}], [\alpha], [\rho]\}$ and they obey the following fusion rule:

$$[\rho^2] = [\mathrm{id}] \oplus [\alpha] \oplus 2[\rho], \quad [\alpha]^2 = [\mathrm{id}], \quad [\alpha][\rho] = [\rho].$$

We have $\mathrm{Ker}(\Phi) = \mathrm{Aut}(\mathscr{M}, \mathscr{N})$, and hence the Loi invariant of α is trivial. Izumi showed that one can take isometries $S_1 \in (\mathrm{id}, \rho^2)$, $S_2 \in (\alpha, \rho^2)$, $S_3, S_4 \in (\rho, \rho^2)$ and a unitary $U \in (\rho, \rho\alpha)$ as

$$S_2 = \alpha(S_1), \ \ \alpha(S_3) = S_3, \ \ \alpha(S_4) = -S_4, \ \ U = S_1S_1^* - S_2S_2^* + S_3S_4^* + S_4S_3^*.$$

Then $\alpha(U) = -U$, and hence $\mathscr{E}(\alpha, \alpha) = -1$. In this case, the graph change does not occur by the orbifold construction.

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