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QUADRATIC GRÖBNER BASES ARISING FROM PARTIALLY ORDERED SETS

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Abstract

The order polytope $\mathscr{O}(P)$ and the chain polytope $\mathscr{C}(P)$ associated to a partially ordered set P are studied. In this paper, we introduce the convex polytope $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ which is the convex hull of $\mathscr{O}(P) \cup (-\mathscr{C}(Q))$, where both P and Q are partially ordered sets with $|P| = |Q| = d$. It will be shown that $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ is a normal and Gorenstein Fano polytope by using the theory of reverse lexicographic squarefree initial ideals of toric ideals.

Introduction

A convex polytope $\mathscr{P} \subset \mathbb{R}^d$ is *integral* if all vertices belong to $\mathbb{Z}^d$. An integral convex polytope $\mathscr{P} \subset \mathbb{R}^d$ is *normal* if, for each integer $N > 0$ and for each $\mathbf{a} \in N\mathscr{P} \cap \mathbb{Z}^d$, there exist $\mathbf{a}_1, \dots, \mathbf{a}_N \in \mathscr{P} \cap \mathbb{Z}^d$ such that $\mathbf{a} = \mathbf{a}_1 + \cdots + \mathbf{a}_N$, where $N\mathscr{P} = \{N\alpha \mid \alpha \in \mathscr{P}\}$. Furthermore, an integral convex polytope $\mathscr{P} \subset \mathbb{R}^d$ is *Fano* if the origin of $\mathbb{R}^d$ is the unique integer point belonging to the interior of $\mathscr{P}$. A Fano polytope $\mathscr{P} \subset \mathbb{R}^d$ is *Gorenstein* if its dual polytope

$$\mathscr{P}^\vee := \{\, \mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{y} \rangle \le 1 \text{ for all } \mathbf{y} \in \mathscr{P} \,\}$$

is integral as well. A Gorenstein Fano polytope is also said to be a reflexive polytope.

In recent years, the study of Gorenstein Fano polytopes has been more vigorous. It is known that Gorenstein Fano polytopes correspond to Gorenstein Fano varieties, and they are related with mirror symmetry (see, e.g., [1], [2]). On the other hand, to find new classes of Gorenstein Fano polytopes is one of the most important problem.

As a way to construct normal Gorenstein Fano polytopes, taking the *centrally symmetric configuration* [11] of an integer matrix is a powerful tool. In [11], it is shown that, for any matrix A with $\operatorname{rank}(A) = d$ such that all nonzero maximal minors of A are ± 1, the integral convex polytope arising from the centrally symmetric configuration of A is normal Gorenstein Fano. Moreover,

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in [12], a way to construct non-symmetric normal Gorenstein Fano polytopes is introduced. In this paper, we treat the integral convex polytopes arising from combining the order polytopes and chain polytopes associated with two partially ordered sets.

Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite partially ordered sets (posets, for short) with $|P| = |Q| = d$. A subset I of P is called a *poset ideal* of P if $p_i \in I$ and $p_j \in P$ together with $p_j \leq p_i$ guarantee $p_j \in I$. Note that the empty set $\emptyset$ as well as P itself is a poset ideal of P. Let $\mathscr{J}(P)$ denote the set of poset ideals of P. A subset A of Q is called an *antichain* of Q if q_i and q_j belonging to A with $i \neq j$ are incomparable. In particular, the empty set $\emptyset$ and each 1-element subset $\{q_j\}$ are antichains of Q. Let $\mathscr{A}(Q)$ denote the set of antichains of Q.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the canonical unit coordinate vectors of $\mathbb{R}^d$. Then, for each subset $I \subset P$ and for each subset J of Q, we define the (0, 1)-vectors $\rho(I) = \sum_{p_i \in I} \mathbf{e}_i$ and $\rho(J) = \sum_{q_j \in J} \mathbf{e}_j$, respectively. In particular $\rho(\emptyset)$ is the origin $\mathbf{0}$ of $\mathbb{R}^d$. Recall that the *order polytope* $\mathscr{O}(P)$, [13, Definition 1.1], is the convex hull of $\{\rho(I) \mid I \in \mathscr{J}(P)\}$ and the *chain polytope* $\mathscr{C}(Q)$, [13, Definition 2.1], is the convex hull of $\{\rho(A) \mid A \in \mathscr{J}(Q)\}$.

Now, we define the convex polytope $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ as the convex hull of $\mathscr{O}(P) \cup -(\mathscr{C}(Q))$, where $-\mathscr{C}(Q) = \{-\beta \mid \beta \in \mathscr{C}(Q)\}$. This is a kind of $(-1, 0, 1)$-polytope, that is, each of its vertices belongs to $\{-1, 0, 1\}^d$. Note that $\dim \Gamma(\mathscr{O}(P), -\mathscr{C}(Q)) = d$. Moreover, since $\rho(P) = \mathbf{e}_1 + \cdots + \mathbf{e}_d \in \mathscr{O}(P)$ and $\rho(\{q_j\}) = \mathbf{e}_j \in \mathscr{C}(Q)$ for $1 \leq j \leq d$, we have that the origin $\mathbf{0}$ of $\mathbb{R}^d$ belongs to the interior of $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$.

$\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ is an integral convex polytope arising from combining the order polytope $\mathscr{O}(P)$ and the chain polytope $\mathscr{C}(Q)$ associated with two posets P and Q. We consider the question when such a polytope is normal Gorenstein Fano. This kind of question has been studied. It is known that $\Gamma(\mathscr{O}(P), -\mathscr{O}(P))$ is normal Gorenstein Fano for any poset P [8] and $\Gamma(\mathscr{O}(P), -\mathscr{O}(Q))$ is normal Gorenstein Fano if P and Q possess a common linear extension [7]. Moreover, it is shown that $\Gamma(\mathscr{C}(P), -\mathscr{C}(Q))$ is normal Gorenstein Fano for any posets P and Q [12].

In this paper, we prove that $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ is normal Gorenstein Fano for any posets P and Q by using the theory of reverse lexicographic squarefree initial ideals of toric ideals. For fundamental material on Gröbner bases and toric ideals, see [5].

1. Squarefree quadratic Gröbner basis

Let, as before, $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite partially ordered sets with the same cardinality. For a poset ideal $I \in \mathscr{J}(Q)$, we write

$\max(I)$ for the set of maximal elements of I. In particular, $\max(I)$ is an antichain of Q. Note that for each antichain A of Q, there exists a poset ideal I of Q such that $A = \max(I)$.

Let $K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}, s]$ denote the Laurent polynomial ring in $2d + 1$ variables over a field K. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$, then $\mathbf{t}^\alpha s$ is the Laurent monomial $t_1^{\alpha_1} \cdots t_d^{\alpha_d} s \in K[\mathbf{t}, \mathbf{t}^{-1}, s]$. In particular $\mathbf{t}^{\mathbf{0}} s = s$. We define the *toric ring* of $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ as the subring $K[\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))]$ of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ which is generated by those Laurent monomials $\mathbf{t}^\alpha s$ with $\alpha \in \Gamma(\mathscr{O}(P), -\mathscr{C}(Q)) \cap \mathbb{Z}^d$. Let

$$K[\mathbf{x}, \mathbf{y}, z] = K\left[\{x_I\}_{\emptyset \neq I \in \mathscr{J}(P)} \cup \{y_{\max(J)}\}_{\emptyset \neq J \in \mathscr{J}(Q)} \cup \{z\}\right]$$

denote the polynomial ring over K and define the surjective ring homomorphism $\pi\colon K[\mathbf{x}, \mathbf{y}, z] \to K[\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))]$ by the following:

- $\pi(x_I) = \mathbf{t}^{\rho(I)} s$, where $\emptyset \neq I \in \mathscr{J}(P)$;
- $\pi(y_{\max(J)}) = \mathbf{t}^{-\rho(\max(J))} s$, where $\emptyset \neq J \in \mathscr{J}(Q)$;
- $\pi(z) = s$.

Then the *toric ideal* $I_{\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))}$ of $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ is the kernel of π.

Let $<$ denote a reverse lexicographic order on $K[\mathbf{x}, \mathbf{y}, z]$ satisfying

- $z < y_{\max(J)} < x_I$;
- $x_{I'} < x_I$ if $I' \subset I$;
- $y_{\max(J')} < y_{\max(J)}$ if $J' \subset J$,

and $\mathscr{G}$ the set of the following binomials:

(i) $x_I x_{I'} - x_{I \cup I'} x_{I \cap I'}$;

(ii) $y_{\max(J)} y_{\max(J')} - y_{\max(J \cup J')} y_{\max(J * J')}$;

(iii) $x_I y_{\max(J)} - x_{I \setminus \{p_i\}} y_{\max(J) \setminus \{q_i\}}$,

where

- $x_\emptyset = y_\emptyset = z$;
- I and I' are poset ideals of P which are incomparable in $\mathscr{J}(P)$;
- J and J' are poset ideals of Q which are incomparable in $\mathscr{J}(Q)$;
- $J * J'$ is the poset ideal of Q generated by $\max(J \cap J') \cap (\max(J) \cup \max(J'))$;
- p_i is a maximal element of I and q_i is a maximal element of J.

First, we have

THEOREM 1.1. *Work with the situation above. Then $\mathscr{G}$ is a Gröbner basis of $I_{\Gamma(\mathscr{O}(P),-\mathscr{C}(Q))}$ with respect to $<$.*

PROOF. It is clear that $\mathscr{G} \subset I_{\Gamma(\mathscr{O}(P),-\mathscr{C}(Q))}$. For a binomial $f = u - v$, u is called the *first* monomial of f and v is called the *second* monomial of f. We note that the initial monomial of each of the binomials (i)–(iii) with respect to $<$ is its first monomial. Let $\mathrm{in}_<(\mathscr{G})$ denote the set of initial monomials of binomials belonging to $\mathscr{G}$. It follows from [10, (0.1)] that, in order to show that $\mathscr{G}$ is a Gröbner basis of $I_{\Gamma(\mathscr{O}(P),-\mathscr{C}(Q))}$ with respect to $<$, we must prove the following assertion: (♣) if u and v are monomials belonging to $K[\mathbf{x}, \mathbf{y}, z]$ with $u \neq v$ such that $u \notin \langle \mathrm{in}_<(\mathscr{G}) \rangle$ and $v \notin \langle \mathrm{in}_<(\mathscr{G}) \rangle$, then $\pi(u) \neq \pi(v)$.

Let $u, v \in K[\mathbf{x}, \mathbf{y}, z]$ be monomials with $u \neq v$. Write

$$u = z^{\alpha} x_{I_1}^{\xi_1} \cdots x_{I_a}^{\xi_a} y_{\max(J_1)}^{\nu_1} \cdots y_{\max(J_b)}^{\nu_b}, \quad v = z^{\alpha'} x_{I'_1}^{\xi'_1} \cdots x_{I'_{a'}}^{\xi'_{a'}} y_{\max(J'_1)}^{\nu'_1} \cdots y_{\max(J'_{b'})}^{\nu'_{b'}},$$

where

- $\alpha \geq 0, \alpha' \geq 0$;
- $I_1, \ldots, I_a, I'_1, \ldots, I'_{a'} \in \mathscr{J}(P) \setminus \{\emptyset\}$;
- $J_1, \ldots, J_b, J'_1, \ldots, J'_{b'} \in \mathscr{J}(Q) \setminus \{\emptyset\}$;
- $\xi_1, \ldots, \xi_a, \nu_1, \ldots, \nu_b, \xi'_1, \ldots, \xi'_{a'}, \nu'_1, \ldots, \nu'_{b'} > 0$,

and where u and v are relatively prime with $u \notin \langle \mathrm{in}_<(\mathscr{G}) \rangle$ and $v \notin \langle \mathrm{in}_<(\mathscr{G}) \rangle$. Note that either $\alpha = 0$ or $\alpha' = 0$. Hence we may assume that $\alpha' = 0$. Thus

$$u = z^{\alpha} x_{I_1}^{\xi_1} \cdots x_{I_a}^{\xi_a} y_{\max(J_1)}^{\nu_1} \cdots y_{\max(J_b)}^{\nu_b}, \quad v = x_{I'_1}^{\xi'_1} \cdots x_{I'_{a'}}^{\xi'_{a'}} y_{\max(J'_1)}^{\nu'_1} \cdots y_{\max(J'_{b'})}^{\nu'_{b'}}.$$

By using (i) and (ii), it follows that

- $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_a$ and $J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_b$;
- $I'_1 \subsetneq I'_2 \subsetneq \cdots \subsetneq I'_{a'}$ and $J'_1 \subsetneq J'_2 \subsetneq \cdots \subsetneq J'_{b'}$.

Furthermore, by virtue of [3] and [6], it suffices to discuss u and v with $(a, a') \neq (0, 0)$ and $(b, b') \neq (0, 0)$.

Since $I_a \neq I'_{a'}$, we may assume that $I_a \setminus I'_{a'} \neq \emptyset$. Then there exists a maximal element p_{i^*} of I_a with $p_{i^*} \notin I'_{a'}$.

Now, suppose that $\pi(u) = \pi(v)$. Then we have

$$\sum_{\substack{I \in \{I_1, \ldots, I_a\} \\ p_i \in I}} \xi_I - \sum_{\substack{J \in \{J_1, \ldots, J_b\} \\ q_i \in \max(J)}} \nu_J = \sum_{\substack{I' \in \{I'_1, \ldots, I'_{a'}\} \\ p_i \in I'}} \xi'_{I'} - \sum_{\substack{J' \in \{J'_1, \ldots, J'_{b'}\} \\ q_i \in \max(J')}} \nu'_{J'},$$

for all $1 \leq i \leq d$, by comparing the degrees of t_i. Since $p_{i^*} \notin I'_{a'}$, one has

$$\sum_{\substack{I \in \{I_1, \ldots, I_a\} \\ p_{i^*} \in I}} \xi_I - \sum_{\substack{J \in \{J_1, \ldots, J_b\} \\ q_{i^*} \in \max(J)}} \nu_J = - \sum_{\substack{J' \in \{J'_1, \ldots, J'_{b'}\} \\ q_{i^*} \in \max(J')}} \nu'_{J'} \leq 0.$$

Moreover, since p_{i^*} belongs to I_a, we also have

$$\sum_{\substack{I \in \{I_1, \ldots, I_a\} \\ p_{i^*} \in I}} \xi_I > 0.$$

Hence there exists an integer c with $1 \leq c \leq b$ such that q_{i^*} is a maximal element of J_c. Therefore we have $x_{I_a} y_{\max(J_c)} \in \langle \mathrm{in}_<(\mathscr{G}) \rangle$, but this is a contradiction.

This theorem guarantees that the toric ideal $I_{\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))}$ possesses a squarefree initial ideal with respect to a reverse lexicographic order for which the variable corresponding to the column vector $[0, \ldots, 0, 1]^t$ is smallest. Therefore, we have the following corollary by using [8, Lemma 1.1].

Corollary 1.2. *For any partially ordered sets P and Q with $|P| = |Q| = d$, $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ is a normal Gorenstein Fano polytope.*

2. Example and remark

For the final part of this paper, we give some examples.

Let $\mathscr{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d. Given integers $t = 1, 2, \ldots$, let $i(\mathscr{P}, t) := \#(t\mathscr{P} \cap \mathbb{Z}^d)$. It is known that $i(\mathscr{P}, t)$ is a polynomial in t of degree d. This polynomial is called the *Ehrhart polynomial* of $\mathscr{P}$. The generating function of $i(\mathscr{P}, t)$ satisfies

$$1 + \sum_{t=1}^{\infty} i(\mathscr{P}, t)\lambda^t = \frac{\delta_{\mathscr{P}}(\lambda)}{(1-\lambda)^{d+1}},$$

where $\delta_{\mathscr{P}}(\lambda) = \sum_{i=0}^{d} \delta_i \lambda^i$ is a polynomial in t of degree $\leq d$. Then the vector $(\delta_0, \ldots, \delta_d)$ is called the *δ-vector* of $\mathscr{P}$. It is known that a Fano polytope $\mathscr{P} \subset \mathbb{R}^d$ is Gorenstein if and only if $\delta_i = \delta_{d-i}$ for all $i = 0, 1, \ldots, d$ (see [4]).

Example 2.1. Let $P = \{p_1 < p_2 < \cdots < p_d\}$ and $Q = \{q_{i_1} < q_{i_2} < \cdots < q_{i_d}\}$ be chains $(1 \leq i_1, \ldots, i_d \leq d)$ and $(\delta_0, \ldots, \delta_d)$ be the δ-vector of $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$. By [9, Corollary 1.5], we have $i(\Gamma(\mathscr{O}(P), -\mathscr{C}(Q)), t) = i(\Gamma(\mathscr{C}(Q), -\mathscr{C}(P)), t)$, where $\Gamma(\mathscr{C}(Q), -\mathscr{C}(P))$ is the convex hull of $\mathscr{C}(Q) \cup$

$(-\mathscr{C}(P))$. Since $\Gamma(\mathscr{C}(Q), -\mathscr{C}(P))$ is the same as the convex hull of the centrally symmetric configuration of $d \times d$ identity matrix, hence $\delta_{\mathscr{P}}(\lambda) = (1+\lambda)^d$. In particular, $\delta_i = \binom{d}{i}$ for all $i = 0, 1, \ldots, d$.

On the other hand, the convex polytope $\Gamma(\mathscr{O}(P), -\mathscr{O}(Q))$ which is the convex hull of $\mathscr{O}(P) \cup (-\mathscr{O}(Q))$ is not Gorenstein Fano [7, Lemma 1.1]. Indeed, the δ-vector of $\Gamma(\mathscr{O}(P), -\mathscr{O}(Q))$ is $(1, d, 0, \ldots, 0)$.

EXAMPLE 2.2. Let $P = \{p_1 < p_3, p_2 < p_4\}$ be a partially ordered set and $Q = \{q_1, q_2, q_3, q_4\}$ be a partially ordered set such that the shape of its Hasse diagram is the same as that of P. If $Q = \{q_1 < q_3, q_2 < q_4\}$ then the δ-vector of $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ is $(1, 12, 28, 12, 1)$. On the other hand, if $Q = \{q_1 < q_2, q_3 < q_4\}$ or $\{q_1 < q_4, q_2 < q_3\}$, then the δ-vector of $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ is $(1, 12, 26, 12, 1)$.

REMARK 2.3. We proved that the convex polytope $\Gamma(\mathscr{C}(P), -\mathscr{C}(Q))$ is normal Gorenstein Fano for all partially ordered sets P, Q with $|P| = |Q|$ ([9, Corollary 1.3]). Moreover, we proved that the Ehrhart polynomial of $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ is the same as that of $\Gamma(\mathscr{C}(P), -\mathscr{C}(Q))$ for all partially ordered sets P, Q with $|P| = |Q|$. In addition, if P and Q possess a common linear extension, these polytopes $\Gamma(\mathscr{O}(P), -\mathscr{O}(Q))$, $\Gamma(\mathscr{O}(P), -\mathscr{C}(Q))$ and $\Gamma(\mathscr{C}(P), -\mathscr{C}(Q))$ have the same Ehrhart polynomial ([9, Theorem 1.1]).

One of the future problems is to determine the δ-vectors of the above polytopes in terms of the partially ordered sets P and Q.

REFERENCES

1. Batyrev, V. V., *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. 3 (1994), no. 3, 493–535.
2. Cox, D. A., Little, J. B., and Schenck, H. K., *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.
3. Hibi, T., *Distributive lattices, affine semigroup rings and algebras with straightening laws*, Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 93–109.
4. Hibi, T., *Algebraic combinatorics on convex polytopes*, Carslaw Publications, Glebe, 1992.
5. Hibi, T. (ed.), *Gröbner bases: Statistics and software systems*, Springer, Tokyo, 2013.
6. Hibi, T., and Li, N., *Chain polytopes and algebras with straightening laws*, Acta Math. Vietnam. 40 (2015), no. 3, 447–452.
7. Hibi, T., and Matsuda, K., *Quadratic Gröbner bases of twinned order polytopes*, European J. Combin. 54 (2016), 187–192.
8. Hibi, T., Matsuda, K., Ohsugi, H., and Shibata, K., *Centrally symmetric configurations of order polytopes*, J. Algebra 443 (2015), 469–478.
9. Hibi, T., Matsuda, K., and Tsuchiya, A., *Gorenstein Fano polytopes arising from order polytopes and chain polytopes*, preprint arXiv:1507.03221 [math.CO], 2015.
10. Ohsugi, H., and Hibi, T., *Quadratic initial ideals of root systems*, Proc. Amer. Math. Soc. 130 (2002), no. 7, 1913–1922.

11. Ohsugi, H., and Hibi, T., *Centrally symmetric configurations of integer matrices*, Nagoya Math. J. 216 (2014), 153–170.
12. Ohsugi, H., and Hibi, T., *Reverse lexicographic squarefree initial ideals and Gorenstein Fano polytopes*, to appear in J. Commut. Algebra.
13. Stanley, R. P., *Two poset polytopes*, Discrete Comput. Geom. 1 (1986), no. 1, 9–23.

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